HARMONIC FUNCTIONS

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by

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CERTIFICATE

This is to certify that the work contained in this project report entitled **"Harmonic Functions"** submitted by **Sayan Das**, **Roll Number: 232123125**, to the Department of Mathematics, Indian Institute of Technology Guwahati towards partial requirement of Master of Science in Mathematics has been carried out by him under my supervision.

It is also certified that this report is a survey work based on the references in the bibliography.

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ABSTRACT

Harmonic functions constitute a central object of study in analysis, with profound applications across mathematics and physics. The proposed project aims to undertake a systematic investigation of the theory of harmonic functions, emphasizing their structural and analytical properties. In particular, attention will be devoted to the development of the associated notions of subharmonic and superharmonic functions, which play a pivotal role in potential theory and related areas.

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Chapter 1

Introduction

Harmonic functions are infinitely differentiable functions that satisfy the Laplace equation, a fundamental partial differential equation in mathematics and physics. Such functions naturally arise in the study of boundary value problems, most notably the Dirichlet problem. They possess several remarkable properties, including the mean value property and the maximum principle, which provide deep insights into their behavior within a region. In this chapter, we present an overview of harmonic functions and discuss their fundamental properties.

1.1 Harmonic Functions and Their Basic Properties

We begin by introducing the notion of harmonic functions.

Definition 1.1.1. Let $G \subset \mathbb{C}$ be an open set. A function $u : G \to \mathbb{R}$ is said to be *harmonic* if u has continuous second-order partial derivatives and satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This equation is called the **Laplace equation**.

Theorem 1.1.2. *A function f defined on a region $G \subset \mathbb{C}$ is analytic if and only if its real and imaginary parts,*

$$f = u + iv, \qquad u = \mathrm{Re}(f),\ v = \mathrm{Im}(f),$$

are harmonic functions on G that satisfy the Cauchy–Riemann equations.

Theorem 1.1.3. *A region $G \subset \mathbb{C}$ is simply connected if and only if for every harmonic function u on G, there exists a harmonic function v on G such that*

$$f = u + iv$$

is analytic on G.

Definition 1.1.4. If $f : G \to \mathbb{C}$ is analytic with $f = u + iv$, then $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ are called *harmonic conjugates*.

Remark 1.1.5. With this terminology, Theorem 1.1.3 can be restated as: *every harmonic function on a simply connected region has a harmonic conjugate.*

More generally, if u is a harmonic function on G and $D \subset G$ is a disk, then there exists a harmonic function v on D such that $u + iv$ is analytic on D. In other words, every harmonic function admits a harmonic conjugate locally.

Proposition 1.1.6. *If $u : G \to \mathbb{R}$ is harmonic, then u is infinitely differentiable on G.*

Proof. Fix $z_0 = x_0 + iy_0 \in G$, and choose $\delta > 0$ such that $B(z_0;\delta) \subset G$. Then u admits a harmonic conjugate v on $B(z_0;\delta)$, so that $f = u + iv$ is analytic there. Since analytic functions are smooth (in fact, real-analytic), it follows that u is infinitely differentiable on $B(z_0;\delta)$. As z_0 was arbitrary, the claim holds on G. □

Theorem 1.1.7 (Mean Value Theorem)**.** *Let $u : G \to \mathbb{R}$ be a harmonic function, and let $\overline{B}(a;r) \subset G$. If γ denotes the circle $|z-a| = r$, then*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u\big(a + re^{i\theta}\big)\, d\theta.$$

Proof. Let D be a disk with $\overline{B}(a;r) \subset D \subset G$, and let f be an analytic function on D such that $u = \operatorname{Re}(f)$.

By the Cauchy integral formula,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\, dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})}{(a+re^{i\theta})-a} \cdot ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\, d\theta. \end{aligned}$$

Taking real parts gives the desired formula. □

Definition 1.1.8. A continuous function $u : G \to \mathbb{R}$ is said to have the

Mean Value Property (MVP) if for every closed disk $\overline{B}(a;r) \subset G$,

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u\big(a + re^{i\theta}\big)\ d\theta.$$

Maximum principle 1.1.9. ***(First Version).*** *Let G be a region, and suppose that u is a continuous real-valued function on G with the Mean Value Property (MVP). If there exists a point $a \in G$ such that $u(a) \geq u(z)$ for all $z \in G$, then u is a constant function.*

