REAL ANALYSIS LECTURE NOTES

MA642: Real Analysis, July-November, 2025
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Contents

1. The Real Numbers

1.1. **Preliminary.** Let $\mathbb{Q}$ be the set of rationals:

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\ \gcd(p, q) = 1 \right\}$$

where $\mathbb{Z}$ is the set of integers.

There are numbers other than rationals.

Consider $(p/q)^2 = 2$, with $\gcd(p, q) = 1$.

$$p^2 = 2q^2 \implies p = 2m \text{ for some } m \in \mathbb{Z}$$
$$\implies 2m^2 = q^2 \implies q = 2n \implies (p, q) \geq 2$$

which is a contradiction. Thus, $\sqrt{2}$ is not a rational number.

Such numbers are called **irrational**. We denote:

$$\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$$

as the set of irrationals.

Remark 1.1. The set of rationals is not complete in the following sense.

1.2. **Bounds.**

Definition 1.2. Let $A \subset \mathbb{R}$. A number $x_0 \in \mathbb{R}$ is called an **upper bound** for A if $a \leq x_0$ for all $a \in A$. Similarly, y_0 is called a **lower bound** for A if $a \geq y_0$ for all $a \in A$.

Definition 1.3. An upper bound x_0 of A is called the **least upper bound** (l.u.b.) or **supremum** ($\sup A$) of A if for any upper bound x of A, implies $x_0 \leq x$. Similarly, the **greatest lower bound** (g.l.b.) or **infimum** ($\inf A$) is defined.

Example 1.4.

$$A = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$$

Show that $\inf A = 0$ and $\sup A = 1$.

Remark 1.5. Every non-empty subset of $\mathbb{R}$ having an upper bound has a l.u.b. (sup), and every non-empty subset of $\mathbb{R}$ having a lower bound has a g.l.b. (inf).

1.3. **Completeness Property of $\mathbb{R}$.** This is known as **completeness property** of $\mathbb{R}$. (For a proof, see Chapter 1, Rudin Principles of Mathematical Analysis.)

Example 1.6.

- If $A(\neq \emptyset) \subseteq \mathbb{R}$ is not bounded above, we write $\sup A = +\infty$.
- If $B(\neq \emptyset) \subseteq \mathbb{R}$ is not bounded below, we write $\inf B = -\infty$.
- If $A = \emptyset$, then we write $\inf A = +\infty$ and $\sup A = -\infty$.

(**Hints:** $\{a\} \subset \{a, b\} \implies \inf\{a\} = a \geq \inf\{a, b\}$
$\therefore \phi \subset \{a\} \implies \inf \phi \geq a$ for all $a \in \mathbb{R}$)

Properties 1.7.

- If $A \subseteq B \subseteq \mathbb{R}$, then $\inf A \geq \inf B$ and $\sup A \leq \sup B$.
- Therefore, $\emptyset \subset Q \subset \mathbb{R} \implies \inf Q \geq q, \forall q \in \mathbb{R}$.

1.4. **Archimedean Property.** Let $x > 0$ and y be any real number. Then there exists a positive integer n such that $nx > y$.

(Implies any two real numbers can be compared.)

Proof. If there does not exist $n \in \mathbb{N}$ such that $nx > y$, then $nx \leq y$ for all $n \in \mathbb{N}$. Thus, y is an upper bound of the set $\{nx : n \in \mathbb{N}\}$.

By completeness property of $\mathbb{R}$, there exists $l \in \mathbb{R}$ such that

$$l = \sup\{nx : n \in \mathbb{N}\}$$

Note that $x \leq l$.

Since l is the least upper bound, there exists $n \in \mathbb{N}$ such that $l - x < nx < l$.
This implies $l < (n+1)x$, which contradicts the fact that l is a supremum. □

Exercise 1.8. Let

$$A = \{r \in \mathbb{Q} : r^2 < 2,\ r > 0\}$$

Show that $\sup A = \sqrt{2}$ (which is not in $\mathbb{Q}$).

Example 1.9. If $x, y \in \mathbb{R}$, then $x < y$ or $x > y$

If $y - x > 0$, by comparing $y - x$ with 1 (using Archimedean property), we get $n(y-x) > 1$
$\implies$ there exist integer m such that $ny > m > nx$
$\implies x < \frac{m}{n} < y$

That is, between any two real numbers, there is a rational.

Similarly,

$$\frac{x}{\sqrt{2}} < \frac{m}{n} < \frac{y}{\sqrt{2}} \implies x < \frac{m}{n}\sqrt{2} < y$$

i.e., between any two real numbers, there is an irrational.

Example 1.10. Find inf and sup of $\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$.

Solution: Let $A = \left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$.
Clearly,
$\left\{\frac{1}{1+n} : n \in \mathbb{N}\right\} \subset A$ and $\frac{1}{1+n}$ approaches to 0 for large n.

So, if $\alpha = \inf A > 0$, then by Archimedean Property, there exists $n \in \mathbb{N}$ such that $(n+1)\alpha > 1 \implies \alpha > \frac{1}{n+1}$, which contradicts that α is $\inf A$.

If $\beta = \sup A < 1$, then $(m+1)(1-\beta) > 1$ (by Archimedean Property), $\implies \beta < \frac{m}{m+1}$, which is a contradiction.

Example 1.11. If $\alpha = \inf A$ and $\beta = \sup A$. Then for $\epsilon > 0$, there exists $x_0, y_0 \in A$ such that $x_0 < \alpha + \epsilon$ and $y_0 > \beta - \epsilon$.

Proof. Suppose for a given $\epsilon > 0$, there does not exist $x \in A$ such that $x < \alpha + \epsilon$. Then $x \geq \alpha + \epsilon$ for all $x \in A$

$\implies x \geq \alpha + \epsilon > \alpha \implies \alpha + \epsilon$ is a lower bound, which contradicts the fact that α is the greatest lower bound.
Similar argument for β works. □

1.5. **Sequence.**

Definition 1.12. A function $f : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called a sequence, and we write $\{f(1), f(2), \ldots, f(n), \ldots\}$ or $\{f_n\}$.

Definition 1.13. A sequence $\{a_n\} \subseteq \mathbb{R}$ is said to be convergent to l if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies |a_n - l| < \epsilon$$

or $a_n \in (l - \epsilon, l + \epsilon)$, for all $n \geq n_0$.

Example 1.14. $a_n = \frac{1}{n} \to 0$. For this, let $\epsilon > 0$,

$$\frac{1}{n} < \epsilon \implies n > \frac{1}{\epsilon} > \left[\frac{1}{\epsilon}\right]$$

Therefore, for all $n > \left[\frac{1}{\epsilon}\right] + 1 = n_0$, $|a_n - 0| < \epsilon$.

Theorem 1.15. *Every convergent sequence is bounded.*

Proof. Let $a_n \to a$. Then for $\epsilon = 1 > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a| < 1 \implies a_n \in (a - 1, a + 1)$ for all $n \geq n_0$.
Let $m = \inf[(a-1, a+1) \cup \{a_1, \ldots, a_{n_0-1}\}]$ and $M = \sup[(a-1, a+1) \cup \{a_1, \ldots, a_{n_0-1}\}]$.
Then $m \leq a_n \leq M$, for all $n \in \mathbb{N}$. □

Theorem 1.16. *If a_n is increasing and bounded above, then a_n is convergent and* $\lim a_n = \sup_{n\geq 1} a_n$.

Proof. Let $\alpha = \sup a_n$. Then for $\epsilon > 0$, there exists a_{n_0} such that $a_{n_0} > \alpha - \epsilon$. $\implies \alpha + \epsilon > a_n \geq a_{n_0} \geq \alpha - \epsilon$, for all $n \geq n_0$. Thus, $a_n \to \alpha = \sup a_n$.

Similarly, if a_n is decreasing and bounded below, then a_n is convergent and $\lim a_n = \inf a_n$. □

1.6. **Nested Interval Theorem.**

Statement: If $I_n \supset I_{n+1} \supset \cdots$ and $\lim(\ell(I_n)) = b_n - a_n = 0$, where $I_n = [a_n, b_n]$, then $\bigcap_{n=1}^{\infty} I_n = \{x\}$.

Proof. It is clear that, a_n is increasing and $< b_1$, and b_n is decresing and $> a_1$. Hence, $\{a_n\}$ and $\{b_n\}$ are convergent. Let $a_n \to a$ and $b_n \to b$.

If $b - a = \lim(b_n - a_n) = 0 \implies a = b$.

Notice that $a_n \leq a$ and $b_n \geq a \implies a_n \leq a \leq b_n \implies a \in \bigcap I_n$.

If $x \in \cap I_n$, then $a_n \leq x \leq b_n \implies x = a$. □

Definition 1.17. If $\{x_n\}$ is a sequence and $n_1 < n_2 < \cdots < n_k < \cdots$, where $n_k \in \mathbb{N}$, then $\{x_{n_k}\}$ is called a *subsequence* of sequence $\{x_n\}$.

Example 1.18. $\{\frac{1}{k^2}\}, \{\frac{1}{2^k}\}$ are subsequences of $\{\frac{1}{n}\}$ with $n_k = k^2$, $n_k = 2^k$ respectively.

Bolzano-Weierstrass Theorem: *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof. Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$. Then there exist $a, b \in \mathbb{R}$ such that $x_n \in [a, b]$, for all $n \in \mathbb{N}$.

Divide $[a, b]$ into two parts, say $[a, b_1]$ and $[b_1, b]$ and Suppose $I_1 = [a, b_1]$ contains infinitely many terms of $\{x_n\}$. Further, choose $x_{n_1} \in I_1$. Further, divide $I_1 = I_2 \cup I_2'$ and suppose I_2 contains infinitely many terms of $\{x_n\}$. Choose $x_{n_2} \in I_2$ such that $n_1 < n_2$.

Then $x_{n_k} \in I_k$ and $I_k \supset I_{k+1} \supset \dots$

Then $\ell(I_k) \to 0$. By Nested Interval Theorem, $\cap I_k = \{x\}$.

Thus for each $\epsilon > 0$, $\exists k_0 \in \mathbb{N}$ such that for all $k \geq k_0 \implies I_k \subset (x - \epsilon, x + \epsilon)$ (How??)

i.e. $x_{n_k} \in (x - \epsilon, x + \epsilon)$ for all $k \geq k_0 \implies x_{n_k} \to x$. ☐

Remark 1.19. Suppose $(x_n) \subset [a, b]$, let $x_{n_k} = \inf_{n \geq k} x_n = \inf\{x_k, x_{k+1}, \dots\}$

Then $x_{n_k} \uparrow$ and $< b \implies x_{n_k} \to \sup_{k \geq 1}(\inf_{n \geq k} x_n)$

i.e. $\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty}(\inf_{n\geq k} x_n) = \underline{\lim} x_n$ (say)

Similarly, $y_{n_k} = \sup_{n \geq k} x_n = \sup\{x_k, x_{k+1}, \dots\}$

Then $y_{n_k} \downarrow$ and $> a \implies y_{n_k} \to \inf_{k\geq 1}(\sup_{n\geq k} x_n)$

i.e. $\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty}(\sup_{n\geq k} x_n) = \overline{\lim} x_n$ (say)

Notice that subsequences (x_{n_k}) and (y_{n_k}) need not be subsequences of (x_n).

Also, $\inf_{n\geq 1} x_n \leq x_{n_k} \leq y_{n_k} \leq \sup_{n \geq 1} x_n$.

Thus, limit of sequence (x_{n_k}) can be thought as lower limit of (x_n) and similarly limit of (y_{n_k}) can be as upper limit of (x_n).

Since both (x_{n_k}) and (y_{n_k}) are convergent, it follows that

$$\lim x_{n_k} \leq \lim y_{n_k}$$

That is, $\underline{\lim} x_n \leq \overline{\lim} x_n$

Example 1.20. $x_n = (-1)^n$, then $\underline{\lim} x_n = -1 < 1 = \overline{\lim} x_n$.

Exercise 1.21. If $x_n \to x$, then show that $\underline{\lim} x_n \geq \overline{\lim} x_n$.

Thus, deduce that a bounded sequence (x_n) is convergent iff $\underline{\lim} x_n = \overline{\lim} x_n$.

Example 1.22. If $X_n = (x_n, y_n) \in \mathbb{R}^2$ is a bounded sequence, then

$$\sqrt{x_n^2 + y_n^2} \leq M, \forall n \geq 1$$
$$\implies |x_n| \leq M \text{ and } |y_n| \leq M, \forall n \geq 1$$

By Bolzano-Weierstrass theorem, there exists (x_{n_k}) such that $x_{n_k} \to x \in \mathbb{R}$.

Now, (y_{n_k}) is also a bounded sequence, hence by Bolzano-Weierstrass theorem, there exists $y_{n_{kl}} \to y$. Thus, $(x_{n_{kl}}, y_{n_{kl}}) \to (x, y) \in \mathbb{R}^2$

Remark 1.23. Similar arguments can be produced for sequences in $\mathbb{R}^n$.

2. Open Sets and Closed Sets

2.1. Open Sets.

Definition 2.1. A set $A \subseteq \mathbb{R}$ is said to be **open** if every point $x \in A$ encloses an open interval in $I_x \subset O$.
i.e., for each $x \in O$, $\exists\, \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset O$.

Thus, **a countable union of open intervals is an open set.**

On the other hand, **any open set in $\mathbb{R}$ can be written as a countable union of disjoint open intervals.**

Theorem 2.2. *Let O be an open set in $\mathbb{R}$, then there exists a disjoint family of countably many open intervals I_n such that*

$$O = \bigsqcup_{n=1}^{\infty} I_n$$

Proof. Since O is open, for $x \in O$, there exists an open interval (a,b) such that $x \in (a,b) \subset O$.
Now, we extract the largest open interval containing x and contained in O.

Let $a_x = \inf\{a : (a,x] \subset O\}$,
and $b_x = \sup\{b : [x,b) \subset O\}$.

Then $I_x = (a_x, b_x)$ will be the largest open interval containing x and contained in O.

Note that $I_x = (a_x, b_y) \subset O$. For this, let $a_x < y < b_y$, then $a_x < y - \epsilon$ for small $\epsilon > 0$
$\implies a_x + \epsilon < y$.
But by definition of infimum, $\exists\, a < a_x + \epsilon$ and $(a, x] \subset O$
$\implies (a_x + \epsilon, x] \subset O$.

Similarly, $[x, b_x - \epsilon) \subset O$
$\implies (a_x + \epsilon, b_x - \epsilon) \subset O$ for small $\epsilon > 0$
$\implies (a_x, b_x) \subset O$.

Now, if $x, y \in O$ and $x \neq y$ then either $I_x \cap I_y = \emptyset$ or $I_x = I_y$.

If $I_x \cap I_y \neq \emptyset$, then $I_x \cup I_y$ is an open interval containing x and y.

Therefore, by maximality of I_x for x and I_y for y, it follows that

$$I_x \cup I_y \subseteq I_x \implies I_y \subseteq I_x$$

Since $y \in I_y \implies I_y = I_x$ ($\because I_y$ is maximal)

Now, $O = \bigsqcup_{x \in O} I_x$. Since I_x and I_y are disjoint (if $x \neq y$), we can assign a distinct rational to each of them. That is, choose $r_x \in I_x$ and $r_y \in I_y$. Then $r_x \neq r_y$

Thus,

$$\{I_x : x \in O\} \xrightarrow{1-1} \mathbb{Q} \text{ (set of rationals) via } I_x \mapsto r_x$$

Hence,

$$O = \bigsqcup_{i=1}^{\infty} I_{r_i} \tag{1}$$

The representation (1) is unique.

Let $O = \bigsqcup_{n=1}^{\infty} I_n = \bigsqcup_{m=1}^{\infty} J_m$.

Then $I_n = I_n \cap O = \bigsqcup_{m=1}^{\infty} (I_n \cap J_m)$.

Since $\{I_n \cap J_m : m \in \mathbb{N}\}$ is a disjoint family and I_n is an open interval, $I_n \subset I_n \cap J_{m_0}$ for some m_0.

But then $I_n \subset J_{m_0}$, and given I_n is maximal, $\implies I_n = J_{m_0}$.

Thus, the representation (1) is unique upto change in order of union. □

2.2. **Closed Sets.**

Definition 2.3. A set $A \subseteq \mathbb{R}$ is said to be **closed** if for each sequence $(x_n) \in F$ with $x_n \to x$, implies $x \in F$.

Theorem 2.4. *A set $F \subseteq \mathbb{R}$ is closed if and only if F^c is open.*

Proof. Let F be a closed set. Suppose F^c is not open. Then for some $x \in F^c$, $\nexists \epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset F^c$.

Take $\epsilon = \frac{1}{n}$, then $x_n \in (x - \frac{1}{n}, x + \frac{1}{n})$ and $x_n \in F$. Thus, $x_n \to x$ and F is closed, implies $x \in F$, which is a contradiction.
Hence, F^c is open.

Conversely, suppose F^c is open. Let $x_n \in F$ and $x_n \to x$.
Claim: $x \in F$.

If $x \notin F$, then $x \in F^c$, which is open. Then $\exists\, r > 0$ such that $(x - r, x + r) \subset F^c$.

Since $x_n \to x$, $\exists\, n_0 \in \mathbb{N}$ such that $n \geq n_0 \implies x_n \in (x - r, x + r) \subset F^c$, which is absurd. Thus $x \in F$. □

Notice that we can define open and closed sets in $\mathbb{R}^n$ in a similar way.

