Compact Self-adjoint Operators:

We know that the spectrum of a compact operator is countable and has only point spectrum (eigen-values), except possibly '0' in the spectrum-set. Hence we can think of expressing a compact operator in terms of orthogonal projections corresponding to eigen-spaces.

Let H be a Hilbert space, and $T \in B(H)$. For $x, y \in H$, the map $x \longmapsto \langle Tx, y\rangle$ is a bounded linear functional on H, and hence by Riesz rep^n thm, $\exists !\ z = T^*(y) \in H$ such that

$$\langle Tx, y\rangle = \langle x, T^*(y)\rangle .$$

Then T^* is linear and bounded, because

$$\sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle Tx, y\rangle| = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle x, T^*y\rangle|$$

$$\Rightarrow \|T\| = \|T^*\| < \infty .$$

Notice that for $x, y \in H$ & $\lambda \in \mathbb{C}$, the relation

$$\langle (T-\lambda I)x, y\rangle = \langle x, (T^*-\bar{\lambda} I)y\rangle \quad \text{———} (*)$$

shows that $R_\lambda = T - \lambda I$ is invertible iff R_λ^* is invertible.

If $R_\lambda^* y = 0$, then $\|y\|^2 = \langle y, y\rangle = \langle R_\lambda R_\lambda^{-1} y, y\rangle = 0$ (when we replaced $x \to R_\lambda^{-1} y$ in $(*)$).

Finally, it is easy to verify that $N(R_\lambda) = R(R_\lambda^*)^\perp$. Since R_λ is one-one, $R(R_\lambda^*)^\perp = \{0\}$, and hence by HBT $\overline{R_\lambda(T^*)H} = H$. But remember that $N(R_\lambda)$ & $R_\lambda(T^*)$ both are closed and

$$H = N(R_\lambda) \oplus R(R_\lambda^*) \quad \text{(verify!)}.$$

Hence $R_\lambda(T^*)H = H$. That is, $R_\lambda(T^*)$ is invertible and vice-versa.

Thus, $\lambda \in \sigma(T)$ iff $\bar\lambda \in \sigma(T^*)$. That is,

$$\sigma(T) = \overline{\sigma(T^*)}.$$

Recall that if $T = T^*$ (self-adjoint) then T is determined by diagonal, and

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Also, if $T = T^*$, then $\langle Tx, x\rangle$ is real and hence all eigen-values of T are real.

We know that $T \in B(H)$ is normal ($T^*T = TT^*$) iff $\|Tx\| = \|T^*x\|$, $\forall x \in H$. By that we can show that spectral radius $r(T) = \|T\|$.

<u>Lemma</u>: let $T \in B(H)$ be normal, then for each $n \in \mathbb{N}$, $\|T^n\| = \|T\|^n$?

<u>proof</u>: $\|T^n x\|^2 = \langle T^n x, T^n x\rangle = \langle T^* T^n x, T^{n-1} x\rangle$

$$\le \|T^*(T^n x)\| \, \|T^{n-1} x\|$$

$$= \|T^{n+1} x\| \, \|T^{n-1} x\| \quad (\because \|T^* y\| = \|Ty\|)$$

$$\Rightarrow \|T^n\|^2 \le \|T^{n+1}\| \, \|T^{n-1}\|.$$

Suppose the assertion is true for n. We have

$\|T\|^{2n} = (\|T\|^{n})^2 = \|T^n\|^2 \leq \|T^{n+1}\|\,\|T^{n-1}\|$

$\Rightarrow \|T\|^{2n} \leq \|T^{n+1}\|\,\|T\|^{n-1}$

$\Rightarrow \|T\|^{n+1} \leq \|T^{n+1}\| \leq \|T\|^{n+1}.$

Since $r(T) = \lim \|T^n\|^{1/n}$, we get $r(T) = \|T\|$.
From this, we may think if $\|T\| \in \sigma(T)$ for T to be a self-adjoint operator. Before that we need the following result.

