# Indian Institute of Technology Guwahati

MA 102 Mathematics-II
Tutorial Sheet-1

**Limit, Continuity and Differentiability**

## Section-I: Tutorial Problems

1. Identify the points, if any, where the following functions fail to be continuous:

   (a) $f(x,y) = \begin{cases} xy, & \text{if } xy \geq 0; \\ -xy, & \text{if } xy < 0. \end{cases}$

   (b) $f(x,y) = \begin{cases} xy, & \text{if } xy \text{ is rational}; \\ -xy, & \text{if } xy \text{ is irrational}. \end{cases}$

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

   Show that the function satisfies the following:

   (a) The iterated limits $\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right]$ and $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]$ exist and equals 0;

   (b) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist;

   (c) $f(x,y)$ is not continuous at $(0,0)$;

   (d) the partial derivatives exist at $(0,0)$.

3. Let $f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and 0, otherwise. Show that $f$ is differentiable at every point of $\mathbb{R}^2$ but the partial derivatives are not continuous at $(0,0)$.

4. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function with $f_x(x,y) = f_y(x,y) = 0$ for all $(x,y)$. Then show that $f(x,y) = c$, a constant.

## Section-II: Practice Problems

1. Examine whether the following limits exist and find their values if they exist.

   $(a)$ $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2}$  $(b)$ $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$  $(c)$ $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$

   $(d)$ $\lim_{(x,y)\to(0,0)} \frac{|x|}{y^2}e^{-|x|/y^2}$  $(e)$ $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$  $(f)$ $\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$

2. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,

   (a) $f(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } 0 < y < x^2, \\ 0 & \text{otherwise}. \end{cases}$

(b) $f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

3. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, if for all $(x,y) \in \mathbb{R}^2$,

   (a) $f(x,y) = \begin{cases} \frac{xy}{x-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$ (b) $f(x,y) = \begin{cases} \frac{xy}{x^2-y^2} & \text{if } x^2 \neq y^2, \\ 0 & \text{if } x^2 = y^2. \end{cases}$

4. Let $f(x,y) = \frac{x^m y^n}{(x^2+y^2)^p}$; if $x^2+y^2 \neq 0$ and $f(0,0) = 0$. Find condition on $(m,n,p)$ for which $f$ is continuous at $(0,0)$. Further, find condition for the function $f$ to be bounded on $\mathbb{R}^2$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(X+Y) = f(X) + f(Y)$ and $f(\alpha X) = \alpha f(X)$, for all $X, Y \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Show that $f$ is continuous on $\mathbb{R}^2$.

6. Examine the differentiability of $f$ at $(0,0)$, where

   (a) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \sqrt{|xy|}$ for all $(x,y) \in \mathbb{R}^2$.

   (b) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = ||x| - |y|| - |x| - |y|$ for all $(x,y) \in \mathbb{R}^2$.

   (c) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$

   (d) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

   (e) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} \frac{\sin(x^2y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

   (f) $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x,y) = \begin{cases} (\sin^2 x + x^2 \sin \frac{1}{x},\ y^2) & \text{if } x \neq 0, \\ (0, y^2) & \text{if } x = 0. \end{cases}$

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be map $f(x,y) = \begin{cases} \sqrt{x^2+y^2}\ \sin\left(\frac{y^2}{x-y}\right) & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$
   Show that $f$ is continuous at $(0,0)$. Does $f$ differentiable at $(0,0)$?

8. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, if for all $(x,y) \in \mathbb{R}^2$,

   (a) $f(x,y) = \begin{cases} x^2+y^2 & \text{if both } x, y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$ (b) $f(x,y) = \begin{cases} x^{4/3}\sin(\frac{y}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

9. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{x}_0$ and $f(\mathbf{x}_0) = 0$. Suppose $g : \mathbb{R}^n \to \mathbb{R}$ is continuous at $\mathbf{x}_0$. Show that $fg : \mathbb{R}^n \to \mathbb{R}$, defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ is differentiable at $\mathbf{x}_0$.

10. Let $\varphi,\ \psi : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x,y) = \varphi(x) + \psi(y)$ for all $(x,y) \in \mathbb{R}^2$, is differentiable.