Example 2.5. The set $A = \{(x, y) : y = \sin\frac{1}{x}, x \neq 0\}$ is neither open nor closed in $\mathbb{R}^2$ (in the usual metric).

Let $x_n = \frac{1}{n\pi}$, $n \in \mathbb{N}$, then $(x_n, y_n) = (\frac{1}{n\pi}, 0) \in A$.
But $\lim(x_n, y_n) = (0, 0) \notin A$.
$B_{\frac{1}{n}}(\frac{1}{\pi}, 0) \not\subset A$

2.3. **Interior of a set.**

Let $A \subseteq \mathbb{R}$, then there exists open set $O \subseteq \mathbb{R}$ such that $A \subset O = \bigsqcup_{n=1}^{\infty} I_n,\ I_n = (a_n, b_n)$.

Let us collect all open intervals which are contained in A.

Interior of A (or A°) = union of all open intervals contained in A
i.e., the interior of A is the largest open set A° contained in A.
(That is, O is open and $O \subset A \implies O \subseteq A^\circ$.)

Example 2.6.

$$\mathbb{N}^\circ = \varnothing, \quad \mathbb{Q}^\circ = (\mathbb{R} - \mathbb{Q})^\circ = \varnothing$$

and

$$\{(x, y) : y = \sin\frac{1}{x}, x \neq 0\}^\circ = \varnothing$$

2.4. **Closure of a set.**

Let $A \subseteq \mathbb{R}$ and $x_n \in A$ such that $x_n \to x$.

Closure of A (or $\overline{A}$) is the collection of x which is the limit of a sequence in $x_n \in A$.

That is, closure of a set A is the smallest set $\overline{A}$ that contains A.
That is, if B is closed and $A \subseteq B \implies \overline{A} \subseteq B$.

Example 2.7. Show that the closure of $A = \{(x, \sin\frac{1}{x}) : x \neq 0\}$ is the set $A \cup (\{0\} \times [-1, 1])$.

Notice that $A \cup (\{0\} \times [-1, 1])$ is a closed set containing A. Hence $\overline{A} \subseteq A \cup (\{0\} \times [-1, 1])$.
Here, $(\frac{1}{n\pi}, 0) \to (0, 0)$ and $(\frac{1}{\pm(2n+1)\frac{\pi}{2}}), \pm 1) \to (0, \pm 1)$.

Hence, $(0, 0), (0, \pm 1) \in \overline{A}$.

Need is to show for $y \in (-1, 1) \setminus \{0\}$. Find a sequence $(x_n) \in \mathbb{R} \setminus \{0\}$ such that $(x_n, \sin\frac{1}{x_n}) \to (0, y)$.

Or $x_n \to 0$ and $\sin\frac{1}{x_n} \to y$ etc.

Definition 2.8. A closed and bounded subset of $\mathbb{R}^n$ is called **compact** in $\mathbb{R}^n$.

Exercise 2.9. The set $\overline{\{(x, y) : y = \sin\frac{1}{x}, x \neq 0\}}$ is closed but not bounded.

Note that if $K \subseteq \mathbb{R}$ there exist an open set $O \subset \mathbb{R}$ exists such that

$$K \subset O = \bigsqcup_{n=1}^{\infty} I_n \quad \text{(open cover)}$$

Using the Bolzano-Weierstrass theorem, it can be deduced that the set $K \subset \mathbb{R}$ is compact if and only if every open cover of K reduces to finite sub-cover, i.e.,

$$K \subset \bigsqcup_{n=1}^{l} I_n$$

Similar arguments hold for K compact subset of $\mathbb{R}^n$.

Example 2.10. A subset $F \subseteq \mathbb{R}$ is closed if and only if $\forall \varepsilon > 0,\ (x - \varepsilon,\ x + \varepsilon) \cap F \neq \emptyset \implies x \in F$

Proof. Suppose F is closed and for all $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) \cap F \neq \emptyset$. Then for $\varepsilon = \frac{1}{n}$, $\exists x_n \in (x - \frac{1}{n}, x + \frac{1}{n}) \cap F$.

$$\implies |x_n - x| < \frac{1}{n}, \quad \text{for all} \quad n \in \mathbb{N} \implies x_n \to x \quad \text{and } F \text{ is closed} \implies x \in F$$

Conversely, let $\forall \varepsilon > 0,\ (x - \varepsilon,\ x + \varepsilon) \cap F \neq \emptyset \implies x \in F$.

Claim: F is closed.

Let $x_n \in F$ and $x_n \to x$. Then for $\varepsilon > 0, \exists n_0 \in \mathbb{N}$,

$$n \geq n_0 \implies x_n \in (x - \varepsilon, x + \varepsilon) \cap F \neq \emptyset \implies x \in F$$

□

2.5. **Dense Set.**

Definition 2.11. Let $A \subseteq \mathbb{R}$ and $x_n \in A$ such that $x_n \to x$. Then,

$$\bar{A} = \{x \in \mathbb{R} : \exists x_n \in A \text{ with } x_n \to x\}$$

If $\bar{A} = \mathbb{R}$, then A is called **dense in** $\mathbb{R}$.

Example 2.12. Let $x \in \mathbb{R}$, then

$$(*) \qquad x = x_0 + \frac{x_1}{10} + \cdots + \frac{x_n}{10^n} + \cdots \quad \text{where} \quad x_i \in \{0, 1, \ldots, 9\}$$

$$\text{Let} \quad S_n = x_1 + \cdots + \frac{x_n}{10^n} \in \mathbb{Q}$$

Then $S_n \to x$, thus $\bar{\mathbb{Q}} = \mathbb{R}$.

Also, $x_n = x + \frac{1}{(1+n^3)^{\frac{1}{3}}} \in \mathbb{R} \setminus \mathbb{Q}$ (??) and $x_n \to x$. Thus $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$.

Note that representation $(*)$ is not unique, e.g. $0.5 = 0.4999\ldots$

Theorem 2.13. *Let $p \in \mathbb{Z}, p \geq 2$ and $0 \leq x \leq 1$.*

Then $\exists$ a sequence of integers (a_n) such that $0 \leq a_n \leq p-1$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

Proof. Choose a_1 to be the largest integer such that

$$\frac{a_1}{p} < x \quad \text{(by Archimedean property)}$$

Since $0 < x \leq 1 \implies a_1 < p$. Given a_1 is an integer, $a_1 \leq p-1$. Also, a_1 is the largest, we must have

$$\frac{a_1}{p} < x \leq \frac{a_1+1}{p}$$

Next, choose a_2 such that

$$\frac{a_1}{p} + \frac{a_2}{p^2} < x$$

$$\implies 0 \leq a_2 \leq p-1 \quad \text{and} \quad [\quad \frac{a_2}{p} < p - a_1 < 1, \text{since } a_1 \text{ is largest }]$$

$$\frac{a_1}{p} + \frac{a_2}{p^2} < x \leq \frac{a_1}{p} + \frac{a_2+1}{p^2}$$

By induction,

$$\frac{a_1}{p} + \cdots + \frac{a_n}{p^n} < x \leq \frac{a_1}{p} + \cdots + \frac{a_n+1}{p^n}$$

$$\implies x = \sum_{n=1}^{\infty} \frac{a_n}{p^n} \quad \text{(p-adic) decimal expansion}$$

□

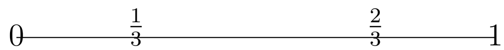
Exercise 2.14. Show that $\left\{\frac{k}{2^n} : k = 0, 1, 2, \ldots, 2^n; n = 1, 2, \ldots\right\}$ is dense in $[0, 1]$.

(Hint: Use binary expansion)

3. Cantor Set

The Cantor set is an **uncountable set** in $[0,1]$ having zero length with many peculiar properties, answering some of the difficult questions related to topology of real line.

Let $C_0 = [0,1]$.



Delete middle one-third open interval $J_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ from C_0. Then

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

Delete one-third open interval from each section of C_1, and let

$$J_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$$

Then,

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

Thus,

- $C_0 = [0,1]$, one closed interval of length 1.
- $C_1 = [0, \frac{1}{3}] \sqcup [\frac{2}{3}, 1]$, two closed disjoint intervals each of length $\frac{1}{3}$.
- $C_2 = [0, \frac{1}{9}] \sqcup [\frac{2}{9}, \frac{1}{3}] \sqcup [\frac{2}{3}, \frac{7}{9}] \sqcup [\frac{8}{9}, 1]$, four closed disjoint intervals each of length $\frac{1}{9}$.

By induction, we can construct C_n with 2^n disjoint closed intervals each of length 3^{-n}.

3.1. Properties of the Cantor Set.

(1) C_n is a sequence of closed and bounded intervals, hence, by nested intervals theorem,

$$\bigcap C_n \neq \emptyset$$

(Hint: use nested intervals theorem for each chain in the construction of C_n).

(2) Let $C = \bigcap_{n=0}^{\infty} C_n$, then C contains all the end points of the deleted open intervals.

(3) $C = [0,1] \setminus J_1 \sqcup J_2 \ldots \sqcup J_n \ldots = [0,1] \setminus \bigsqcup_{n=1}^{\infty} J_n$

(4) Since $C \subset C_n$, $\forall n \geq 0$,

$$l(C) \leq l(C_n) = 2^n \cdot \frac{1}{3^n} \to 0$$

Thus, the total length $C = 0$. This shows that the set C is "small". On the other hand, we shall see that C is uncountable.

(5) The Cantor ternary set C (later we just say Cantor set) is nowhere dense. i.e. $(\bar{C})^\circ = C^\circ = \emptyset$. If not, then for $x \in C^\circ \implies \exists \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset C^\circ \subset C \implies l((x-\epsilon, x+\epsilon)) \leq l(C) = 0 \implies 2\epsilon \leq 0$, which is a contradiction. Hence C is nowhere dense.

(6) C is totally disconnected (i.e. connected sets in C are singletons only). (We shall prove it later!)

(7) Every point of C is a limit point of C itself (i.e., C is a perfect set).
Let $x \in C = \bigcap C_n \implies x \in C_n, \forall n \geq 0$. Then x must belong to one of the closed intervals that constitute to C_n. That is, $x \in [x_n, y_n]$ with $y_n - x_n = \frac{1}{3^n}$.

$$\implies |x_n - x| \leq |y_n - x_n| = \frac{1}{3^n} \to 0$$

Note that x_n and y_n are end points of the deleted open intervals J_n's. Hence, $x_n, y_n \in C$. Thus, if E denotes the set of all end points, then $\bar{E} = C$. Since E is countable (being subset of rationals), C is separable (we define later).

3.2. **Representation of Cantor's set.** Consider the end pt $\frac{1}{3} \in C$. We can write

$$\frac{1}{3} = \frac{0}{3} + \frac{2}{3} + \frac{2}{3^2} + \cdots = (0.022\ldots)_3$$

Similarly,

$$\frac{2}{3} = (0.2)_3$$

Inductively, it can be shown that any end point $x \in E$ can be expressed as

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots, \quad a_i \in \{0, 2\}$$

Since each $x \in [0, 1]$ has ternary representation, consider the set

$$F = \left\{ x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, a_i \in \{0, 1, 2\} \right\}$$

If $x \in F$, then x is not an end point, and

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots, \quad a_i \in \{0, 1, 2\}$$

Notice that $a_1 = 1$ iff $x \in \left(\frac{1}{3}, \frac{2}{3}\right)$ iff $x \notin C_1$.

Next, $a_1 \neq 1, a_2 = 1$ iff $x \in \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$ iff $x \notin C_2$.

Thus, $a_{i_0} = 1$ for some i_0 iff $x \notin C_{i_0}$.

Now, let $x \in C = \cap C_n$ and $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$. Suppose some of $a_i = 1$, then $x \notin C_i \implies x \notin C \implies$ all the $a_i \in \{0, 2\}$.

That is,

$$C \subseteq \left\{ x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \ a_i \in \{0, 2\} \right\} \tag{*}$$

On the other hand, let $x \notin C$, then $x \notin C_{i_0}$ for some i_0. This completes $a_{i_0} = 1$.

That means, $x \notin$ RHS of (*)

Thus,

$$C = \left\{ x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \ a_i = 0, 2 \right\}$$

This implies Cantor set loses only one decimal index taken from $\{0, 1, 2\}$. Can it thought some light about uncountability of Cantor set?

3.3. **Representation is Unique.** For every $x \in C$, there exist unique sequence (a_n) from $\{0,2\}$ such that

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \tag{1}$$

Suppose

$$x = \sum_{i=1}^{\infty} \frac{b_i}{3^i}, \quad b_i \in \{0,2\} \tag{2}$$

Then claim $a_i = b_i,\ \forall i$.

If not, let i_0 be the smallest integer such that $a_{i_0} \neq b_{i_0}$. Then $a_i = b_i$ for $i = 1, 2, \ldots, i_0 - 1$.

Now, without loss of generality, we can take $i_0 = 1$. That is, $a_1 \neq b_1 \implies a_1 = 0$ and $b_1 = 2$ (or otherwise).

From (1), $x \in [0, \frac{1}{3}]$ and from (2), $x \in [\frac{2}{3}, 1]$, which is absurd.

Exercise 3.1. Conclude without assuming $i_0 = 1$.

Cantor set is uncountable:

Define $f : C \to [0,1] = \left\{x = \sum_{i=1}^{\infty} \frac{b_i}{2^i} : \ b_i \in \{0,1\}\right\}$ by

$$f(x) = f\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i}\right) = \sum_{i=1}^{\infty} \frac{\frac{a_i}{2}}{2^i}$$

then $b_i = \frac{a_i}{2} \in \{0,1\}$ an d$f(x) \in [0,1]$.

Since each $x \in C$ has unique representation, the map f is well defined.

f is not one-one:

$$f\left(\frac{1}{3}\right) = f\left((0.022\ldots)_3\right) = (0.011\ldots)_2 = (0.1)_2 = \frac{1}{2}$$

and

$$f\left(\frac{2}{3}\right) = f\left((0.2)_3\right) = (0.1)_2 = \frac{1}{2}$$

$$\Rightarrow f\left(\frac{1}{3}\right) = f\left(\frac{2}{3}\right)$$

Exercise 3.2. Show that $f(x) = f(y)$ iff x, y are end points of one of the deleted open interval.

f is an onto map:

Here $f : C \to [0,1]$ and let $y \in [0,1]$ such that

$$f(x) = y = \sum_{i=1}^{\infty} a_i \frac{1}{2^i}$$

Let

$$x = \sum \frac{2a_i}{3^i}$$

then $f(x) = y$ holds.

Hence C is an uncountable set.

<u>f is monotone increasing:</u>

Let $x, y \in C$ and $x < y$. Since ternary representation of C is unique, $\exists$ the least positive integer $n \in \mathbb{N}$ such that $a_n < b_n$. Hence $a_i = b_i, i = 1, 2, \dots, n-1$. Thus, while comparing $f(x)$ and $f(y)$, we can ignore the first $n-1$ terms. Therefore, WLOG, we can assume $n = 1$.

That is, $a_1 < b_1 \implies a_1 = 0, b_1 = 2$.

$$\therefore f(x) \leq \frac{0}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{1}{2}$$

and

$$f(y) = \frac{1}{2} + \frac{\frac{b_2}{2}}{2^2} + \frac{\frac{b_3}{2}}{2^3} + \cdots \geq \frac{1}{2}$$
$$\Rightarrow f(x) \leq f(y)$$

Notice that $f(\frac{1}{3}) = f(\frac{2}{3}) = \frac{1}{2}$. Hence, we can extend f to $[0,1]$ by keeping it constant on the deleted intervals.

Thus, $\tilde{f} : [0,1] \to [0,1]$ is defined by

$$\tilde{f}|_C = f \quad \text{and} \quad \tilde{f}([0,1] \setminus C) = \{\alpha_i\}$$

where α_i is the common value of f at the end point of deleted interval.

Thus, $\tilde{f} : [0,1] \to [0,1]$ is a monotone increasing onto function. Hence $\tilde{f}$ continuous (Why?) (We will see later.)

Now, define $g : [0,1] \to [0,2]$ by

$$g(x) = \tilde{f}(x) + x$$

Then g is strictly monotone increasing and onto function.

If $x < y$ then $g(x) = \tilde{f}(x) + x \leq \tilde{f}(y) + x < \tilde{f}(y) + y = g(y) \implies g(x) < g(y)$

Hence,

$$g(0) = 0 \quad \text{and} \quad g(1) = 2$$

$(\because g(1) = f(1) + 1 = f\left(\sum \frac{2}{3^i}\right) + 1 = 2)$

Since g is continuous on $[0,1]$, by Intermediate Value Theorem,

$$g([0,1]) = [0,2]$$

Exercise 3.3. Show that g^{-1} is monotone and continuous.

4. Limit and Continuity

Let f be a real valued function, which is defined in an open neighbourhood (nbd) of a point a, and may not be necessarily at a.

A number L is called left limit of f at a if for each $\varepsilon > 0$, $\exists\delta > 0$ such that

$$\text{for} \quad x \in (a-\delta, a) \implies |f(x) - L| < \varepsilon$$

or simply, we write

$$L = \lim_{x\to a^-} f(x) = f(x-)$$

Similarly, right limit if for $\varepsilon > 0$, $\exists\delta > 0$ such that

$$\text{for} \quad x \in (a, a+\delta) \implies |f(x) - M| < \varepsilon$$

or

$$M = \lim_{x\to a^+} f(x) = f(x+)$$

Moreover, if f is defined in nbd of a and a, then f is said to be continuous at a if for any $\varepsilon > 0$, there exist $\delta > 0$ such that

$$x \in (a-\delta, a+\delta) \implies |f(x) - f(a)| < \epsilon$$

or $f(x^-) = f(x) = f(x^+)$.