Proof. Let the set A is defined by

$$A = \{z \in G : u(z) = u(a)\}.$$

Since u is continuous the set A is closed in G. Let $z_0 \in A$ and r be chosen such that $\overline{B}(z_0;r) \subset G$. Again choose $0 < \rho < r$, Then by **MVP**

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\theta})\, d\theta \Rightarrow \frac{1}{2\pi} \int_0^{2\pi} \{u(z_0) - u(z_0 + \rho e^{i\theta})\}\, d\theta = 0$$

Since $u(z_0) - u(z_0 + \rho e^{i\theta}) \geq 0$ and u is continuous we have

$$u(z_0) = u(z_0 + \rho e^{i\theta}) \quad \forall \theta \in [0, 2\pi]$$

But ρ is arbitrary in $(0, r)$. So $B(z_0;r) \subset A$ and A is also open. By the connectedness of G, $A = G$. □

Maximum principle 1.1.10. ***(Second Version).*** *Let G be a region and let u and v be two continuous real valued function on G that have the* ***MVP****.*

If for each point a in the extended boundary $\partial_\infty G$,

$$\limsup_{z\to a} u(z) \le \liminf_{z\to a} u(z)$$

then either $u(z) < v(z)$ for all $z \in G$ or $u = v$.

Proof. Fix a in $\partial_\infty G$ and for each $\delta > 0$ let $G_\delta = G \cap B(a;\delta)$. Then according to the hypothesis,

$$0 \ge \lim_{\delta\to 0}\left[\sup\{u(z) : z \in G_\delta\} - \inf\{v(z) : z \in G_\delta\}\right]$$

$$= \lim_{\delta\to 0}\left[\sup\{u(z) : z \in G_\delta\} + \sup\{-v(z) : z \in G_\delta\}\right]$$

$$\ge \lim_{\delta\to 0}\sup\{u(z) - v(z) : z \in G_\delta\}.$$

So $\limsup[u(z) - v(z)] \le 0$ for each $a \in \partial_\infty G$. So it is sufficient to prove the theorem under the assumption that $v(z) = 0$ for all $z \in G$. That is, assume

$$\limsup_{z\to a} u(z) \le 0 \tag{1.1.1}$$

for all $a \in \partial_\infty G$ and show that either $u(z) < 0$ for all $z \in G$ or $u \equiv 0$. In virtue of the first version of the Maximum Principle, it suffices to show that $u(z) \le 0$ for all $z \in G$.

Suppose that u satisfies 1.1.1 and there is a point $b \in G$ with $u(b) > 0$. Let $\epsilon > 0$ be chosen so that $u(b) > \epsilon$ and let $B = \{z \in G : u(z) \ge \epsilon\}$. If $a \in \partial_\infty G$ then 1.1.1 implies there is a $\delta = \delta(a)$ such that $u(z) < \epsilon$ for all $z \in G \cap B(a;\delta)$. Using the Lebesgue Covering Lemma, a δ can be found which is independent of a. That is, there is a $\delta > 0$ such that if $z \in G$ and

$d(z, \partial_\infty G) < \delta$ then $u(z) < \epsilon$. Thus,

$$B \subset \{z \in G : d(z, \partial_\infty G) \geq \delta\}.$$

This gives that B is bounded in the plane; since B is clearly closed, it is compact. So if $B \neq \emptyset$, there is a point $z_0 \in B$ such that $u(z_0) \geq u(z)$ for all $z \in B$. Since $u(z) < \epsilon$ for $z \in G - B$, this gives that u assumes a maximum value at a point in G. So u must be constant. But this constant must be $u(z_0)$ which is positive and this contradicts 1.1.1. □

Corollary 1.1.11. *Let G be a bounded region and suppose that $w : G^- \to \mathbb{R}$ is a continuous function that satisfies the **MVP** on G. If $w(z) = 0$ for all $z \in \partial G$, then $w(z) = 0$ for all $z \in G$.*

Proof. First we take $w = u$ and $v = 0$ in Theorem 1.1.10 so $w(z) < 0$ for all z or $w(z) \equiv 0$. Now take $w = v$ and $u = 0$ in 1.1.10; So either $w(z) > 0$ for all z or $w(z) \equiv 0$. Since both of these hold, $w \equiv 0$. □

Remark 1.1.12. Even though Theorem 1.1.9 is called the Maximum Principle, it is also a Minimum Principle For the sake of completeness, a Minimum Principle corresponding to Theorem 1.1.9 is stated below. It can be proved by considering the function $-u$ and appealing to 1.1.9.