<u>Theorem:</u> If $T \in B(H)$ is a self-adjoint operator, then $\sigma(T) = \sigma_{app}(T)$ (set of approxi: spectrum).
That is, $\lambda \in \sigma(T)$ iff $\inf\limits_{\|x\|=1} \|(\lambda I - T)x\| = 0$.

Proof: Suppose $\lambda \in \rho(T)$, then $(\lambda I - T)^{-1}$ is invertible, and

$$\|x\| = \|(\lambda I - T)^{-1}(\lambda I - T)x\| \leq \|(\lambda I - T)^{-1}\|\,\|(\lambda I - T)x\|$$

$$\Rightarrow \inf_{\|x\|=1} \|(\lambda I - T)x\| \geq \|(\lambda I - T)^{-1}\|^{-1}.$$

Conversely, if $\inf\limits_{\|x\|=1} \|(\lambda I - T)(x)\| = \alpha > 0$, then

$$\|(\lambda I - T)x\| \geq \alpha \|x\|,\ \forall x \in H.$$

Hence $(\lambda I - T)$ is one-one & $R(\lambda I - T)$ is closed.

If $R(\lambda I - T) \neq H$, then $\exists\, y_0 \in H$, $y_0 \neq 0$ such that $\langle (\lambda I - T)x, y_0 \rangle = \langle x, (\bar{\lambda} I - T) y_0 \rangle = 0$ for all $x \in H$. Hence $T y_0 = \bar{\lambda} y_0$. That is, y_0 is an eigen-vector of T. Since $T = T^*$, $\bar{\lambda} = \lambda$.
Hence $T y_0 = \lambda y_0 \Rightarrow (\lambda I - T) y_0 = 0$, for $y_0 \neq 0$, is a contradiction.

Theorem: Let $T \in B(H)$ be self-adjoint. Then $\sigma(T) \subset \mathbb{R}$. If we write $\mu = \inf_{\|x\|=1} \langle Tx, x\rangle$ and $\nu = \sup_{\|x\|=1} \langle Tx, x\rangle$. Then $\sigma(T) \subset [\mu, \nu]$ and $\mu, \nu \in \sigma(T)$.

proof: Suppose H is a complex Hilbert space, and $\lambda = \alpha + i\beta \in \sigma(T)$. Then

$$\langle \lambda x - Tx, x\rangle - \langle x, \lambda x - Tx\rangle$$
$$= \lambda\|x\|^2 - \langle Tx, x\rangle - \bar{\lambda}\|x\|^2 + \langle x, Tx\rangle$$
$$= (\lambda - \bar{\lambda})\|x\|^2 = 2i\beta\|x\|^2$$

for $x \in S_H = \{x \in H : \|x\| = 1\}$.

$$2|\beta| \le |\langle \lambda x - Tx, x\rangle| + |\langle x, \lambda x - T\rangle|$$
$$\le 2\|\lambda x - Tx\|$$
$$\Rightarrow \inf_{\|x\|=1} \|(\lambda I - T)x\| \ge |\beta| = 0 \quad \text{(by previous Thm)}$$

Hence $\sigma(T) \subset \mathbb{R}$.

Note that if $S = T + \gamma I$, then $\sigma(S) = \sigma(T) + \gamma$.
($\lambda \in \rho(S)$ iff $(\lambda I - S)^{-1} \in B(H)$ iff $((\lambda - \gamma)I - T)^{-1} \in B(H)$
$\therefore \lambda \in \sigma(S)$ iff $\lambda - \gamma \in \sigma(T)$.)

Hence, w.l.g, we can assume that $0 \le \mu \le \nu$.
Since $\nu = \|T\|$ and $\sigma(T) \subset \mathbb{R}$, we get
$$\sigma(T) \subset [-\nu, \nu].$$
We claim that $\lambda = \mu - a \notin \sigma(T)$, $\forall a > 0$.
Here, $\langle (T - \lambda I)x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle \ge \mu - \lambda = a$, if $\|x\| = 1$.

$\Rightarrow 0 < a \le \|(T-\lambda I)x\| \, \|x\|, \ \forall \ \|x\| = 1.$

Hence $\inf\limits_{\|x\|=1} \|(T-\lambda I)x\| \geq a > 0$ is a contradiction. Thus, $\sigma(T) \subset [\mu, \nu]$.