In case, when $f(x^-)$ and $f(x^+)$ exists and are unequal, we say f has jump discontinuity at a.

4.1. **Monotone Function.**

We shall see that a monotone function is continuous except on a countable set and it is also known that such functions are very close to differentiable function. We skip here the later one property.

Theorem 4.1. *Let $f : (a, b) \to \mathbb{R}$ be a monotone function, then for $c \in (a, b)$, $f(c^+)$ and $f(c^-)$ both exist.*

Proof. Let f be an increasing function.

$$f(c^-) = \sup\{f(x) : a < x < c\} = L \leq f(c)$$
$$f(c^+) = \inf\{f(x) : c < x < b\} = M \geq f(c)$$

[*]

For $\epsilon > 0$, there exists $x_0 \in (a, c)$ such that $f(x_0) > L - \epsilon$. Let $\delta = c - x_0$, then for $x \in (c-\delta, c)$,

$$L + \epsilon > f(x) \geq f(x_0) > L - \epsilon \quad \text{(since } f \text{ is increasing)}$$

i.e., for $x \in (c-\delta, c) \implies |f(x) - L| < \epsilon$. Hence,

$$f(c^-) = \sup\{f(x) : a < x < c\} = L$$

Similarly,

$$f(c^+) = \inf\{f(x) : c < x < b\} = M$$

Notice from [*] that if $c, d \in (a, b)$ and $c < d$, then $f(c^+) \leq f(d^-)$.

Hence either $(f(c^-), f(c^+))$ and $(f(d^-), f(d^+))$ both coincide or disjoint.

Choose rational r_c and r_d from the above intervals. Then these intervals have one-one correspondence with the set of rationals. Hence, the set of discontinuities of a monotone function is atmost countable. □

Example 4.2. If $f : [a,b] \to [c,d]$ is monotone and onto, then f is continuous.

Proof. Let f be an increasing function. Then $f(a) = c$ and $f(b) = d$.

If $f(a) > c$, then for $y \in [c, f(a))$, *there is no* $x \in [a,b]$ such that $f(x) = y$. If so, then $f(x) = y < f(a) \implies x < a$ (since f is increasing).

Further, if possible, let $f(c^-) < f(c)$.

Then $y \in (f(c^-), f(c))$ has no pre-image.

On contrary, if there exist $x_0 \in (a,c)$ such that $f(x_0) = y$. Then

$$L = \sup\{f(x) : a < x < c\} = f(c^-) < y = f(x_0) < f(c)$$

which contradicts the fact that L is supremum on (a,c).

Thus, $f(c^-) = f(c) = f(c^+)$. Hence, f is continuous. □

Example 4.3. If $f : (a,b) \to (c,d)$ is monotone and onto, then f is continuous.

(Proof is similar to the above case).

Observe that if f is monotone onto, then f need not be one-one.

(For example, Cantor function.)

$\tilde{f} : [0,1] \to [0,1]$ is monotone and onto but not one-one.

However, if $f : (a,b) \to (c,d)$ is strictly monotone and onto, then

$$f^{-1} : (c,d) \to (a,b)$$

is continuous, because, in this case, f^{-1} is also strictly monotone.

For this, if f is increasing function, then for $y_1 < y_2$

$$\implies f^{-1}(y_1) < f^{-1}(y_2).$$

If not, then for $y_1 = f(x_1)$ and $y_2 = f(x_2)$, it follows that $x_1 \geq x_2$ (since $f^{-1}(y_1) \geq f^{-1}(y_2)$), but then $f(x_1) = y_1 < y_2 = f(x_2)$ is a contradiction to the fact that f is strictly increasing.

Notice that $f : C([a,b]) \xrightarrow{onto} C([c,d])$need not be continuous if f is monotone, else $f([a,b])$ is compact.

Finally, if $f : \mathbb{R} \to \mathbb{R}$ is one-one and onto, then f and f^{-1} both are continuous.

Example 4.4. If I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a monotone function, then

$$E_\alpha = \{x \in I : f(x) > \alpha\} = I' \text{ or } \varnothing,$$

where I' is an interval.

Let f be an increasing function. If $x' \in E_\alpha$, then for $x < x' \leq b, \implies f(b) \geq f(x) \geq f(x') > \alpha \implies [x', b] \subset E_\alpha$.

Let $x_0 = \inf\{x \in I : f(x) > \alpha\} = \inf E_\alpha$.

(i) If $x_0 = a$, then for $x \in I$, there exists $x_1 \in E_\alpha$ such that $x_1 \leq x$ and $f(x) \leq f(x_1) > \alpha \implies x \in E_\alpha$. So $I = E_\alpha$.

(ii) If $a < x_0 \leq b$, then for $x > x_0$, there exists $x_1 \in E_\alpha$ such that $x_0 < x_1 < x$ and $f(x) \geq f(x_1) > \alpha \implies (x_0, b] \subset E_\alpha$.

(ii) If $x < x_0$, then $f(x) \leq \alpha \implies x \notin E_\alpha \implies (x_0, b] \subset E_\alpha \subset [x_0, b]$.

This proves the claim that E_α is an interval.

4.2. **Construction of Monotone Function.** Let D be a countable set in $\mathbb{R}$, then we can construct a monotone increasing function which is discontinuous only on D.

Let $D = \{x_1, x_2, \ldots\}$ and $0 < \epsilon_n < 1$ be a sequence such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Let us define

$$f(x) = \sum_{x_n \le x} \epsilon_n,$$

where the sum is on the set $\{n : x_n \le x\} = A_x$(say) and $f(x) = 0$ if the set $A_x = \varnothing$.

If $x < y$, then

$$f(y) = \sum_{x_n \le y} \epsilon_n = \sum_{x_n \le x} \epsilon_n + \sum_{x < x_n \le y} \epsilon_n \ge f(x).$$

Note that for $x = x_k < y$, we get

$$f(y) = f(x_k) + \sum_{x_k < x_n \le y} \epsilon_n$$

Then

$$f(x_k^+) = f(x_k) + \lim_{y \to x_k^+} \sum_{x_k < x_n \le y} \epsilon_n = f(x_k)$$

Since $\sum_{n=N}^{\infty} \epsilon_n \to 0$ as $N \to \infty$.

And when $x < x_k = y \Rightarrow$

$$f(x_k) = f(x) + \sum_{x < x_n \le x_k} \epsilon_n \ge f(x) + \epsilon_k$$

Then

$$\lim_{x \to x_k^-} f(x) = f(x_k) - \lim_{x \to x_k^-} \sum_{x < x_n \le x_k} \epsilon_n = f(x_k) - \epsilon_k$$

So

$$f(x_k^-) = f(x_k) - \epsilon_k$$

Thus,

$$f(x_k^+) - f(x_k^-) = \epsilon_k.$$

The proof of f is continuous at each point of $\mathbb{R} \setminus D$ is similar to the above.

Let $x \in \mathbb{R} \setminus D$. Then $x \ne x_n$ for any n.

For $x < y$,

$$f(y) = f(x) + \sum_{x < x_n \le y} \epsilon_n.$$

When $y \to x^+$, then

$$\sum_{x < x_n \le y} \epsilon_n \to 0 \quad (\because \{n : x < x_n \le y\} \to \varnothing)$$

If $y < x$, then

$$f(x) = f(y) + \sum_{y < x_n < x} \epsilon_n.$$

Hence,

$$f(x) = \lim_{y \to x^-} f(y) + \lim_{y \to x^-} \sum_{y < x_n < x} \epsilon_n = f(x^-) + 0 \quad (\because \{n : y < x_n < x\} \to \varnothing)$$

Example 4.5. Let $D = \mathbb{Z}$, then

$$f(x) = \sum_{n \le x} \epsilon_n$$

For $x \in (0,1)$,

$$f(x) = \sum_{n \le 0} \epsilon_n = C.$$

$\Rightarrow$ Constant on each open interval $(n, n+1)$

Example 4.6. Let D be the of end points of deleted open intervals in the construction of Cantor set. Find appropriate sequence $0 < \epsilon_n < 1$ define Cantor function via

$$f(x) = \sum_{x_n \le x} \epsilon_n, \quad x_n \in D.$$

Example 4.7. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = x + \sum_{n=0}^{n_x} 2^{-n}, \quad n_x = \left\lfloor \frac{1}{1-x} \right\rfloor \text{ if } x < 1$$

and $f(1) = 3$.

Show that f is strictly increasing and discontinuous on $\left\{1 - \frac{1}{k} : k \in \mathbb{N}\right\}$.

5. Metrics and Norms

Let X be a non-empty set. A map $d : X \times X \to \mathbb{R}^+ = [0, \infty)$ such that

(i) $d(x, y) = 0$ iff $x = y$, $\quad x, y \in X$
(ii) $d(x, y) = d(y, x)$ $\quad$ (symmetric)
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ $\quad$ (triangle inequality)

is called a **metric** on X, and the pair (X, d) is called a **metric space**.

Example 5.1. If $X = \mathbb{R}^n$, then for $x, y \in \mathbb{R}^n$, $1 \leq p < \infty$,

(1) $d_p(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p}, \quad x = (x_1, \dots, x_n)$
is a metric on $\mathbb{R}^n$ (we prove it later).
(2) $d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$ is a metric on $\mathbb{R}^n$. (It follows easily.)

Example 5.2. Let (X, d) be a metric space. Show that $d'(x, y) = \min\{1, d(x, y)\}$ defines a metric.

Example 5.3. If $X = C[0, 1]$, the space of continuous functions on $[0, 1]$, then for $f, g \in X$,

$$d_\infty(f, g) = \sup_{0 \leq t \leq 1} |f(t) - g(t)|$$

defines a metric on $\mathbb{R}$.

(*Hint:* f is continuous on $[0, 1]$, so f is bounded and $|f(t) - h(t)| \leq |f(t) - g(t)| + |g(t) - h(t)|$)

Example 5.4. For $f, g \in C[0, 1]$, define

$$\rho(f, g) = \int_0^1 \min\{|f(t) - g(t)|, 1\}\, dt$$

Then ρ is a metric on $C[0, 1]$.

Example 5.5. If $X \neq \emptyset$, then for $x, y \in X$,

$$d_0(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

defines a metric on X and called the **discrete metric**. Thus, every non-empty set has a metric.

Note that for $d(x, z) \leq d(x, y) + d(y, z)$ to hold, we need to verify three cases:

(1) $x = y$, $y \neq z$
(2) $x \neq y$, $y = z$
(3) all of x, y, z are distinct

Question 5.6. If (X, d) is a metric space and $f : [0, \infty) \to [0, \infty)$ is a map, does it imply that $f \circ d$ is a metric on X?

Example 5.7. Let $f(t) = \frac{t}{1+t}$, then $f'(t) = (1 - \frac{1}{1+t})' \implies f'(t) = \frac{1}{(1+t)^2} > 0$ for all $t \in [0, \infty)$. Hence f is strictly increasing.

And $f''(t) = -\frac{2}{(1+t)^3} < 0$, hence concave.
Also, $f(t) = 0$ iff $t = 0$.

Note that
$$\frac{t+s}{1+t+s} \leq \frac{t}{1+t} + \frac{s}{1+s}$$
Let $s = d(x,y)$, $t = d(x,z)$, $r = d(y,z)$, then $r \leq s+t$,
$$f(r) \leq f(s+t) \leq f(s) + f(t)$$
$$\implies (f \circ d)(x,z) \leq (f \circ d)(x,y) + (f \circ d)(y,z)$$
Thus, $f \circ d$ is a metric on X.

This result is true for a large class of concave function.

Example 5.8. Let $f : [0,\infty) \to [0,\infty)$ be concave and $f(0) \geq 0$. Then
$$f(x+y) \leq f(x) + f(y) \qquad \text{(sub-additive)}$$
Hence,
$$\frac{y}{x+y} f(0) + \frac{x}{x+y} f(x+y) \leq f\left(\frac{y}{x+y} \cdot 0 + \frac{x}{x+y}(x+y)\right) \implies \frac{x}{x+y} f(x+y) \leq f(x) \ (\because f \text{ is concave})$$
Replacing $x \to y$, we get $\frac{y}{x+y} f(x+y) \leq f(y) \implies f(x+y) \leq f(x) + f(y)$.

Result: Let (X,d) be a metric space and $f : [0,\infty) \to [0,\infty)$ be a monotone increasing function with $f(t) = 0$ iff $t = 0$. If f is concave, then $f \circ d$ is a metric on X.

(Hint: Conclude from the example and the previous result.

Example 5.9. Let H^∞ (**Hilbert cube**) be the space of sequences $x = (x_n) = (x_1, x_2, \ldots, x_n, \ldots)$ such that $|x_n| \leq 1$. Then
$$d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}$$
defines a *metric* on H^∞.

(i) $d(x,y) \leq \sum \frac{2}{2^n} < \infty$

(ii) $|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n|$
$$\Rightarrow \sum_{n=1}^{k} \frac{|x_n - z_n|}{2^n} \leq \sum_{n=1}^{k} \frac{|x_n - y_n|}{2^n} + \sum_{n=1}^{k} \frac{|y_n - z_n|}{2^n}$$
$$\leq d(x,y) + d(y,z) < \infty$$
Since LHS is an increasing sequence which is bounded above, it follows that
$$\lim_{k\to\infty} \sum_{n=1}^{k} \frac{|x_n - z_n|}{2^n} \leq d(x,y) + d(y,z)$$
$$\implies d(x,z) \leq d(x,y) + d(y,z).$$

Exercise 5.10. Show that $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ defines a metric on $(0,\infty)$.

(*Hint:* $|\frac{1}{x} - \frac{1}{z}| = |\frac{1}{x} - \frac{1}{y} + \frac{1}{y} - \frac{1}{z}| \leq |\frac{1}{x} - \frac{1}{y}| + |\frac{1}{y} - \frac{1}{z}|$)

Definition 5.11. $B_r(x) = \{y \in X : d(y,x) < r\}$ is called an open ball in the metric space (X,d).
$B_r[x] = \{y \in X : d(y,x) \leq r\}$ is called closed ball in (X,d).

Definition 5.12. A set O in a metric space (X, d) is called open if for each $x \in O$, $\exists\, r > 0$ such that $B_r(x) \subseteq O$.

Let $\mathcal{J}$ be the collection of all open sets in X with respect to d. Then

(i) $\emptyset, X \in \mathcal{J}$ (why?)
(ii) $\bigcup_{i\in I} O_i \in \mathcal{J}$, for $O_i \in \mathcal{J}$ and for any index set I
(iii) $\bigcap_{i=1}^{n} O_i \in \mathcal{J}$, for $O_i \in \mathcal{J}$

(*Hint:* Follows from definition of open set.)

Definition 5.13. A function $f : (X, d) \to \mathbb{R}$ is said to be continuous at $x \in X$, if for any $\epsilon > 0$, there exist $\delta > 0$ such that

(*) $$d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$$

If it happen for each $x \in X$, we say that f is continuous on X.
From (*), it follows that

$$y \in B_\delta(x) \implies f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$$

i.e. $B_\delta(x) \subseteq f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$.
Since $x \in$ RHS, it follows that RHS is open around x.

Result: A function $f : (X, d) \to \mathbb{R}$ is continuous iff $f^{-1}(O) \in \mathcal{J}$ for each open set O in $\mathbb{R}$.

Proof. Suppose f is continuous. Let $O \subset \mathbb{R}$ be open.
Claim: $f^{-1}(O)$ is open in X. Let $x \in f^{-1}(O)$, then $f(x) \subset O$.

Hence, $\exists$ some $\epsilon_0 > 0$ such that

$$(f(x) - \epsilon_0, f(x) + \epsilon_0) \subseteq O$$

Given f is continuous at x. For $\epsilon_0 > 0$, $\exists\, \delta > 0$ such that

$$B_\delta(x) \subseteq f^{-1}((f(x) - \epsilon_0, f(x) + \epsilon_0)) \subseteq f^{-1}(O)$$

$\implies f^{-1}(O)$ is open in X.

Conversely, let $f^{-1}(O) \in \mathcal{J}$ for each open set O in $\mathbb{R}$.
For $\epsilon > 0$, it follows that $x \in f^{-1}((f(x) - \epsilon, f(x) + \epsilon)) \in \mathcal{J}$.
Since $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is open in X, it follows that $\exists\, \delta > 0$ such that

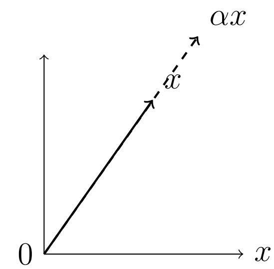
$$y \in B_\delta(x) \subset f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$$

i.e. $d(x, y) < \delta \implies f(y) \in f(B_\delta(x)) \subset (f(x) - \epsilon, f(x) + \epsilon) \implies |f(x) - f(y)| < \epsilon$

For a metric space (X, d), we call $(X, \mathcal{J})$ the topology of X generated by d. □

5.1. **Normed Linear Space.** A **normed linear space** is eventually mixing of linear structure of a space with its some topological structure.