Minimum principle 1.1.13. *Let G be a region, and suppose that u is a continuous real-valued function on G with the Mean Value Property (MVP). If there exists a point $a \in G$ such that $u(a) \geq u(z)$ for all $z \in G$, then u is a constant function.*

1.2 Harmonic Functions on a Disk

The **Poisson kernel** plays a fundamental role in the study of harmonic functions. In this section, we study the Poisson kernel and the behavior of harmonic functions on the open unit disk

$$D = \{z : |z| < 1\},$$

and subsequently extend these results to arbitrary disks. We also introduce a framework for convergence of continuous and harmonic functions.

1.2.1 The Poisson Kernel

Definition 1.2.1. The function

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}, \quad 0 \leq r \leq 1, \ -\infty < \theta < \infty,$$

is called the **Poisson kernel**.

Let $z = re^{i\theta}$ with $0 \leq r < 1$. Then

$$\begin{aligned} \frac{1 + re^{i\theta}}{1 - re^{i\theta}} &= (1 + z)(1 + z + z^2 + \cdots) \\ &= 1 + 2\sum_{n=1}^{\infty} z^n \\ &= 1 + 2\sum_{n=1}^{\infty} r^n e^{in\theta}. \end{aligned}$$

Hence,

$$\mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = P_r(\theta).$$

Equivalently, we can write

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}. \tag{1.2.1}$$

Proposition 1.2.2. *The Poisson kernel satisfies:*

(a) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1$*;*

(b) $P_r(\theta) > 0$ *for all* θ*,* $P_r(-\theta) = P_r(\theta)$*, and* P_r *is* 2π*-periodic;*

(c) If $0 < \delta < |\theta| \leq \pi$*, then* $P_r(\theta) < P_r(\delta)$*;*

(d) For each $\delta > 0$*,* $\lim_{r\to 1^-} P_r(\theta) = 0$ *uniformly for* $\delta \leq |\theta| \leq \pi$*.*

1.2.2 The Dirichlet Problem on the Unit Disk

Theorem 1.2.3. *Let* $D = \{z : |z| < 1\}$ *and let* $f : \partial D \to \mathbb{R}$ *be continuous. Then there exists a unique continuous function* $u : \overline{D} \to \mathbb{R}$ *such that*

(a) $u|_{\partial D} = f$*,*

(b) u *is harmonic in* D*,*

given by

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})\,dt, \quad 0 \leq r < 1.$$

Corollary 1.2.4. *If* $u : \overline{D} \to \mathbb{R}$ *is continuous and harmonic in* D*, then*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) u(e^{it})\,dt,$$

and $u = \mathrm{Re}(f)$ *where*

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(e^{it})\, dt.$$

Corollary 1.2.5. *Let* $B(a;\rho) = \{z : |z - a| < \rho\}$ *and* h *be continuous on* $\partial B(a;\rho)$. *Then there exists a unique continuous* $w : \overline{B(a;\rho)} \to \mathbb{R}$ *harmonic in* $B(a;\rho)$ *with* $w|_{\partial B(a;\rho)} = h$.

1.2.3 Poisson Kernel for an Arbitrary Disk

For a disk of radius $R > 0$, the Poisson kernel is obtained by scaling:

$$P_R(r,\theta) = \frac{R^2 - r^2}{R^2 - 2rR\cos\theta + r^2}, \quad 0 \le r < R. \tag{1.2.2}$$

Then for u continuous on $\overline{B(a;R)}$ and harmonic in $B(a;R)$:

$$u(a + re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_R(r, \theta - t) u(a + Re^{it})\, dt.$$

Theorem 1.2.6 (Harnack's Inequality)**.** *Let* $u \ge 0$ *be continuous on* $\overline{B(a;R)}$ *and harmonic in* $B(a;R)$. *Then for* $0 \le r < R$,

$$\frac{R-r}{R+r} u(a) \le u(a + re^{i\theta}) \le \frac{R+r}{R-r} u(a), \quad \forall\theta.$$

1.2.4 The Space of Continuous Functions

Definition 1.2.7. Let $G \subset \mathbb{C}$ be open. Denote by $C(G,\mathbb{R})$ the set of all continuous functions $f : G \to \mathbb{R}$.

Definition 1.2.8. Define a metric ρ on $C(G, \mathbb{R})$ by

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(f, g)}{1 + \rho_n(f, g)},$$

where

$$\rho_n(f, g) = \sup\{|f(z) - g(z)| : z \in K_n\},$$

and $\{K_n\}$ is a sequence of compact subsets of G satisfying:

(a) $K_n \subset \text{int}(K_{n+1})$,

(b) Every compact $K \subset G$ is contained in some K_n,

(c) Every component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus G$.