Note that $\exists \ x_n \in S_H$ such that $\langle Tx_n, x_n \rangle \to \nu$.

Therefore,
$$\|(\nu I - T)x_n\|^2 = \nu^2 \|x_n\|^2 + \|Tx_n\|^2 - 2\nu \langle Tx_n, x_n \rangle.$$
$$\le 2\nu^2 - 2\nu \langle Tx_n, x_n \rangle \to 0$$

$\Rightarrow \nu \in \sigma(T). \longrightarrow (*)$

Further, $\sigma(T) \subset [\mu, \nu]$

$\Rightarrow \sigma(T) - \nu I \subset [\mu - \nu, 0]$

$\Rightarrow \sigma(\nu I - T) \subset [0, \nu - \mu]$

By invoking $(*)$, we get $\nu - \mu \in \sigma(\nu I - T)$

$\Rightarrow \mu \in \sigma(T)$.

Cor: If $\exists \ x_0, y_0 \in S_H$ such that $\mu = \langle Tx_0, x_0 \rangle$ and $\nu = \langle Ty_0, y_0 \rangle$. Then x_0 & y_0 are eigen-vectors corresponding to eigen-values μ & ν respectively.

Proof: For $\alpha \in \mathbb{C}$ and $y \in H$, by def^n of μ

$$\langle T(x_0 + \alpha y), x_0 + \alpha y \rangle \geq \mu \langle x_0 + \alpha y, x_0 + \alpha y \rangle$$

$$\Rightarrow \langle Tx_0, x_0 \rangle + 2\alpha \, Re \langle Tx_0, y \rangle + |\alpha|^2 \langle Ty, y \rangle$$
$$\geq \mu \langle x_0, x_0 \rangle + \mu |\alpha|^2 \langle y, y \rangle + 2\mu \, Re\alpha \langle x_0, y \rangle.$$

Since $\mu = \langle Tx_0, x_0 \rangle$, letting $\alpha = \gamma \langle (T - \mu I)x_0, y \rangle$, $\gamma \in \mathbb{R}$, we infer that $\langle y, (T - \mu I)x_0 \rangle = 0, \ \forall \ y \in H$.

By replacing $T \longrightarrow -T$, we get the other part.

Lemma: $T \in B(H)$ be a self-adjoint operator on H, and M be a closed T-invariant subspace of H. Then $N = M^{\perp}$ is T-invariant. If $T_1 = T|_M$ & $T_2 = T|_N$, then T_1 & T_2 are self adjoint; $T(H) = T_1(M) \oplus T_2(N)$ and

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2).$$

Pf: Let $y \in N$. Since $T(M) \subseteq M$, for each $x \in M$,

$0 = \langle Tx, y \rangle = \langle x, Ty \rangle \Rightarrow T(N) \subset N$.

Clearly T_1 & T_2 are self adjoint on M & N resp. and $T(H) = T(M \oplus N) = T(M) \oplus T(N)$.

Now, let $\lambda \in \sigma(T_1)$, then $\exists$ $x_n \in S_M$ s.t

$$\|\lambda x_n - T_1 x_n\| = \|\lambda x_n - T x_n\| \longrightarrow 0$$

$$\Rightarrow \lambda \in \sigma(T) \Rightarrow \sigma(T_1) \subset \sigma(T).$$

Similarly, $\sigma(T_2) \subset \sigma(T)$.

If $\lambda \notin \sigma(T_1) \cup \sigma(T_2)$, then $\exists$ $k > 0$ s.t. for $x \in M$ & $y \in N$,

$$\|\lambda x - Tx\| \geq k\|x\| \text{ \& } \|\lambda y - Ty\| \geq k\|y\|.$$

Let $z \in H$. Then $z = x + y$, $x \in M$, $y \in N$.