Let $(X, +, \cdot)$ be a linear space over the field F $(= \mathbb{R}$ or $\mathbb{C})$. Let (X, J) be the topological structure given by some metric d on X. Now, the question is: how to mix linear structure with topological structure?



Note that a linear space is mainly concerned about two maps:

$$\text{(i)} \quad (x, y) \mapsto x + y \qquad (X \times X \to X)$$

$$\text{(ii)} \quad (\alpha, x) \mapsto \alpha x \qquad (F \times X \to X)$$

Therefore, a linear space X can be thought of made by these two maps.

Topology is all about continuity of maps. Thus, we can think of continuity of "+" and " ·" on $X \times X$ and $F \times X$ respectively, in their respective product topology $J \times J$ and $U \times J$, where U is the usual topology on $\mathbb{R}$ or $\mathbb{C}$.

A linear space with such property is called a *topological vector (linear) space*.

Note that an open set in $J \times J$ is a union of sets of the form $O_1 \times O_2$, where $O_1, O_2 \in J$. And open set in $U \times J$ is a union of sets $O_1 \times O_2$, with $O_1 \in U$, $O_2 \in J$.

Now, because of linearity and homogeneity of the space X, we can opt for a sense of distance that should satisfy the following set of rules:

$$(i) \quad \text{dist}(0, \alpha x) = |\alpha| \, \text{dist}(0, x)$$

$$(ii) \quad \text{dist}(0, x + y) \leq \text{dist}(0, x) + \text{dist}(0, y)$$

$$(iii) \quad \text{when } \alpha = 0, \ \text{dist}(0, 0) = 0$$

Let $p := \text{dist} : X \to [0, \infty)$ be defined by $p(x) = \text{dist}(0, x)$. Then

$$(i) \quad p(x) = 0 \quad \text{for } x = 0$$

$$(ii) \quad p(\alpha x) = |\alpha| \, p(x) \quad \text{(absolute homogeneity)}$$

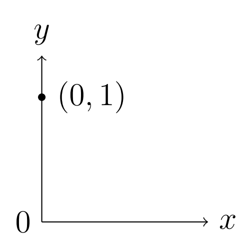
$$(iii) \quad p(x + y) \leq p(x) + p(y) \quad \text{(triangle inequality)}$$

Here, p is known as a *semi-norm*, because it is little away from the natural sense of usual distance.

Example 5.14.

$$p : \mathbb{R}^2 \to [0, \infty), \quad p(x_1, x_2) = |x_1|.$$

Then p is a semi-norm and $p(0, 1) = 0$.



That is, points on the y-axis are at zero distance from the origin. This is not convincing as long as usual distance is concerned.

Let $\|\cdot\| : X \to [0,\infty)$ be a map such that

(i) $\|x\| \geq 0$ for each $x \in X$, and $\|x\| = 0$ iff $x = 0$,
(ii) $\|\alpha x\| = |\alpha|\,\|x\|$ for each $(\alpha, x) \in F \times X$ (absolute homogeneity)
(iii) $\|x+y\| \leq \|x\| + \|y\|$ for each $x, y \in X$ (triangle inequality).

The map $\|\cdot\|$ is called a *norm* on X.

Note that the norm $\|\cdot\|$ induces a metric on X by $d(x,y) = \|x-y\|$, that produces a topology on X. For $r > 0$, $x \in X$, the open ball

$$B_r(x) = \{y \in X : \|x-y\| < r\}$$

Hence,open sets can be defined accordingly.

Note that every metric on a linear space need not produce a norm.

For example, the discrete metric on any linear space is not normable, because it fails to satisfy the absolute homogeneity property.

For $x, y \in X$, define

$$d_0(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

If we write $\|x\| = d(0,x)$, then for $\alpha \in \mathbb{F}$, $\|\alpha x\| \neq |\alpha|\|x\|$ $(x \neq 0)$ unless $|\alpha| = 1$.

However, if d is a metric on a linear space X such that $d(x,y) = d(x-y,0)$ and $d(\alpha x, \alpha y) = |\alpha| d(x,y)$, then $d(x,0) = \|x\|$ defines a norm on X.

(1) $\|x\| = 0 \iff d(x,0) = 0 \iff x = 0$
(2) $\|\alpha x\| = d(\alpha x, 0) = |\alpha| d(x,0) = |\alpha|\|x\|$
(3) $\|x+y\| = d(x+y,0) = d(x,-y) \leq d(x,0) + d(-y,0) = \|x\| + \|y\|$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if

$$f(t_1x_1 + \cdots + t_nx_n) \leq t_1f(x_1) + \cdots + t_nf(x_n)$$

where $0 \leq t_i \leq 1$ and $x_i \in \mathbb{R}^n$.

Example 5.15. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function satisfying $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{R}$, for all $x \in \mathbb{R}^n$. Prove that

(i) $f(x+y) \leq f(x) + f(y)$
(ii) $f(0) = 0$
(iii) $f(-x) \geq -f(x)$
(iv) $f(t_1x_1 + \cdots + t_nx_n) \leq t_1f(x_1) + \cdots + t_nf(x_n)$

Further, what requires to make f a norm on $\mathbb{R}^n$?

5.2. **Convergence of Sequence in Metric Space.** A sequence (x_n) in a metric space (X,d) is said to be converging to $x \in X$, if for any $\epsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $n \geq N_0$ $\implies d(x_n, x) < \epsilon$.

Example 5.16. Let $X = (0, \infty)$ and $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. Then $x_n = n$ does not converge to any point of X.

However, this sequence is not so bad as $x_n = n \to 0$, which is not in X. Such sequences can be classified as Cauchy sequences.

5.3. **Cauchy Sequences.**

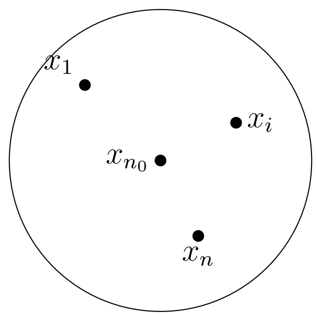
Definition 5.17. A sequence (x_n) in (X, d) is said to be a *Cauchy sequence* if for any $\epsilon > 0$, there exist $N_0 \in \mathbb{N}$ such that $\forall m, n \geq N_0$, $d(x_n, x_m) < \epsilon$.

Example 5.18. Show that every Cauchy sequence in a metric space is bounded.

Proof. A set $A \subset X$ is said to be *bounded* if $A \subseteq B_r(x)$ for some fixed x and $r > 0$.)

$x_n \in B_\epsilon(x_{n_0})$ for $n \geq n_0$.

Let $r = \max\{\epsilon, d(x_{n_0}, x_i) : i = 1, 2, \ldots, n_0 - 1\}$. Then $x_n \in B_r(x_{n_0})$, for all $n \geq 1$.

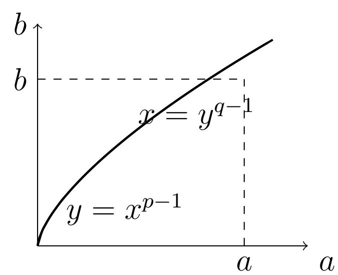


We need certain inequalities to deal with sequence spaces. □

5.4. **Young's Inequality.** Let $1 < p < \infty$ and $a, b > 0$. Then for $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{*}$$

Proof: Let $y = x^{p-1}$,then $x = y^{q-1}$ ($\because p - 1 = \frac{1}{q-1}$ by $\frac{1}{p} + \frac{1}{q} = 1$).



Now, from the figure, it is clear that

$$ab \leq \int_0^a x^{p-1}\, dx + \int_0^b y^{q-1}\, dy = \frac{a^p}{p} + \frac{b^q}{q}$$

Note that equality in $(*)$ holds iff $a^p = b^q$ (or $a = b^{q-1}$).

For this, consider

$$ab = \frac{a^p}{p} + \frac{b^q}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Replace $a \to a^{\frac{1}{p}}$, $b \to b^{\frac{1}{q}}$ and $\frac{1}{p} = \alpha$.

Then, we get

$$a^{\alpha} b^{1-\alpha} = \alpha a + (1-\alpha) b$$

or

$$t^{\alpha} - \alpha t - (1-\alpha) = 0 \quad \text{if} \quad t = a/b.$$

Let

$$f(t) = t^{\alpha} - \alpha t - (1-\alpha), \quad t \in (0, \infty).$$

Then $f(1) = 0$ and

$$f'(t) = \alpha t^{\alpha-1} - \alpha = \alpha(t^{\alpha-1} - 1) = 0 \iff t = 1.$$

Since $f'(t) < 0$ if $t > 1$ and $f'(t) > 0$ for $0 < t < 1$,
Hence, f is strictly increasing in $(0, 1)$ and strictly decreasing in $(1, \infty)$. Thus, $t = 1$ is the point of absolute maximum of f.
Therefore, $f(t) \leq f(1) = 0$, which is another proof of the inequality. On the other hand, $f(t) = 0$ iff $t = 1$. This completes the proof.

Example 5.19. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Write

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

Then $(\mathbb{R}^n, \|\cdot\|_1)$ is a normed linear space (n.l.s.).

If

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

then by Cauchy-Schwarz inequality, $(\mathbb{R}^n, \|\cdot\|_2)$ is a n.l.s.

For

$$\|x\|_{\infty} = \sup_i |x_i|,$$

$(\mathbb{R}^n, \|\cdot\|_{\infty})$ is a normed linear space.

For $1 \leq p < \infty$, write

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

Then $l_n^p := (\mathbb{R}^n, \|\cdot\|_p)$ will be a normed linear space.

5.5. **Space of Sequences.** Let $1 \leq p < \infty$ and let l^p denote the space of all sequences that satisfy

$$\sum_{i=1}^{\infty} |x_i|^p < \infty; \qquad x = (x_1, x_2, \ldots, x_n, \ldots)$$

Then $(l^p, \|\cdot\|_p)$ or simply l^p, will be a normed linear space.

If $p = \infty$,

$$\|x\|_{\infty} = \sup_{1 \leq i < \infty} |x_i| < \infty,$$

then $(l^{\infty}, \|\cdot\|_{\infty})$ is a normed linear space (follows from definition of supremum).

For $1 \leq p < \infty$, showing l^p is a normed linear space required the following inequalities.

5.6. **Hölder's Inequality.** Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for $x \in l^p$ and $y \in l^q$, it follows that

$$x \cdot y (= x_1 y_1 + \ldots + x_n y_n + \ldots) \in l^1,$$

and

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q \qquad \cdots (*)$$

(where $\frac{1}{\infty} = 0$ adopted.)

When $p = 1$, $q = \infty$. In this case $(*)$,

$$\|x \cdot y\|_1 = \sum_{i=1}^{\infty} |x_i y_i| \leq \sum |x_i| \cdot \sup |y_i| = \|x\|_1 \|y\|_\infty$$

Now, let $1 < p < \infty$, then $1 < q < \infty$.

Substitute $a = a_j = \frac{|x_j|}{\|x\|_p}$ and $b = b_j = \frac{|y_j|}{\|y\|_q}$ in the Young's Inequality. Then

$$\sum_{i=1}^{n} \frac{|x_j y_j|}{\|x\|_p \|y\|_q} \leq \sum_{i=1}^{n} \left(\frac{|x_j|^p}{p\|x\|_p^p} + \frac{|y_j|^q}{q\|y\|_q^q} \right) \leq \left(\frac{\|x\|_p^p}{p\|x\|_p^p} + \frac{\|y\|_p^p}{q\|y\|_q^q} \right) = \frac{1}{p} + \frac{1}{q} = 1$$

That is,

$$\sum_{j=1}^{n} |x_j y_j| \leq \|x\|_p \|y\|_q, \quad \text{for all} \quad n \geq 1$$

Since LHS is an increasing sequence which is bounded above, hence

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q$$

Notice that if $\|x\|_p = 1 = \|y\|_q$, then $\|x \cdot y\|_1 \leq 1$,
and equality holds iff $|y_j|^p = |x_j|^q$, $\forall j$.

This follows from Young's equality. For

$$ab = \frac{a^p}{p} + \frac{b^q}{q},$$

we must have $a^p = b^q$.

5.7. **Minkowski's Inequality.** Let $1 \leq p \leq \infty$. Then for $x, y \in l^p$, $x + y \in l^p$, and

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

Proof. For $b = 1$ or ∞, the proof is trivial.

Let $1 < p < \infty$ .Then

$$\|x + y\|_p = \left(\sum_{i=1}^{\infty} |x_i + y_i|^p \right)^{1/p}$$

$$\leq \left(\sum_{i=1}^{\infty} (|x_i| + |y_i|)^p \right)^{1/p} \tag{1}$$

Since

$$(|x_i| + |y_i|)^p = (|x_i| + |y_i|)(|x_i| + |y_i|)^{p-1}$$

By Hölder's inequality,

$$\sum(|x_i|+|y_i|)^{p-1}|x_i| \leq \left(\sum(|x_i|+|y_i|)^{(p-1)q}\right)^{1/q}\left(\sum|x_i|^p\right)^{1/p}$$

Thus,

$$\sum(|x_i|+|y_i|)^p \leq \left(\sum(|x_i|+|y_i|)^p\right)^{1/q}(\|x\|_p+\|y\|_p)$$

That is

$$\left(\sum(|x_i|+|y_i|)^p\right)^{1-\frac{1}{q}} \leq \|x\|_p+\|y\|_p$$

From (1), we get

$$\|x+y\|_p \leq \left(\sum(|x_i|+|y_i|)^p\right)^{1/p} \leq \|x\|_p+\|y\|_p$$

□

Remark 5.20. Equality in $\|x+y\|_p \leq \|x\|_p+\|y\|_p$ holds iff $x = \frac{\|x\|_p}{\|y\|_p}y$.
(*Hint:* Consider $\|x\|_p = 1 = \|y\|_p$ etc.)

Example 5.21. Since we know that any convergent sequence is bounded, it follows that the space c of all convergent sequences is a normed linear space under the norm

$$\|x\| = \sup|x_i| < \infty;$$

where $x = (x_1, x_2, \ldots, x_n, \ldots)$.

Further, the space c_0 of all sequences converging to "zero" is also a normed linear space. That is, $x = (x_1, x_2, \ldots, x_n, \ldots)$,

$$\lim_{n\to\infty}|x_n| = 0.$$

Thus, $(c_0, \|\cdot\|_\infty)$ is a linear subspace of $(c, \|\cdot\|_\infty)$.

Exercise 5.22. Show that the following strict inclusions hold:

$$\ell^1 \subsetneq \ell^2 \subsetneq c_0 \subsetneq c \subsetneq \ell^\infty$$

(*Hint:* $x = (x_n) \in \ell^1$, then $\lim x_n = 0 \implies x \in \ell^\infty$, $\sum|x_n|^2 \leq \sum\|x\|_\infty|x_n| \implies \|x\|_2^2 \leq \|x\|_\infty\|x\|_1$.)

Exercise 5.23. For $x = (x_1, ..., x_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$), show that:

$$\|x\|_\infty \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty$$

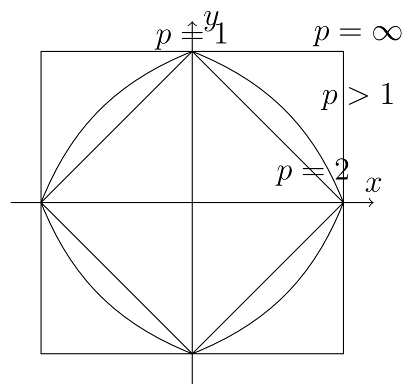
5.8. **Geometry of Spheres in ($\mathbb{R}^n$, $\|\cdot\|_p$).** For $0 \leq p \leq \infty$ and $x \in \mathbb{R}^n$, write

$$\|x\|_p = \left(\sum|x_i|^p\right)^{1/p}$$

Then $\|\cdot\|_p$ is a norm for $1 \leq p < \infty$, and for $0 < p < 1$, $\|x\|_p^p = d_p(0,x)$ with $d_p(x,y) = \|x-y\|_p^p$ is a metric. (We see later)

Let $S_1^p(0) = \{x : d_p(0,x) = 1\}$.

Then the following figure can be plotted for different values of $p; 0 < p < \infty; p = \infty$



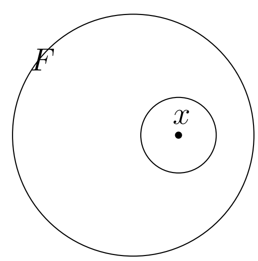
Shapes for $0 < p < 1$ would look like star-shaped curves (not shown).

5.9. **Closed sets in** (X, d)**.**

Definition 5.24. A set $F \subset (X, d)$ is said to be closed if F^c is open. i.e., for all $x \in F^c = X \setminus F$, $\exists\, \epsilon > 0$ such that $B_\epsilon(x) \subseteq F^c$.

On the other hand, if for each $\epsilon > 0$,

$$B_\epsilon(x) \cap F \neq \emptyset \implies x \in F.$$



Theorem 5.25. *Let (X, d) be a metric space and $F \subset X$. Then the following are equivalent (F.A.E):*

(1) F is a closed set (F^c open).
(2) $\forall \epsilon > 0$, $B_\epsilon(x) \cap F \neq \emptyset \implies x \in F$.
(3) $\forall$ sequence $(x_n) \in F$ such that $x_n \to x \implies x \in F$.