Proposition 1.2.9. *A sequence $\{f_n\}$ in $C(G, \mathbb{R}, \rho)$ converges to f if and only if $\{f_n\}$ converges to f uniformly on every compact subset of G.*

Proposition 1.2.10. *The space $C(G, \mathbb{R})$ equipped with the metric ρ is complete.*

Proposition 1.2.11. *Let $\{u_n\}$ be a sequence of harmonic functions on a region $G \subset \mathbb{C}$, i.e., $u_n \in Har(G)$ for all n. If $u_n \to u$ uniformly on every compact subset of G, then the limit function u satisfies the **Mean Value Property**.*

Definition 1.2.12. Let $G \subset \mathbb{C}$ be an open set. Denote by $\text{Har}(G)$ the space of harmonic functions on G. Since $\text{Har}(G) \subset C(G, \mathbb{R})$, it is equipped with the metric inherited from $C(G, \mathbb{R})$.

1.3 Harnack's Theorem

We are now ready to state an important result regarding harmonic functions.

Theorem 1.3.1 (Harnack's Theorem)**.** *Let G be a region.*

(a) The metric space $\mathrm{Har}(G)$ *is complete.*

(b) If $\{u_n\}$ is a sequence in $\mathrm{Har}(G)$ *such that $u_1 \leq u_2 \leq \cdots$, then either $u_n(z) \to \infty$ uniformly on compact subsets of G, or $\{u_n\}$ converges in* $\mathrm{Har}(G)$ *to a harmonic function.*

Proof. (a) To show completeness, it suffices to prove that $\mathrm{Har}(G)$ is a closed subspace of $C(G, \mathbb{R})$. Let $\{u_n\} \subset \mathrm{Har}(G)$ and $u_n \to u$ in $C(G, \mathbb{R})$. By Lemma IV.2.7, u satisfies the Mean Value Property, and hence, by Theorem 2.11, u is harmonic.

(b) Assume $u_1 \geq 0$ (otherwise consider $\{u_n - u_1\}$). Define

$$u(z) = \sup_{n \geq 1} u_n(z), \quad z \in G.$$

Set

$$A = \{z \in G : u(z) = \infty\}, \quad B = \{z \in G : u(z) < \infty\}.$$

Then $G = A \cup B$ and $A \cap B = \emptyset$. We show both A and B are open.

Let $a \in G$ and choose $R > 0$ such that $\overline{B}(a; R) \subset G$. By Harnack's inequality:

$$\frac{R - |z - a|}{R + |z - a|} u_n(a) \leq u_n(z) \leq \frac{R + |z - a|}{R - |z - a|} u_n(a), \quad z \in B(a; R),\ n \geq 1.$$

If $a \in A$, then $u_n(a) \to \infty$, and the left-hand inequality shows $u_n(z) \to \infty$ in $B(a; R)$, so A is open. Similarly, B is open.

Since G is connected, either $A = G$ or $B = G$. If $A = G$, Harnack's inequality implies $u_n \to \infty$ uniformly on compact subsets. If $B = G$, a similar argument using Harnack's inequality shows $\{u_n\}$ is Cauchy uniformly on compact subsets, hence convergent to a harmonic function by part (a). □

1.4 Subharmonic and Superharmonic Functions

Definition 1.4.1. Let G be a region and $\varphi : G \to \mathbb{R}$ continuous. Then φ is *subharmonic* if, for every closed disk $\overline{B}(a; r) \subset G$,

$$\varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + re^{i\theta})\, d\theta.$$

Definition 1.4.2. Let G be a region and $\varphi : G \to \mathbb{R}$ continuous. Then φ is *superharmonic* if, for every closed disk $\overline{B}(a; r) \subset G$,

$$\varphi(a) \geq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + re^{i\theta})\, d\theta.$$

Remark 1.4.3.

- φ is superharmonic iff $-\varphi$ is subharmonic.
- Every harmonic function is both subharmonic and superharmonic.
- u is harmonic iff it is both subharmonic and superharmonic.
- Nonnegative linear combinations of subharmonic functions are subhar-

monic.