Note that $\lambda x - Tx \in M$ & $\lambda y - Ty \in N$. Hence

$$\begin{aligned}\|\lambda z - Tz\|^2 &= \|\lambda x - Tx - (\lambda y - Ty)\|^2 \\ &= \|\lambda x - Tx\|^2 + \|\lambda y - Ty\|^2 \\ &\geq k\|x\|^2 + k\|y\|^2 = k\|z\|^2.\end{aligned}$$

$$\Rightarrow \lambda \in \sigma(T) \Rightarrow \sigma(T) = \sigma(T_1) + \sigma(T_2).$$

Note that the eigen-vectors corresponding to distinct eigen-values of a self-adjoint operator on a Hilbert space are orthogonal. Also, eigen-spaces, in this case, are operator invariant.

Further, if T is a compact self-adjoint operator on H, then $\sigma_p(T) \neq \Phi$. If $T = 0$, then 0 is an eigen-value, and if $T \neq 0$, then $\|T\| \in \sigma(T) = \sigma_p(T) \cup \{0\}$ ($\because T$ is cpt)

Hence $\|T\| \in \sigma_p(T)$.

<u>Theorem:</u> Let T be a compact self-adjoint operator on an infinite dim. Hilbert space H. Then set of eigen-vectors of T forms an ONB for T. Moreover, for $x \in H$,

$$Tx = \sum \lambda_n \langle x, e_n \rangle e_n,$$

where $Te_n = \lambda_n e_n$ and $|\lambda_n| \to 0$.

proof: Let $T \neq 0$. Since $\dim H = \infty$, $0 \in \sigma(T)$.

Also, $\|T\| \in \sigma(T) = \sigma_p(T) \cup \{0\}$

$$\Rightarrow \quad 0 \neq \|T\| \in \sigma_p(T).$$

Hence, T has a non-zero eigen-value.

Since T is cpt & self adjoint, $\sigma_p(T)$ is countable. Let $\lambda_n \in \sigma_p(T) = \{\lambda_n : n = 1, 2, \cdots\}$ and write $N_n = \mathrm{Ker}(\lambda_n I - T)$.

Remember that $\dim N_n < \infty$ ($\because T$ is cpt & $\lambda_n \neq 0$).

We can form an ONB say B_n for each N_n. Now, let $B = \bigcup_{n=1}^{\infty} B_n$. Then B is an orthonormal set. (Since eigen-vectors corresponding to distinct eigen-values of T are orthonormal).
We claim $\overline{\text{span}}(B) = H$. If $\overline{\text{span}}(B) \neq H$, then write $G = \overline{\text{span}}(B)^{\perp}$.
Since B_n is T-invariant, G will also be T-invariant. Hence by the previous result,

$$\sigma(T) = \sigma(T/\overline{\text{span}}(B)) + \sigma(T/G).$$

Since T/G is compact & self-adjoint, T/G has a non-zero eigen-value. Thus, T/G has an eigen-vector $y \in G$. But then y is also eigen-vector for T. Hence

$y \in G \cap \overline{\text{span}}(B)$, which is a contradiction. Thus, $\overline{\text{span}}(B) = H$.

Now, let $\{e_n\}$ be an ONB that satisfies $Te_n = \lambda_n e_n$. Notice that the series

$\sum \lambda_n \langle x, e_n \rangle e_n$ is absolutely conv.

$$\left\| \sum_{n=k}^{l} \lambda_n \langle x, e_n \rangle e_n \right\|^2 \le \|T\| \sum_{n=k}^{l} |\langle x, e_n \rangle|^2 \to 0.$$

Consider $Sx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n$. Then

$$\|Sx\|^2 \le \|T\|^2 \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = \|T\|^2 \|x\|^2.$$

$\Rightarrow \|Sx\| \le \|T\| \|x\|, \ \forall x \in H.$

Since $Se_n = \lambda_n e_n = Te_n \Rightarrow S = T.$

Further, if $\lambda_n \not\to 0$, then $\exists\ \epsilon > 0$ such that $|\lambda_n| \geq \epsilon$ for ∞ly many n. Then

$\|Te_n - Te_m\|^2 = |\lambda_n|^2 + |\lambda_m|^2 \geq 2\epsilon^2.$

Hence (Te_n) has no conv. subseq^n. Thus, $(\lambda_n) \to 0$.