Proof. **(1) $\implies$ (2):** Suppose F is closed.

Claim: $B_\epsilon(x) \cap F \neq \emptyset, \forall \epsilon > 0 \implies x \in F$.

Notice that if $x \notin F \implies x \in F^c$ and F^c is open $\implies \exists \epsilon_0 > 0$ s.t.

$$B_{\epsilon_0}(x) \subset F^c \implies B_{\epsilon_0}(x) \cap F = \emptyset,$$

which is a contradiction.

(2) $\implies$ (3): Let $(x_n) \subset F$ and $x_n \to x$. Then for each $\epsilon > 0$, $x_n \in B_\epsilon(x)$ for all $n \geq n_0$.

$$\implies x_n \in B_\epsilon(x) \cap F \neq \emptyset, \quad \forall \epsilon > 0 \implies x \in F$$

(3) $\implies$ (1):

Claim: F^c is open. Let $x \in F^c$.

Then $x \notin F$. By (3), $\exists \epsilon_0 > 0$ such that

$$B_{\epsilon_0}(x) \cap F = \emptyset \implies B_{\epsilon_0}(x) \subset F^c.$$

□

Example 5.26. Let $f : (X, d) \to \mathbb{R}$ be a function. Then f is continuous at $x \in X$ iff for every sequence $x_n \in X$ with $x_n \to x \implies f(x_n) \to f(x)$.

Proof. Suppose f is continuous at x (ϵ-δ definition).

Let $x \in X$ and $x_n \in X$ such that $x_n \to x$.

Since f is continuous at x, for each $\epsilon > 0$, $\exists \delta > 0$ such that

$$d(y, x) < \delta \implies |f(x) - f(y)| < \epsilon.$$

Given $x_n \to x$. For $\delta > 0$, $\exists n_0 \in \mathbb{N}$ such that $n \geq n_0, d(x_n, x) < \delta \implies |f(x_n) - f(x)| < \epsilon$.

[That is, for each $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $n \geq n_0 \implies |f(x_n) - f(x)| < \epsilon \implies f(x_n) \to f(x)$.]

Conversely, suppose for each sequence $x_n \in X$ with $x_n \to x \implies f(x_n) \to f(x)$.

$d(x_n, x) \to 0 \implies |f(x_n) - f(x)| \to 0$.

$$\text{[That is, for each } \epsilon > 0, \exists n_0 \in \mathbb{N} \text{ and } \delta > 0 \text{ such that}$$
$$n \geq n_0 \implies d(x_n, x) < \delta \implies |f(x_n) - f(x)| < \epsilon]$$

If f is not continuous at x, then $\exists \epsilon_0 > 0$ such that for each $\delta > 0$, there exist y such that

$$d(x, y) < \delta \text{ but } |f(x) - f(y)| \geq \epsilon_0.$$

Let $\delta = 1/n$, then $\exists y_n \in X$ such that

$$d(x, y_n) < \frac{1}{n} \text{ but } |f(x) - f(y_n)| \geq \epsilon_0$$

i.e., $y_n \to x$ but $f(y_n) \not\to f(x)$, is a contradiction. □

Exercise 5.27. If $f : (X, d) \to \mathbb{R}$ is continuous and $f(x_0) \neq 0$ for some $x_0 \in X$, then $\exists \delta > 0$ such that

$$f(x) \neq 0 \ \forall x \in B_\delta(x).$$

(*Hint:* take $\epsilon_0 = \frac{1}{2}|f(x_0)| > 0$, $\exists \delta > 0$ etc.)

Example 5.28. Show that if $f : (X, d) \to \mathbb{R}$ is continuous, then $A = \{x : f(x) > 0\}$ is open (without using the complement should be closed).

(*Hint:* Let $x \in A$, then for $\epsilon = \frac{1}{2}f(x) > 0$, $\exists \delta > 0$ such that $d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$)

5.10. **Interior in** (X, d)**.** Let $A \subset X$. Then interior(A) or Int(A) or A° is the largest open set contained in A.

$$\text{i.e.} A^\circ = \bigcup \{O \subset X : O \text{ open}, O \subseteq A\}$$
$$= \bigcup \{B_\epsilon(x) \subset A : \text{ for } x \in A \text{ and some } \epsilon > 0\}$$

= union of all open balls contained in A.

5.11. **Closure in (X,d).** The closure of set $A \subset X$ is the smallest closed set containing A.

$$\text{i.e.}\overline{A} = \bigcap \{F \subset X : F \text{ closed and } A \subset F\}$$
$$= \{x \in X : \exists x_n \in A \text{ with } x_n \to x\}$$

= collection of limits of all convergent sequences in A (limit need not be in the set A).

Example 5.29. $A = \{(n, \frac{1}{n}) : n \in \mathbb{N}\}$. Then closure of A in $(\mathbb{R}, u)$ is $\overline{A} = A$ and $A^\circ = \emptyset$ (Why?).

Result: Let $A \subset (X,d)$. Then $x \in \overline{A} \iff B_\epsilon(x) \cap A \neq \varnothing, \quad \forall \epsilon > 0$.

Proof. Let $x \in \overline{A}$. Suppose $\exists\, \epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \cap A = \varnothing$. Then $A \subset (B_{\epsilon_0}(x))^c$, a closed set. By definition of $\overline{A}$, $\overline{A}$ is the smallest closed set containing A. Hence,

$$\overline{A} \subset (B_{\epsilon_0}(x))^c.$$

Since $x \in \overline{A}$, but $x \notin (B_{\epsilon_0}(x))^c$, this is a contradiction.

Conversely, suppose $B_\epsilon(x) \cap A \neq \varnothing$ for all $\epsilon > 0$. By the previous result, $x \in \overline{A}$ ($\because \overline{A}$ is closed). □

Result: $x \in \overline{A}$ if and only if there exists a sequence (x_n) with $x_n \in A$ such that $x_n \to x$.

Proof. If $x \in \overline{A}$, then for all $n \in \mathbb{N}$, $B_{1/n}(x) \cap A \neq \varnothing$. So, $\exists\, x_n \in B_{1/n}(x) \cap A$. Thus,

$$d(x_n, x) < \frac{1}{n}, \forall n \in \mathbb{N} \implies x_n \to x.$$

Conversely, if there exists $x_n \in A$ with $x_n \to x$. Then for $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq n_0$, $\implies x_n \in B_\epsilon(x) \cap A \neq \varnothing$ for all $\epsilon > 0$. Thus $x \in \overline{A}$ (by previous result). □

Definition 5.30. A set $A \subset (X,d)$ is said to be *dense* in X if $\overline{A} = X$.

5.12. **Space of Finite Sequences.** Space of finite sequences play a vital role similar to the space of all polynomials.

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n$$
$$\Rightarrow (a_0, a_1, \ldots, a_n) \sim (a_0, a_1, \ldots, a_n, 0, 0, 0, \ldots)$$

Let

$$c_{00} = \{x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) : x_i \in F\}.$$

Then obviously, x is a bounded sequence, and

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| < \infty$$

defines a norm on c_{00}.

Notice that the space of all finite sequences c_{00} is dense in all ℓ^p; $1 \leq p < \infty$ (which will see later). However, the closure of c_{00} is c_0, which is a closed proper subspace of ℓ^∞.

For

$$x_n = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, ...\right) \in c_{00},$$

and

$$x = \left(1, \frac{1}{2}, ..., \frac{1}{n}, \frac{1}{n+1}, ...\right),$$

then

$$\|x - x_n\|_\infty = \sup_{k \geq n} \frac{1}{k+1} = \frac{1}{n+1} \to 0,$$

but $x \notin c_{00}$. Hence c_{00} is not a closed subspace of ℓ^∞.

In addition c_{00} is not open in ℓ^∞.
For this, let $\epsilon > 0$,be arbitrarily small. Then for $B_\epsilon(0) \subset \ell^\infty$, $(\epsilon/2, \epsilon/2, ...) \in B_\epsilon(0)$ but $(\epsilon/2, \epsilon/2, ...) \notin c_{00}$. Hence, $B_\epsilon(0) \not\subseteq c_{00}$ for any $\epsilon > 0$.

For $1 \leq p < \infty$, $c_{00} \subsetneq \ell^p$ and c_{00} is neither closed nor open in ℓ^p. For this, let

$$x_n = \left(\frac{\epsilon^p}{2^{n+1}}\right)^{1/p}, \ 1 \leq p < \infty,$$

and consider $x = (x_1, x_2, ...)$. Then $x \in B_\epsilon(0) \subset \ell^p$, but $x \notin c_{00}$. Now write $x_n = (x_1, ..., x_n, 0, ...) \in c_{00}$. Then

$$\|x - x_n\|_p^p = \sum_{k=n+1}^{\infty} \frac{\epsilon^p}{2^{k+1}} \to 0,$$

but $x \notin c_{00}$.

Example 5.31. Let M be a non-open subspace of a normed linear space (n.l.s.) X. Show that $M = X$.

(*Hint:* $O \in M \implies B_\epsilon(0) \subset M \subset X$. Since M is linear, $\alpha B_\epsilon(0) \subset M \subset X$ for all $\epsilon > 0 \implies B_{\epsilon_1}(0) \subset M$ for some $\epsilon_1 > 0$. If $y \in X$, then $y \in B_{\epsilon_1}(0) \subset M \subset X$ for some $\epsilon_1 > 0$.)

Notice that for $x = (x_1, x_2, ..., x_n, ...) \in \ell^p; 1 \leq p < \infty$, $x_n = (x_1, ..., x_n, 0, ...) \in c_{00}$. And

$$\|x - x_n\|_p^p = \sum_{k=n+1}^{\infty} |x_k|^p \to 0 \quad (\because x \in \ell^p).$$

Hence $x_n \to x$ in ℓ^p. Thus, $\overline{c_{00}} = \ell^p$.

However, c_{00} is not dense in ℓ^∞, but $\overline{c_{00}} = c_0$. For this, let

$$X = (x_1, x_2, \dots) \in c_0.$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

For $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|x_n| < \epsilon/2$. Now, write

$$X_n = (x_1, \dots, x_n, 0, 0, \dots).$$

Then $X_n \in c_{00}$ and for $n \geq n_0$,

$$\|X - X_n\|_\infty = \sup_{i \geq n+1} |x_i| < \epsilon/2.$$

Therefore, $X_n \to X$.

Remark 5.32. $\overline{c_{00}} = c_0 \subsetneq \ell^\infty$. That is, c_{00} is not dense in ℓ^∞.

Example 5.33. Let $f\colon \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) be a continuous function. Suppose $\lim_{|x|\to\infty} f(x) = 0$. Then for $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x)| < \epsilon$ for $|x| > \frac{1}{\delta}$.

Since f is continuous, it follows that f is bounded. Let $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)| < \infty$. Then

$$C_0 = \left\{ f\colon \mathbb{R} \to \mathbb{R} \text{ is continuous, } \lim_{|x|\to\infty} |f(x)| = 0 \right\}$$

is a normed linear space.

For any function $f\colon \mathbb{R} \to \mathbb{R}$, define

$$\operatorname{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$$

called the support of f.

Let

$$C_C = \{f\colon \mathbb{R} \to \mathbb{R} \text{ continuous} \quad \text{and } \operatorname{supp}(f) \text{ is compact}\}.$$

Then $f \in C_c$ is a bounded function and

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in\operatorname{supp}(f)} |f(x)| < \infty.$$

Let $K = \operatorname{supp}(f)$ be compact. Then $(C_C, \|\cdot\|_\infty)$ is a dense subspace of $(C_0, \|\cdot\|_\infty)$.

For this, let $f \in C_0$, then for $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x)| < \epsilon$ for $|x| > \frac{1}{\delta}$. Write $K = \{x : |x| \leq \frac{1}{\delta\}}$.

Let O be a bounded open set with $K \subset O$.

Define

$$g(x) = \frac{d(x, O^c)}{d(x, O^c) + d(x, K)}$$

Then g is continuous on $\mathbb{R}$, $0 \leq g(x) \leq 1$ and $g(x) = 1$ for $x \in K$ and $g(O^c) = \{0\}$.

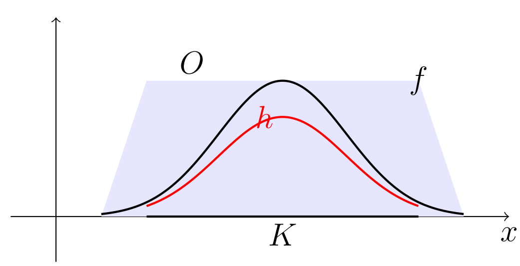
Let $h = f \cdot g$. Then $h \in C_C$ and

$$\|f - h\|_\infty = \|f(1-g)\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|(1 - g(x)) \leq \epsilon.$$

Hence, C_C is dense in C_0.

Note that $d(x, A) = \inf_{y\in A} |x - y|$.

Diagram:



5.13. **Complete Metric Spaces.** We have seen that there are Cauchy sequences whose limits need not necessarily belong to the space.

For example, the sequence $\frac{1}{n} \in ((0, 1), u)$ under the usual metric, is a Cauchy sequence but the limit $\frac{1}{n} \to 0 \notin (0, 1)$.

It is always possible to enlarge the space so that limits of all Cauchy sequences can be accommodated. This process is known as the completion of metric spaces, we shall see later. However, there are many spaces which do accommodate limits of their Cauchy sequences.

Definition 5.34. A metric space (X, d) is called complete if every Cauchy sequence in X has its limit in X.

Example 5.35. $(\mathbb{R}, u)$ is a complete space.

Let (x_n) be a Cauchy sequence in $\mathbb{R}$. Then it is bounded. And by the Bolzano–Weierstrass theorem, there exists a subsequence $x_{n_k} \to x \in \mathbb{R}$. For any $\epsilon > 0$, there exists a natural number k_0 such that

$$|x_{n_k} - x| < \epsilon \quad \text{for all} \quad k \geq k_0 \tag{1}$$

But the sequence (x_n) is Cauchy, so for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ for all $n, m \geq n_0$.

Let $m \geq n_0$ and $m \geq n_{k_0}$. Then

$$|x_n - x_{n_k}| < \epsilon \quad \text{for any } n \geq n_0 \text{ and } k \geq k_0. \tag{2}$$

From (1) and (2), it follows that:

$$\begin{aligned} |x_n - x| &\leq |x_n - x_{n_k}| + |x_{n_k} - x| \\ &< 2\epsilon \end{aligned}$$

for $n \geq n_0$ and $n_k \geq n_{k_0}$.

Thus, for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies |x_n - x| < \epsilon.$$

Notice that the above discussion can be used to prove the following result.

Result: Let (x_n) be a Cauchy sequence in a metric space (X, d). If (x_n) has a convergent subsequence $x_{n_k} \to x$, then $x_n \to x$.
(Proof is similar to the above.)

Example 5.36. $(\mathbb{R}^n, \|\cdot\|_p)$ is complete for $1 \leq p \leq \infty$.

Let $1 \leq p < \infty$, and $x^k = (x_1^k, \ldots, x_n^k)$ be a Cauchy sequence in $(\mathbb{R}^n, \|\cdot\|_p)$. Then for $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for all $k, l \geq k_0$,

$$\begin{aligned} \|x^k - x^l\|_p = \left(\sum_{j=1}^{n} |x_j^k - x_j^l|^p \right)^{1/p} &< \epsilon \\ \implies |x_j^k - x_j^l| < \epsilon \quad &\text{for all } k, l \geq k_0 \\ \implies (x_j^k) \text{ is a Cauchy sequence in } &(\mathbb{R}, u). \end{aligned}$$

Hence $x_j^k \to x_j$ for all j. Then for $\epsilon > 0$, there exists $m_j \in \mathbb{N}$ such that $k \geq m_j \implies |x_j^k - x_j| < \epsilon$.

Let $m_0 = \max_j\{m_j\}$. Then, for $x = (x_1, \ldots, x_n)$,

$$\|x^k - x\|_p < \epsilon \qquad \text{for } k \geq m_0.$$

Notice that the case $p = \infty$ is similar. We skip its proof here.

Example 5.37. Let $1 \leq p \leq \infty$. Then $(\ell^p, \|\cdot\|_p)$ is complete.

Let $1 \leq p < \infty$, and let $x^k = (x_1^k, x_2^k, \ldots)$ be a Cauchy sequence in $(\ell^p, \|\cdot\|_p)$. Then for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall k, l \geq n_0 \implies \|x^k - x^l\|_p < \epsilon$$

$$\implies \sum_{j=1}^{n} |x_j^k - x_j^l|^p < \epsilon^p \tag{1}$$

For each fixed n, this reduces to $(\mathbb{R}^{\cdot}\|\cdot\|_p)$, which we know is complete. Hence $x_j^k \to x_j$; $j = 1, 2, \ldots, n$. Thus, letting $k \to \infty$ in (1), it follows that

$$\sum_{j=1}^{n} |x_j^l - x_j|^p < \epsilon^p, \quad \forall l \geq n_0 \tag{2}$$

But the left-hand side of (2) is an increasing sequence and bounded above, hence, letting $n \to \infty$, we get

$$\sum_{j=1}^{\infty} |x_j^l - x_j|^p < \epsilon^p$$

$$\|x^l - x\|_p \leq \epsilon, \quad \forall l \geq n_0$$

where $x = (x_1, x_2, \ldots, x_n, \ldots)$.