Theorem 1.4.4 (Maximum Principle for Subharmonic Functions)**.** *Let G be a region and $\varphi : G \to \mathbb{R}$ subharmonic. If φ attains a maximum in G, then φ is constant.*

Theorem 1.4.5 (Comparison Principle)**.** *Let φ be subharmonic and ψ superharmonic on G. If*

$$\limsup_{z\to a} \varphi(z) \le \liminf_{z\to a} \psi(z), \quad a \in \partial_\infty G,$$

then either $\varphi < \psi$ in G, or $\varphi = \psi$ and is harmonic.

Theorem 1.4.6. *A continuous function $\varphi : G \to \mathbb{R}$ is subharmonic iff for every subregion $G_1 \subset G$ and every harmonic function u_1 on G_1, the function $\varphi - u_1$ satisfies the Maximum Principle on G_1.*

Corollary 1.4.7. *The maximum of finitely many subharmonic functions is subharmonic.*

Corollary 1.4.8. *If φ is subharmonic on G and $\overline{B}(a;r) \subset G$, the function φ' defined by*

$$\varphi'(z) = \begin{cases} \varphi(z), & z \in G - B(a;r), \\ \text{harmonic extension of } \varphi \text{ in } B(a;r), & z \in B(a;r) \end{cases}$$

is subharmonic.

1.4.1 Perron Function

Definition 1.4.9. Let G be a region and $f : \partial_\infty G \to \mathbb{R}$ continuous. The *Perron family* $\mathscr{P}(f, G)$ consists of all subharmonic functions $\varphi : G \to \mathbb{R}$ such that

$$\limsup_{z \to a} \varphi(z) \le f(a), \quad a \in \partial_\infty G.$$

Definition 1.4.10. The *Perron function* associated with f is

$$u(z) = \sup\{\varphi(z) : \varphi \in \mathscr{P}(f, G)\}, \quad z \in G.$$

Theorem 1.4.11. *Let G be a region and $f : \partial_\infty G \to \mathbb{R}$ continuous. Then the Perron function u is harmonic on G.*

Proof. Let $G \subset \mathbb{C}$ be a region and let $f : \partial_\infty G \to \mathbb{R}$ be bounded. Consider the Perron family

$$\mathscr{P}(f, G) = \{\varphi : G \to \mathbb{R} \mid \varphi \text{ is subharmonic in } G, \limsup_{z \to a} \varphi(z) \le f(a) \text{ for all } a \in \partial_\infty G\},$$

and define the Perron function

$$u(z) = \sup\{\varphi(z) : \varphi \in \mathscr{P}(f, G)\}, \quad z \in G.$$

For each $z \in G$, we can choose a sequence $\{\varphi_n\} \subset \mathscr{P}(f, G)$ such that $\varphi_n(z) \nearrow u(z)$ as $n \to \infty$. Define

$$\Phi_n(z) := \max\{\varphi_1(z), \ldots, \varphi_n(z)\}.$$

Since the maximum of finitely many subharmonic functions is subharmonic, each $\Phi_n \in \mathscr{P}(f, G)$, and $\Phi_n \nearrow u$ pointwise.

Fix any point $z_0 \in G$ and choose a closed disk $\overline{B}(z_0; r) \subset G$. Let Φ'_n denote the harmonic function in $B(z_0; r)$ which agrees with Φ_n on $\partial B(z_0; r)$. By the Maximum Principle, $\Phi_n(z) \leq \Phi'_n(z)$ for all $z \in B(z_0; r)$. The sequence $\{\Phi'_n\}$ is monotone increasing and bounded above, so by Harnack's theorem, $\{\Phi'_n\}$ converges uniformly on compact subsets of $B(z_0; r)$ to a harmonic function v in $B(z_0; r)$:

$$v(z) := \lim_{n \to \infty} \Phi'_n(z), \quad z \in B(z_0; r).$$

Since $\Phi_n \leq \Phi'_n$ and $\Phi_n \nearrow u$, we have $u(z) \leq v(z)$ in $B(z_0; r)$. Conversely, for any $\varphi \in \mathscr{P}(f, G)$, $\varphi \leq \Phi'_n$ for sufficiently large n, which implies $u(z) \geq v(z)$. Therefore, $u(z) = v(z)$ for all $z \in B(z_0; r)$.

Because $z_0 \in G$ was arbitrary, u coincides locally with a harmonic function in a neighborhood of every point in G, so u is harmonic on all of G. Moreover, the uniform convergence of Φ'_n on compact subsets implies that u is continuous in G.

Hence, the Perron function u is harmonic in G, completing the proof. □

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