Notice that

$$\|x\|_p \leq \|x - x^{n_0}\|_p + \|x^{n_0}\|_p < \epsilon + \|x^{n_0}\|_p < \infty \implies x \in \ell^p.$$

Result: Every closed subset of a complete metric space is complete.

Proof. Let F be a closed subset of a complete metric space (X, d). Then $(x_n) \subset F$ is a Cauchy sequence, it follows that (x_n) is a Cauchy sequence in X. Hence $x_n \to x \in X$. But F is closed, it implies that $x \in F$.

In fact, if (X, d) is complete, then F is closed if and only if F is complete. (*Hint:* it follows easily.) □

Example 5.38. Show that c_0 is a proper closed subspace of $(\ell^\infty, \|\cdot\|_\infty)$.

We know that $c_0 \subsetneq \ell^\infty$.

Now, let $x^k = (x_1^k, \ldots, x_j^k, \ldots)$ be a sequence in c_0 such that $x^k \to x = (x_1, \ldots, x_j, \ldots)$.

That is, for every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\forall k > k_0 \implies \|x^k - x\|_\infty < \epsilon$$

which implies

$$|x_j^k - x_j| < \epsilon \quad \text{for each } j \geq 1\text{and} \quad \forall k > k_0. \tag{1}$$

Since $x_j^k \in c_0 \implies \lim_{j\to\infty} x_j^k = 0$ for each k

For $\epsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that

$$|x_j^k| < \epsilon \quad \forall j \geq j_0 \quad \text{and} \quad k \geq k_0. \tag{2}$$

It follows from 1 and 2 that

$$|x_j| < |x_j^{k_0} - x_j| + |x_j^{k_0}| < 2\epsilon \quad \forall j > J_0,$$

i.e., $|x_j| < 2\epsilon$ for all $j > J_0$, which means $\lim_{j\to\infty} x_j = 0$.

Hence c_0 is a closed subspace of ℓ^∞. Thus, c_0 is complete in its own right.

Example 5.39. The space $(C[a,b], \|\cdot\|_\infty)$ is a complete normed linear space.

Let (f_n) be a Cauchy sequence in $(C[a,b], \|\cdot\|_\infty)$.

Then for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, m \geq n_0 \implies \|f_n - f_m\|_\infty < \epsilon$$

which implies

$$|f_n(t) - f_{n_0}(t)| < \epsilon \quad \forall n \geq n_0, \forall t \in [a,b]. \tag{1}$$

So $(f_n(t))$ is a Cauchy sequence in $(\mathbb{R}, u)$ for each fixed $t \in [a,b]$. Hence $f_n(t) \to f(t)$

Letting $n \to \infty$ in (1), we get

$$|f(t) - f_{n_0}(t)| \leq \epsilon \quad \forall t \in [a,b].$$

(Notice that n_0 is free of choice of t)

Since f_{n_0} is continuous, for each fixed t and $\epsilon > 0$, there exists $\delta > 0$ such that $|s-t| < \delta$ implies $|f_{n_0}(s) - f_{n_0}(t)| < \epsilon$.

Hence,

$$\begin{aligned}|f(s) - f(t)| &< |f(s) - f_{n_0}(s)| + |f_{n_0}(s) - f_{n_0}(t)| + |f_{n_0}(t) - f(t)| \\ &< 3\epsilon\end{aligned}$$

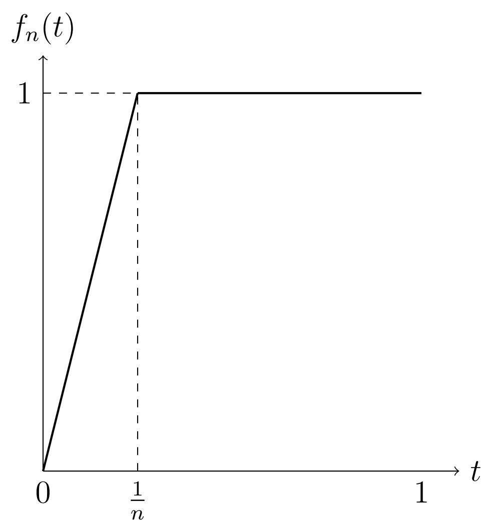
So f is continuous on $[a,b]$.

However, the space $(C[a,b], \|\cdot\|_1)$ is not complete.

For this, we consider the following:

Consider

$$f_n(t) = \begin{cases} nt & 0 \leq t < \frac{1}{n} \\ 1 & \frac{1}{n} \leq t \leq 1 \end{cases}$$



It is easy to see that for $\frac{1}{m} < \frac{1}{n}$,

$$\begin{aligned}\|f_n - f_m\|_1 &= \left(\int_0^{1/m} + \int_{1/m}^{1/n} + \int_{1/n}^{1}\right) |f_n(t) - f_m(t)| dt \\ &= \int_0^{1/m} (mt - nt) dt + \int_{1/m}^{1/n} (1 - nt) dt + \int_{1/n}^{1} (1-1) dt\end{aligned}$$

$$= \frac{1}{2}\left(\frac{1}{m} - \frac{1}{n}\right) \to 0 \text{ as } n < m \to \infty$$

Thus (f_n) is a Cauchy sequence in $(C[0,1], \|\cdot\|_1)$.

But the pointwise limit:

$$f(t) = \lim_{n\to\infty} f_n(t) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & t = 0 \end{cases}$$

(*Hint:* $f_n(0) = 0$ and $f_n(1) = 1$ for all n, so $f(0) = 0$ and $f(1) = 1$. For $0 < t_0 < 1$, we can find large n such that $0 < \frac{1}{n} < t_0 < 1$. Hence $f_n(t_0) = 1$ for large n. Thus $f(t_0) = 1$.)

However, f is not continuous, hence $(C[0,1], \|\cdot\|_1)$ is not complete.

6. Sequences of Functions

Notice that in the previous exercises, we have seen that $(C([0,1]), \|\cdot\|_\infty)$ is complete. That is, if $\|f_n - f_m\|_\infty \to 0$, then there exists $f \in C([0,1])$ such that $\|f_n - f\|_\infty \to 0$. But then,

$$|f_n(t) - f(t)| < \|f_n - f\|_\infty \to 0, \quad \forall t \in [0,1],$$

i.e., $f_n(t) \to f(t)$ for each $t \in [0,1]$.

We say that $f_n \to f$ uniformly if

$$\sup_t |f_n(t) - f(t)| \to 0.$$

But there are sequence of functions which converge pointwise but not uniformly.

Example 6.1. Let $f_n(t) = t^n, \ \ t \in [0,1]$.

Then,

$$f(t) = \lim_{n\to\infty} f_n(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & t = 1 \end{cases}$$

So,

$$\sup_t |f_n(t) - f(t)| = 1 \not\to 0.$$

Example 6.2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by

$$f_n(t) = e^{-nt^2}, \quad n \in \mathbb{N}$$

Then,

$$f(t) = \lim_{n\to\infty} f_n(t) = \begin{cases} 1 & t = 0 \\ 0 & |t| > 0 \end{cases}$$

Notice that for $t = 0$,

$$|f_n(0) - f(0)| = |1 - 1| = 0 < \epsilon, \quad \forall n \in \mathbb{N}$$

If $|t_0| > 0$, $t_0^2 > 0$. Then for $|f_n(t_0) - 0| < \epsilon$, we get

$$e^{-nt_0^2} < \epsilon \implies n > \frac{\log 1/\epsilon}{t_0^2}$$

Let $n_0 = \left\lceil \frac{\log 1/\epsilon}{t_0^2} \right\rceil + 1$. Then,

$$|f_n(t_0) - f(t_0)| < \epsilon \text{ for } n \ge n_0$$

Notice that $n_0 = n_0(\epsilon, t_0)$ and n_0 is large for $|t_0|$ close to zero. Thus, n_0 cannot be free from t_0. Therefore, $f_n \to f$ pointwise but not uniformly.

Also,

$$\|f_n - f\|_\infty = \sup_{t\in\mathbb{R}} e^{-nt^2} = 1 \not\to 0$$

If $f_n(t) = e^{-nt}$ for $t \in [1,\infty)$, then

$$\sup_t |f_n(t) - 0| = e^{-n} \to 0 \implies e^{-nt} \xrightarrow[{[1,\infty)}]{\text{unif.}} 0$$

Exercise 6.3. Let $f_n, f : A(\subseteq \mathbb{R}) \to \mathbb{R}$ be such that $f_n \to f$ uniformly on A. Then for $|f_n(t)| \le M_n$ (i.e. f_n's are bounded), that implies f is bounded.

(*Hint:* $|f(t)| \le |f_{n_0}(t) - f(t)| + |f_{n_0}(t)| < \epsilon + M_{n_0} < \infty \quad \forall t \in A$)

We shall see later that uniform convergent sequences is a good carrier for many underline properties.

Result: Let $f, f_n : A(\subset \mathbb{R}) \to \mathbb{R}$ be such that $f_n \to f$ uniformly. Then f is continuous if f_n's are continuous (i.e. the uniform limit of a sequence of continuous functions is continuous).

Proof. For $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{t \in A} |f_{n_0}(t) - f(t)| < \epsilon$$

Thus,

$$|f_{n_0}(t) - f(t)| < \epsilon, \quad \forall t \in A$$

Since f_{n_0} is continuous on A, for fixed t and for $\epsilon > 0$, there exists $\delta > 0$ such that if $|t - s| < \delta \implies |f_{n_0}(t) - f_{n_0}(s)| < \epsilon$.

Thus,

$$|f(s) - f(t)| < |f(s) - f_{n_0}(s)| + |f_{n_0}(s) - f_{n_0}(t)| + |f_{n_0}(t) - f(t)| < 3\epsilon$$

□

Result: Let $\mathcal{R}[a, b]$ denote the space of all Riemann integrable functions on $[a, b]$. Let $f_n, f \in \mathcal{R}[a, b]$ and $f_n \to f$ uniformly. Then,

$$\int_a^b f_n \to \int_a^b f$$

that is,

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n$$

Proof.

$$\left| \int_a^b (f_n - f) \right| \le \int_a^b |f_n - f| \le \|f_n - f\|_\infty (b - a) \to 0$$

□

Corollary 6.4. *If $f_n \in \mathcal{R}[a, b]$ such that $S_n = f_1 + f_2 + \ldots + f_n$ converges uniformly to S, then*

$$\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n$$

(Obvious from the previous result)

Result: Let $f_n \in C^1[a, b]$ be such that $f_n' \to g$ uniformly. If there exists $x_0 \in [a, b]$ such that $f_n(x_0)$ converges, then there exists $f \in C^1[a, b]$ such that $f_n \to f$ uniformly and $f' = g$.

Proof. Since $f_n' \to g$ uniformly and f_n is continuous, g will be continuous. Define

$$f : [a, b] \to \mathbb{R} \quad \text{by} \quad f(x_0) = \lim_{n\to\infty} f_n(x_0)$$

and

$$f(x) = \begin{cases} f(x_0) + \int_{x_0}^{x} g(t)\, dt, & \text{if } x > x_0 \\ f(x_0) - \int_{x}^{x_0} g(t)\, dt, & \text{if } x < x_0 \end{cases}$$

Then $f'(x) = g(x)$ for every $x \in [a, b]$. Hence, $f \in C^1[a, b]$.

Now,

$$f_n(x) - f_m(x) = f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0)) + (f_n(x_0) - f_m(x_0))$$
$$= (x - x_0)(f_n'(t) - f_m'(t)) + (f_n(x_0) - f_m(x_0))$$

Therefore,

$$\|f_n - f_m\|_\infty \le (b-a)\|f_n' - f_m'\|_\infty + |f_n(x_0) - f_m(x_0)| \to 0,$$

as $n, m \to \infty$.

Hence, (f_n) is a Cauchy sequence in $(C[a,b], ||\cdot||_\infty)$. Therefore, f_n converges uniformly.

Again, since $f_n' \to g = f'$ uniformly, it follows that

$$\int_{x_0}^{x} f_n'(t)\,dt \to \int_{x_0}^{x} f'(t)\,dt.$$

$$\lim_{n\to\infty} [f_n(x) - f_n(x_0)] = f(x) - f(x_0)$$
$$\lim_{n\to\infty} f_n(x) = f(x) \quad (\because \lim_{n\to\infty} f_n(x_0) = f(x_0))$$

□

Remark 6.5. Convergence of $(f_n(x_0))$ for some point is necessary. Consider

$$f_n(t) = \sqrt{t+n}, \qquad t \in [0,1]$$

Then f_n does not converge at any point of $[0,1]$, but

$$f_n'(t) = \frac{1}{2\sqrt{t+n}} \xrightarrow{\text{unif.}} 0$$

Since

$$\sup_{t\in[0,1]} |f_n'(t) - 0| = \sup_{t\in[0,1]} \frac{1}{2\sqrt{t+n}} = \frac{1}{2\sqrt{n}} \to 0.$$

Exercise 6.6. Let $f_n : \mathbb{R} \to \mathbb{R}$. Check for uniform convergence of f_n to some f:

(1) $f_n(t) = \frac{\sin(nt)}{\sqrt{n}}$
(2) $f_n(t) = n^2 t(1-t^2)^n$
(3) $f_n(t) = te^{-nt}$

Also, verify for term-by-term integration and differentiation for each of the above.

Theorem 6.7. *Let $E \subseteq \mathbb{R}$, and $f_n \to f$ uniformly on E. For a limit point x of E, suppose*

(*) $$\lim_{t\to x} f_n(t) = A_n \quad \text{(finite)}$$

Then (A_n) is convergent and

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n.$$

That is,

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

Proof. Since $f_n \to f$ uniformly on E. For each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

(*) $$|f_n(t) - f_m(t)| < \epsilon, \quad \forall n, m \ge n_0, \ \forall t \in E$$

By $(*)$, it implies that

$$|A_n - A_m| < \epsilon, \quad \forall n, m \ge n_0$$

So (A_n) is Cauchy, hence convergent $\implies A_n \to A$ (Say).

Now,

$$\begin{aligned}|f(t)-A| &= |f(t)-f_n(t)+f_n(t)-A_n+A_n-A| \\ &\le |f(t)-f_n(t)|+|f_n(t)-A_n|+|A_n-A| \\ &< \epsilon+\epsilon+\epsilon\end{aligned}$$

for $t \in (x-\delta, x+\delta)\setminus x$ and $n \ge n_0$ (free of t)

$$\lim_{t\to x} f(t) = A = \lim_{n\to\infty} A_n$$

$$\text{Thus,}\quad \lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

□

Theorem 6.8. *Let $f_n : [a,b] \to \mathbb{R}$ be such that (f_n') converges uniformly. If there exists $x_0 \in [a,b]$ such that $(f_n(x_0))$ is convergent, then (f_n) is uniformly convergent, and*

$$\lim_{n\to\infty} f_n'(x) = \Big(\lim_{n\to\infty} f_n(x)\Big)'$$

(i.e. limit and derivative commute)

Proof. The first part of the proof is as earlier. By the Mean Value Theorem, it follows that

$$|f_n(x)-f_m(x)| \le (b-a)\|f_n'-f_m'\| + |f_n(x_0)-f_m(x_0)|$$

Since f_n' converges uniformly and $f_n(x_0)$ is convergent, it follows that $f_n \to f$ (say) uniformly.

Claim: $\lim_{n\to\infty} f_n'(x) = f'(x)$.

Notice that f_n' need not be continuous, hence Fundamental Theorem of Calculus cannot be applied.

Therefore, we need to exploit the differentiability of f.

For $x \in [a,b]$, define

$$\varphi_n(t) = \frac{f_n(x)-f_n(t)}{x-t}, \qquad t \in [a,b]\setminus\{x\}$$

Then

$$\lim \varphi_n(t) = \frac{f(x)-f(t)}{x-t} =: \varphi(t)$$

Notice that

$$\lim_{t\to x}\varphi_n(t) = f_n'(x) \quad \text{(finite)}$$

Also,

$$|\varphi_n(t)-\varphi_m(t)| = |f_n'(x)-f_m'(x)| < \epsilon \quad \text{(by MVT)}$$

for $n, m \ge n_0$ and for all $t \in [a,b]\setminus\{x\}$.

Thus,

$$\varphi_n \to \varphi \text{ uniformly on } [a,b]\setminus\{x\}.$$

Apply previous theorem with $E = [a,b]$.

Then,

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty}\lim_{t\to x}\varphi_n(t) = \lim_{t\to x}\lim_{n\to\infty}\varphi_n(t) = \lim_{t\to x}\varphi(t) = f'(x).$$

Thus,

$$\lim_{n\to\infty} f_n'(x) = \Big(\lim_{n\to\infty} f_n(x)\Big)'$$

□

6.1. **Term-by-term differentiation.** Let $S_n = f_1 + f_2 + \cdots + f_n$, where each $f_i : [a, b] \to \mathbb{R}$ such that $S_n' \xrightarrow{\text{unif}} S$ and $S_n(x_0) \to L$. Then,

$$\lim(S_n') = (\lim S_n)'$$

i.e.,

$$f_1' + f_2' + \cdots + f_n' + \cdots = (f_1 + f_2 + \cdots + f_n + \cdots)'.$$

This raises a very fundamental question: When does

$$\left(\int_a^x f(t)\,dt\right)' = \int_a^x f'(t)dt \tag{**}$$

hold?

Notice that if f' is continuous then for

$$F(x) = \int_a^x f'(t)\,dt,$$

by the Fundamental Theorem of Calculus, $F'(x) = f'(x)$.

$$(F - f)' = 0$$

By the Mean Value Theorem, $F - f$ is constant. So $F(x) = f(x) - f(a)$ ($\because F(a) = 0$).

However, if f' is not continuous, i.e. $f' \in \mathcal{R}[a, b]$, then $(**)$ need not be true.

7. Uniform Continuity

Definition 7.1. A function $f : A(\subset (X,d)) \to \mathbb{R}$ is said to be *uniformly continuous* on A if for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$,

$$d(x,y) < \delta \implies |f(x) - f(y)| < \epsilon$$

Notice that δ is free of choice of locations of points $x, y \in A$; it only depends on their separation.

Example 7.2. For $x_0 \in X$, let $f(x) = d(x, x_0)$. Then f is uniformly continuous on X.

(Hint: $d(x,x_0) \le d(x,y) + d(y,x_0) \implies f(x) - f(y) < d(x,y)$.)

Similarly, by replacing x with y, it follows.

Example 7.3. For $x \in X$, $A \subset X$, define

$$d(x,A) = \inf\{d(x,a) : a \in A\},$$

which is called the *distance of A from x*, and is uniformly continuous as a function of x.

(Hint: $d(x,a) \le d(x,y) + d(y,a)$.)

Thus,

$$d(x,A) \le d(x,y) + d(y,A)$$

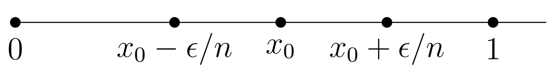
and so,

$$|f(x) - f(y)| \le d(x,y) \quad (\because x \leftrightarrow y)$$

Example 7.4. The function $f : (0,1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is continuous on $(0,1)$, but not uniformly continuous.

7.1. **Pointwise continuity of f.** Let $x_0 \in (0,1)$. Then for $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$(x_0 - \epsilon/n,\ x_0 + \epsilon/n) \subset (0,1)$$



Suppose

$$\left|\frac{1}{x_0} - \frac{1}{y}\right| < \epsilon$$

for $y \in (x_0 - \epsilon/n, x_0 + \epsilon/n) =: I_{x_0}$.

Then $|x_0 - y| < \epsilon x_0 y$.

Let $\delta = \min_{y \in I_{x_0}}\{\epsilon x_0 y\} = \epsilon x_0(x_0 - \epsilon/n) > 0$.

If $|x_0 - y| < \delta$. Then

$$\left|\frac{1}{x_0} - \frac{1}{y}\right| = \frac{|x_0 - y|}{x_0 y} < \frac{\delta}{x_0 y} \le \frac{\epsilon x_0(x_0 - \epsilon/n)}{x_0 y} < \epsilon$$

Hence, f is continuous at each $x_0 \in (0,1)$.

f is not uniformly continuous:

Let $\epsilon = 1/2$, $x = \frac{1}{n}$, $y = \frac{1}{n+1}$, $n \in \mathbb{N}$.

Then for any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|x - y| = \left|\frac{1}{n} - \frac{1}{n+1}\right| < \delta$$

but

$$|f(x) - f(y)| = 1 \not< \frac{1}{2}.$$

Hence, f is not uniformly continuous on $(0, 1)$.

From the above argument, we can prove the following result.

Theorem 7.5. *Let $f : A(\subset (X, d)) \to \mathbb{R}$. Then f is uniformly continuous on A if and only if for every pair of sequences $x_n, y_n \in A$ with $d(x_n, y_n) \to 0$, implies $|f(x_n) - f(y_n)| \to 0$.*

Proof. Suppose f is uniformly continuous on A. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon. \tag{1}$$

Let $x_n, y_n \in A$ such that $d(x_n, y_n) \to 0$. Then for $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$d(x_n, y_n) < \delta \implies |f(x_n) - f(y_n)| < \epsilon \quad \text{(from (1))},$$

That is, if $d(x_n, y_n) \to 0$, then $|f(x_n) - f(y_n)| \to 0$.

Conversely, suppose that f is not uniformly continuous.

Then there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ there exist $x, y \in A$ with $d(x, y) < \delta$ but $|f(x) - f(y)| \geq \epsilon_0$.

Now, let $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$. Then there exist $x_n, y_n \in A$ such that

$$d(x_n, y_n) < \frac{1}{n}\ , \forall n \in \mathbb{N}, \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0.$$

That is, $d(x_n, y_n) \to 0$ but $\underline{\lim}|f(x_n) - f(y_n)| \geq \epsilon_0$, is a contradiction. Hence, f is uniformly continuous. ☐

Exercise 7.6. Show that a uniformly continuous function on a metric space (X, d) sends Cauchy sequences to Cauchy sequences.
(*Hint:* If $f : (X, d) \to \mathbb{R}$ is uniformly continuous, so for $d(x_n, x_m) \to 0 \implies |f(x_n) - f(x_m)| \to 0$.)

Result: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous.

Proof. On contrary, suppose f is not uniformly continuous on $[a, b]$. Then there exists $\epsilon_0 > 0$ such that for every $\delta > 0$, there exist $x, y \in [a, b]$ with $|x - y| < \delta$ but

$$|f(x) - f(y)| \geq \epsilon_0.$$

For $\delta = \frac{1}{n}$, there exist $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$.

By the Bolzano–Weierstrass theorem, x_n, y_n have convergent subsequences, say $x_{n_k} \to x$ and $y_{n_k} \to y$.

Now,

$$|x - y| = \lim_{k\to\infty} |x_{n_k} - y_{n_k}| \leq \lim_{k\to\infty} \frac{1}{n_k} = 0,$$

so $x = y$. Since f is continuous, $f(x_{n_k}) - f(y_{n_k}) \to f(x) - f(y) = 0$, but $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon_0$, contradiction. ☐

Example 7.7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that

$$\lim_{|x|\to\infty} f(x) = 0.$$

Then f is uniformly continuous.

Proof. For $\epsilon > 0$, there exists $[-a, a]$ such that $|f(x)| < \epsilon/2$ if $x \in [-a, a]^c$.

Hence, if $x, y \in [-a, a]^c$, then

$$|f(x) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad (1)$$

Since f is uniformly continuous on $[-a, a]$. For $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon \qquad (2)$$

Since (1) holds true for x, y with $|x - y| < \delta$. It follows that for $\epsilon > 0$, we get $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ (for any $x, y \in \mathbb{R}$).

Hence, f is uniformly continuous on $\mathbb{R}$. □

Notice that if $f \in C_0(\mathbb{R})$, that is f is continuous and $\lim_{|x|\to\infty} f(x) = 0$ and hence f is uniformly continuous.

But if f is continuous and bounded, then f need not be uniformly continuous on $\mathbb{R}$.

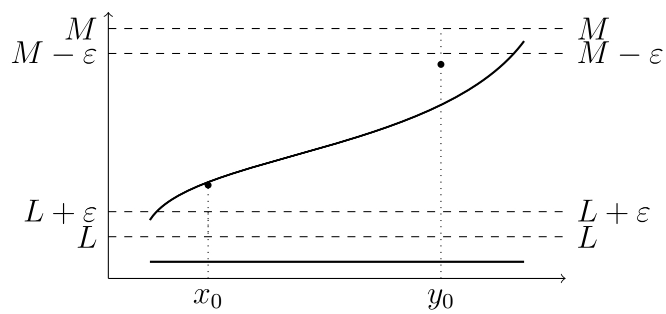
Example 7.8. $f(x) = \sin x^2$, which is continuous and bounded but not uniformly continuous on $\mathbb{R}$.

Example 7.9. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. If f is monotone, then f is uniformly continuous on $\mathbb{R}$.

Since f is bounded, let

$$\inf f(x) = L, \quad \sup f(x) = M.$$

For $\epsilon > 0$, there exist $x_0, y_0 \in \mathbb{R}$ such that $f(x_0) < L + \epsilon$ and $f(y_0) > M - \epsilon$.



If f is monotone increasing, then for $x, y \in [x_0, y_0]^c$ and $x, y \geq y_0$

$$f(y) - f(x) \leq M - f(y_0) < M - (M - \varepsilon) = \varepsilon.$$

Similarly, if $x, y \leq x_0$ then

$$f(y) - f(x) \leq L + \varepsilon - f(x_0) < L + \varepsilon - L = \varepsilon.$$

Thus, for $x, y \in [x_0, y_0]^c$, we get

$$|f(x) - f(y)| < \varepsilon \qquad (1)$$

Since f is continuous on $[x_0, y_0]$, f is uniformly continuous on $[x_0, y_0]$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in [x_0, y_0],\ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \qquad (2)$$

Notice that (1) also holds for $x, y \in [x_0, y_0]^c$ with $|x-y| < \delta$. Thus, we get single $\delta > 0$ such that

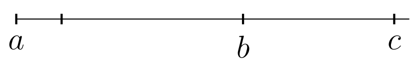
$$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Exercise 7.10. If $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function then for f monotone, it follows that

$$\lim_{x\to-\infty} f(x) = \text{finite}, \qquad \lim_{x\to+\infty} f(x) = \text{finite}.$$

(*Hint:* For any sequence $x_n \to \infty$, $f(x_n)$ is bounded and $\lim_{n\to\infty} f(x_n) = \sup_n f(x_n)$, for f is increasing.)

Example 7.11. Let $f : (a, b] \to \mathbb{R}$ and $f : (b, c) \to \mathbb{R}$ be uniformly continuous. Then $f : (a, c) \to \mathbb{R}$ is uniformly continuous.



Proof. Since f is uniformly continuous on $(a, b]$ and (b, c), for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in (a, b]$ or $x, y \in (b, c)$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Now, let $x, y \in (a, c)$, with $|x - y| < \delta$.

Then $|x - b| < \delta$ and $|y - b| < \delta$. Hence,

$$|f(x) - f(y)| < |f(x) - f(b)| + |f(b) - f(y)| < 2\varepsilon.$$

Thus, f is uniformly continuous on (a, c). □

We see that a uniformly continuous function can be extended uniformly to the closure of the set.

Theorem 7.12. *Let $f : A(\subset \mathbb{R}) \to \mathbb{R}$ be uniformly continuous on A. Then f can be extended uniformly to $\overline{A}$, and this extension is unique.*

Proof. Let $x \in \overline{A}$. Then there exists $x_n \in A$ such that $x_n \to x$. Now, $f(x_n)$ is a bounded sequence in $\mathbb{R}$. Hence, by Bolzano-Weierstrass theorem, $f(x_n)$ has a convergent subsequence. WLOG we can assume that $f(x_n)$ is convergent.

Let $\tilde{f}(x) = \lim f(x_n)$ ($\because \lim f(x_n)$ exists)

Notice that $\tilde{f}$ is well defined, because f is uniformly continuous on A. If $x_n, y_n \to x$,

then $x_n - y_n \to 0 \implies f(x_n) - f(y_n) \to 0$ i.e. $\lim f(x_n) = \lim f(y_n)$ ($\because \lim f(x_n)$ and $\lim f(y_n)$) both exist).

Hence $\tilde{f} : \overline{A} \to \mathbb{R}$ is well defined. Suppose $x, y \in \overline{A}$ and they are close enough to each other. Then there exist $x_n, y_n \in A$ such that $x_n \to x$ and $y_n \to y$.

Hence,

$$\tilde{f}(x) - \tilde{f}(y) = \tilde{f}(x) - f(x_n) + f(x_n) - f(y_n) + f(y_n) - \tilde{f}(y)$$

$$\implies |\tilde{f}(x) - \tilde{f}(y)| \leq |\tilde{f}(x) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - \tilde{f}(y)|$$

Notice that $|\tilde{f}(x) - f(x_n)| < \varepsilon$ and $|\tilde{f}(y) - f(y_n)| < \varepsilon$ for $n \geq n_0$ (say).

Let $|x - y| < \delta$ (small enough). Then there exists $n' \in \mathbb{N}$ such that $|x_n - y_n| < \delta$ for $n \geq n'$.

Since f is uniformly continuous on A, it follows that $|f(x_n) - f(y_n)| < \epsilon$ for $n \geq n'$.

Thus for sufficiently large $n \geq \max(n_0, n')$.

$$|\tilde{f}(x) - f(y)| \leq 3\epsilon, \quad \text{where } |x-y| < \delta.$$

Hence, $\tilde{f}$ is uniformly continuous on $\overline{A}$.

This extension of f is unique:

If there exists $\tilde{g} : \overline{A} \to \mathbb{R}$ which is uniformly continuous and $\tilde{g} = f$ on A, then for $x \in \overline{A}$, there is a sequence $x_n \in A$ such that $x_n \to x$.

Hence,

$$\tilde{f}(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = \tilde{g}(x)$$

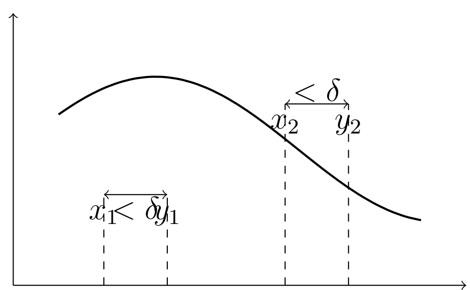
($\because$ g is uniformly continuous extension). □

Next, we shall see that uniformly continuous function grows slower than a straight line.

Theorem 7.13. *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous,then there exist constants $A, B \geq 0$ such that $|f(x)| \leq A|x| + B$ for all $x \in \mathbb{R}$.*

Proof. For any $\epsilon > 0$, there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < 1$.

We divide the proof into two parts: one is near "0" and other is away from "0".



Let $a > 0$. Then $|f(x)| \leq A < \infty$ for $x \in [-a, a]$.

Now, consider $f : [a, \infty) \to \mathbb{R}$.Then for $x \in [a, \infty)$, we can find $n \in \mathbb{N}$ such that $x \in [a + n\delta,\, a + (n+1)\delta]$.

Then,

$$\begin{aligned} f(x) - f(a) &= f(x) - f(a+n\delta) + f(a+n\delta) - f(a) \\ &= f(x) - f(a+n\delta) + \sum_{j=1}^{n} [f(a+j\delta) - f(a+(j+1)\delta)] \\ &\Rightarrow |f(x)| < 1 + n + |f(a)| \end{aligned}$$

$$\Rightarrow \left|\frac{f(x)}{x}\right| < \frac{(n+1) + |f(a)|}{a + n\delta} < \frac{(n+1) + |f(a)|}{n\delta} < \left(1 + \frac{1}{n}\right)\frac{1}{\delta} + \frac{|f(a)|}{n\delta} \leq B < \infty$$

Notice that B is independent of n, hence B is independent of x.

That is, $|f(x)| \leq B|x|$ if $x > a$.

Hence, we can summarize that

$$|f(x)| \leq B|x| + A \qquad \text{for all } x \in \mathbb{R}.$$

□

Example 7.14. Notice that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$, as it cannot satisfies the conclusion of the above theorem.

Example 7.15. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and its derivative is bounded. Then f is uniformly continuous on $\mathbb{R}$.

For any $x, y \in \mathbb{R}$, by the Mean Value Theorem,

$$|f(x) - f(y)| = |f'(t)(x - y)| \leq M|x - y|$$

where t is between x and y, and M is an upper bound for $|f'(t)|$.

However, $f(x) = \sqrt{x}$ for $x \in (0, \infty)$ is uniformly continuous, but its derivative is $f'(x) = \frac{1}{2\sqrt{x}}$, is not bounded.

8. Completeness

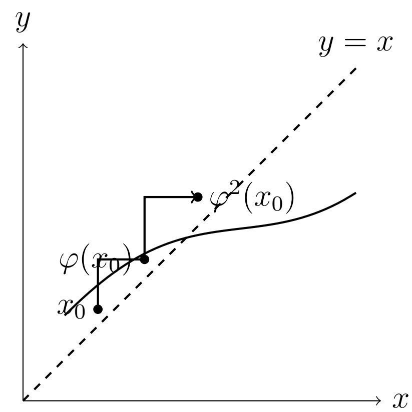
8.1. **Fixed Points.** Fixed point searching is an idea to solve equation of the form $\varphi(x) = x$. This helps solving a range of problems, including approximation theory, differential equations etc.

Fixed points can be obtained via iterations, i.e. if the function "shrinks nicely", then we get fixed points via iteration.

That is, if x_0 is a point in the space X, then

$$x_0 \to \varphi^1(x_0) \to \varphi^2(x_0) \to \cdots$$

where φ^n denotes n-times composition of φ.



If the sequence $(\varphi^n(x_0))$ is convergent and φ is continuous, then $\varphi^n(x_0) \to x$ and thus $\varphi(x) = \varphi(\lim_{n\to\infty} \varphi^n(x_0)) = x$.

However, if the space is complete, we only need to verify $\varphi^n(x_0)$ to be a Cauchy sequence.

Nicely shrinking function, we mean here with contraction mapping.

Definition 8.1. A function $\varphi : (X, d) \to (X, d)$ is called contraction if there exists $0 < \alpha < 1$ such that

$$d(\varphi(x), \varphi(y)) \leq \alpha\, d(x, y), \quad \forall x, y \in X.$$

Theorem 8.2. *Let (X, d) be a complete metric space. If $\varphi : (X, d) \to (X, d)$ is a contraction, then φ has a unique fixed point.*

Proof. Let $0 < \alpha < 1$ be such that

$$d(\varphi(x), \varphi(y)) \leq \alpha\, d(x, y), \quad \forall x, y \in X.$$

For a point $x_0 \in X$, let

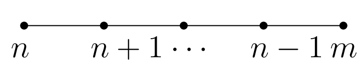
$$\varphi^0(x_0) = x_0, \quad \varphi^1(x_0) = \varphi(x_0) \quad etc.$$

Then

$$d(\varphi^{n+1}(x_0), \varphi^n(x_0)) \leq \alpha\, d(\varphi^n(x_0), \varphi^{n-1}(x_0)) \leq \alpha^n d(\varphi(x_0), x_0).$$

We show that $\varphi^n(x_0)$ is a Cauchy sequence.

Let $m > n$:



Then

$$\begin{aligned} d(\varphi^n(x_0), \varphi^m(x_0)) &\le (\alpha^n + \cdots + \alpha^{m-1})\, d(\varphi(x_0), x_0) \\ &\le \frac{\alpha^n}{1-\alpha} d(\varphi(x_0), x_0) \qquad (\because 0 < \alpha < 1) \\ &\to 0 \quad \text{as } n \to \infty \end{aligned}$$

Since (X, d) is complete, $\varphi^n(x_0) \to x \in X$ (say).

$$\begin{aligned} \implies \varphi(x) = \varphi\left(\lim_{n\to\infty} \varphi^n(x_0)\right) &= \lim_{n\to\infty} \varphi^{n+1}(x_0) \\ \implies \varphi(x) &= x \end{aligned}$$

If $\exists y \in X$ such that $\varphi(y) = y$, then

$$\begin{aligned} d(x, y) = d(\varphi(x), \varphi(y)) &\le \alpha d(x, y) \\ \iff x = y &\quad (\because 0 < \alpha < 1) \end{aligned}$$

This establishes that φ has unique fixed point. □

Notice that completeness property of the space is a sufficient condition for existence of fixed point.

For example,

$$\begin{aligned} \varphi : (0, \infty) &\to (0, \infty) \\ \varphi(x) &= \frac{1}{2}(x + \frac{a}{x}), \qquad a > 0 \end{aligned}$$

satisfies $\varphi(\sqrt{a}) = \sqrt{a}$.

Also, contraction is a sufficient condition for existence of fixed point. Notice that φ above is not a contraction mapping, since

$$|\varphi(x) - \varphi(y)| = \frac{1}{2}\, |1 - \frac{a}{xy}|\, |x - y|$$

because the function $|1 - \frac{a}{xy}|$ is not bounded near zero.

Exercise 8.3. If (X, d) is a complete metric space and $f : X \to X$ is such that f^k is a contraction, then show that f has a unique fixed point.

(*Hint:* do for $k = 2$, use the fact that f^k cannot have two fixed points.

If $f^2(x_0) = x_0$ and $y_0 = f(x_0)$(say), implies that $f(y_0) = y_0 \implies y_0 = x_0$)

Exercise 8.4. Let $T : C[0, 1] \to C[0, 1]$ be defined by

$$T(f)(x) = \int_0^x f(t)dt$$

Show that T^2 is a contraction but T is not a contraction.

Notice that the above fact in these example is also clear from the fact that in the convergence of $\varphi^n(x_0)$, we can ignore finitely many steps.

Now, we shall try to understand the existence and uniqueness of the initial value problem:

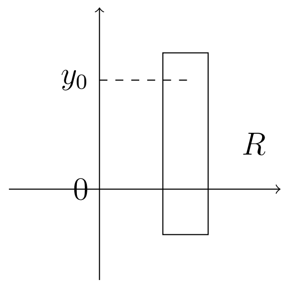
$$(*) \qquad \begin{cases} y' = f(x, y) \\ y(0) = y_0 \end{cases}$$

with the help of fixed point theorem.

Suppose f is a continuous function in some rectangle containing the interval $(0, y_0)$ in its interior, and f is Lipschitz in the second variable, i.e.,

$$|f(x, y_1) - f(x, y_2)| \le K|y_1 - y_2|,$$

where K is a fixed constant.



Then the equation $(*)$ has a unique solution in some neighborhood of $x = 0$.

Notice that solving $(*)$ is equivalent to solve

$$\int_0^x y'(t)dt = \int_0^x f(t, y(t))dt$$

i.e.,

$$(**) \qquad y(x) = y_0 + \int_0^x f(t, y(t))dt$$

That is, we want $y(t)$ such that $(**)$ holds.
In other words, we want to get fixed point for the map $\varphi \mapsto F(\varphi)$, where

$$F(\varphi)(x) = y_0 + \int_0^x f(t, \varphi(t))dt,$$

with $\varphi \in C[-\delta, \delta]$ for some $\delta > 0$, which we get very soon.

Now,

$$\begin{aligned} |F(\varphi)(x) - F(\psi)(x)| &\le \int_0^x |f(t, \varphi(t)) - f(t, \psi(t))|dt, \\ &\le K \int_0^x |\varphi(t) - \psi(t)|dt \\ &\le K \cdot 2\delta \cdot \|\varphi - \psi\|_\infty. \end{aligned}$$

Thus, $F : C[-\delta, \delta] \to C[-\delta, \delta]$ is a contraction as long as $2K\delta < 1$, i.e. if $\delta < \frac{1}{2K}$.
Hence F has a unique fixed point in $C[-\frac{1}{2K}, \frac{1}{2K}]$.

That is, $(*)$ has a unique solution in $|x| < \frac{1}{2K}$.

Example 8.5. Consider

$$y' = 2x(1 + y), \qquad y(0) = 0.$$

Then

$$\varphi(x) = \int_0^x 2t(1 + \varphi(t))dt.$$

With the initial guess $\varphi^0 \equiv 0$, we get

$$\varphi^1(x) = \int_0^x 2t(1 + 0)\, dt = x^2,$$

$$\varphi^2(x) = \int_0^x 2t(1 + t^2)\, dt = x^2 + \frac{x^4}{2},$$

$$\varphi^3(x) = x^2 + \frac{x^4}{2} + \frac{x^6}{6}.$$

Thus, by induction,

$$(*) \qquad \varphi^n(x) = \sum_{k=1}^{n} \frac{x^{2k}}{k!} \longrightarrow e^{x^2} - 1,$$

and $\varphi(x) = e^{x^2} - 1$ is a solution, which is same as method of separation of variables.

Notice that the series $(*)$ converges uniformly on every interval $[-a, a]$, or on any interval $[a, b]$.

On the other hand, $\varphi'(x) = 2x(1 + \varphi(x))$ has unique solution in neighborhood of any point x_0, i.e., $[x_0 - \delta, x_0 + \delta]$ with $\delta < \frac{1}{4}$.

(*Hint:* Lipschitz constant $= 2$.)

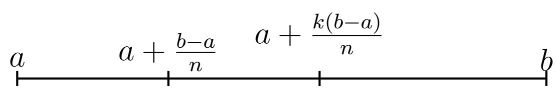
8.2. **Totally Bounded Set.** Suppose A be a bounded set in $\mathbb{R}$ and, without loss of generality, $A \subset (0, 1)$. Then for $\epsilon = \frac{1}{n} > 0$:

$$A \subset \bigcup_{k=1}^{n} \left(\frac{k-1}{n}, \frac{k}{n} \right]$$

That is, A can be covered by finitely many intervals of arbitrarily small length.

A similar argument can be produced for a bounded set $A \subset \mathbb{R}^m$ (or in finite dimensional spaces).

Notice that if A is bounded in $\mathbb{R}$, then $A \subset [a, b]$ ($a = \inf A, b = \sup A, b - a < \infty$).



Hence,

$$A \subseteq \bigcup_{k=1}^{n} \left[a + \frac{(k-1)(b-a)}{n},\ a + \frac{k(b-a)}{n} \right]$$

Notice that, with small perturbation of the intervals, A can be covered by open intervals of arbitrarily small length $\epsilon > 0$.

However, if the dimension of the space X is infinite, then the above property need not be inherited for an arbitrary bounded set.

For example, let $X = \ell^1$, $e_n = (0, 0, \ldots, 1, 0, \ldots)$ (with 1 in the nth position):

$$\|e_n - e_m\|_1 = 2, \quad \text{if } n \neq m$$

$$\Rightarrow A = \{e_n : n \in \mathbb{N}\} \subset B_1[0] \subset B_2[0]$$

This means A is bounded.

Notice that for any ϵ, $0 < \epsilon < 1$, if

$$A \subset \bigcup_{n=1}^{\infty} B_\epsilon(e_n)$$

But A cannot be covered by finitely many balls of arbitrarily small radius, i.e. if

$$A \subset \bigcup_{i=1}^{n} B_\epsilon(f_i) \quad \forall f_i \in \ell^1$$

Then, for $\epsilon < 1$, each ball $B_\epsilon(f_i)$ can contain exactly one point of A ($\because \|e_n - e_m\|_1 = 2$).

Also, notice that A has no convergent subsequence. Since ℓ^1 is complete, it is equivalent to say that A has no Cauchy subsequence.

Definition 8.6. $A \subseteq (X, d)$ is said to be totally bounded if for every $\epsilon > 0$, there exist $x_1, x_2, \ldots, x_n \in X$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$$

We can show that centers of these balls can be taken from some points of A, since

$$A \subseteq \bigcup_{i=1}^{n} B_{\epsilon/2}(x_i)$$

Also, we can assume that $A \cap B_{\epsilon/2}(x_i) \neq \emptyset$, for all $i = 1, 2, \ldots, n$. Then there exists $y_i \in A \cap B_{\epsilon/2}(x_i)$.

And it is easy to see that

$$A \subset \bigcup_{i=1}^{n} B_\epsilon(y_i)$$

(*Hint:* $x \in A \implies d(x, x_i) < \epsilon/2$ for some i and $y_i \in A \cap B_{\epsilon/2}(x_i) \implies d(x, y_i) < d(x, x_i) + d(x_i, y_i) < \epsilon$.

Moreover, if A is totally bounded, then we can replace balls with sets in A with arbitrarily small diameter.

Result: A in (X, d) is totally bounded if and only if for every $\epsilon > 0$, there exist sets $A_1, \cdots, A_n \subset A$ with $\delta(A_i) < \delta$ such that

$$A \subseteq \bigcup_{i=1}^{n} A_i$$

Proof. Let A be totally bounded. Then for every $\epsilon > 0$, there exist points $x_1, \ldots, x_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$$

Set $A_i = A \cap B_\epsilon(x_i) \subseteq A$ and $\delta(A_i) \leq 2\epsilon$.

i.e.

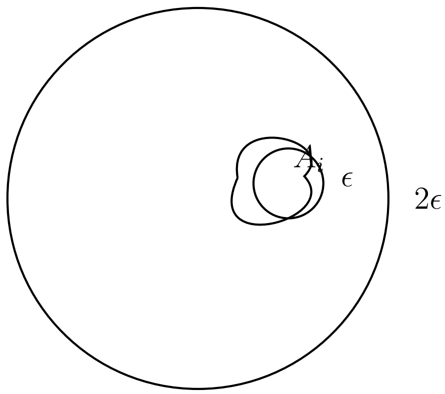
$$A = \bigcup_{i=1}^{n} A_i, \quad \delta(A_i) \le 2\epsilon.$$

Conversely, suppose for all $\epsilon > 0$, there exist $A_i \subset A$ such that $A \subseteq \bigcup_{i=1}^{n} A_i$, with $\delta(A_i) < \epsilon$.

Let $x_i \in A$, then $A_i \subseteq B_{2\epsilon}(x_i)$.

Since $\epsilon > 0$ is arbitrary, we get

$$A \subseteq \bigcup_{i=1}^{n} B_{2\epsilon}(x_i)$$



Notice that if $A \subseteq \bigcup_{i=1}^{n} B_i$, $B_i \subseteq X$, with $\delta(B_i) < \epsilon$, then for $A_i = A \cap B_i \subset A$,

(*) $$A = \bigcup_{i=1}^{n} A_i; \qquad \delta(A_i) < \delta$$

It is easy to see that if A is totally bounded in (X, d), then A is bounded.

Also, every finite set $A = \{x_1, x_2, \ldots, x_n\}$ is totally bounded because $A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$. □

Notice that total boundedness of a set solely depends upon the metric.

In fact, in discrete metric space, (X, d_0), $A \subset X$ is totally bounded if and only if A is finite.

(*Hint:* If $A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$, $x_i \in A$, then for $0 < \epsilon < \frac{1}{2}$, each $B_\epsilon(x_i) = \{x_i\}$)

However, if $X = \ell^1$, $\|e_n - e_m\|_1 = 2$ for $n \neq m$, $A = \{e_n : n \in \mathbb{N}\}$ cannot be covered by finitely many balls of radius < 2.

In fact, $A = \{e_n : n \in \mathbb{N}\}$ with

$$d(e_n, e_m) = \begin{cases} 2 & n \neq m \\ 0 & \text{otherwise} \end{cases}$$

(in its own discrete metric) is not totally bounded.

Exercise 8.7. Every subset of a totally bounded set is totally bounded.

Exercise 8.8. $A \subset \mathbb{R}$ is totally bounded if and only if A is bounded.

Exercise 8.9. A is totally bounded iff A is covered by finitely many closed sets of arbitrarily small diameters.

(*Hint:* $A \subset \bigcup_{i=1}^n A_i$; $\delta(A_i) < \epsilon$ but $\delta(\overline{A_i}) = \delta(A_i) < \epsilon$ and $A \subset \bigcup_{i=1}^n \overline{A_i}$)

Exercise 8.10. A is totally bounded iff $\overline{A}$ is totally bounded.

If A is totally bounded, then $A \subseteq \bigcup_{i=1}^n A_i$, $\delta(A_i) < \epsilon$. So $\overline{A} \subseteq \bigcup_{i=1}^n \overline{A_i}$, $\delta(\overline{A_i}) < \epsilon$, so $\overline{A}$ is totally bounded.

On the other hand, if $\overline{A}$ is totally bounded, then for $\epsilon > 0$, $\exists x_1, \ldots, x_n \in X$ such that

$$A \subseteq \overline{A} \subseteq \bigcup_{i=1}^n B_i, \qquad \delta(B_i) < \epsilon$$

Exercise 8.11. If $A \subset \mathbb{R}^n$ is bounded, then A is totally bounded.

Result: Let (x_n) be a sequence in (X, d) and let $A = \{x_n : n \in \mathbb{N}\}$ (range of (x_n)).
(i) If (x_n) is Cauchy sequence, then A is totally bounded.
(ii) If A is totally bounded, then (x_n) has a Cauchy subsequence.

Proof. **(i)** Since (x_n) is a Cauchy sequence, for $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$d(x_n, x_N) < \epsilon \qquad \forall n \geq N$$
$$\implies \delta\{x_n : n \geq N\} \leq \epsilon$$

Let

$$A = \{x_i : i = 1, \ldots, N-1\} \cup \{x_n : n \geq N\}$$
$$A \subseteq \bigcup_{i=1}^{N-1} B_\epsilon(x_i) \cup B_\epsilon(x_N)$$

which shows A is totally bounded.

(ii) If A is finite, then trivial.

Suppose A is an infinite set and totally bounded.
Then A can be covered by finitely many sets of diameter < 1. And one of them, say A_1, will contain infinitely many points of A. But A_1 is also totally bounded, and hence covered by finitely many sets of diameter $< \frac{1}{2}$. Let A_2 be one of them having infinitely many points from A. Thus,

$$A_1 \supset A_2 \cdots \supset A_k \supset A_{k+1} \supset \cdots$$

where each A_k is an infinite set with $\delta(A_k) < \frac{1}{k}$.

Choose $x_{n_k} \in A_k$. Then

$$\delta\{x_{n_k} : n \geq k\} \leq \delta(A_k) < \frac{1}{k}$$

($\because$ A_k are decreasing).

Thus, x_{n_k} is a Cauchy sequence. □

Example 8.12. Let $x_n = (-1)^n$ has Cauchy subsequences, as it is totally bounded.

Example 8.13. Let $e_n \in \ell^2$, $e_n = (0, 0, \ldots, 1, 0, \ldots)$. Then (e_n) has no Cauchy subsequence.

Theorem 8.14. *A set $A \subset (X, d)$ is totally bounded if and only if every sequence in A has a Cauchy subsequence.*

Proof. Let A be a totally bounded set in X, and (x_n) be a sequence in A. Then (x_n) is totally bounded and by the previous result (x_n) has a Cauchy subsequence.

For the other implication, suppose A is not totally bounded. Then, there exists $\epsilon > 0$ such that

$$A \neq \bigcup_{i=1}^{n} B_\epsilon(x_i)$$

for every choice of finite set $\{x_1, \cdots, x_n\}$.

Thus, for each $n \geq 1$, there exists $y_n \in A$ such that

$$d(y_n, x_i) \geq \epsilon \quad \forall i = 1, 2, \ldots, n.$$

Notice that the y_n's must be distinct (or an infinite set), else A will be covered by finitely many balls of radius ϵ.

Also, notice that (y_n) cannot be a Cauchy sequence, else A would be covered by finitely many ϵ-balls and therefore A is covered by ϵ-balls.

This implies that (y_n) has no Cauchy subsequence (as y_n's are distinct).

Therefore, A must be totally bounded. □

Corollary 8.15. ***(The Bolzano-Weierstrass Theorem)*** *Every bounded infinite subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$.*

Proof. Let A be an infinite bounded set in $\mathbb{R}$. Then there exist distinct sequence $x_n \in A$. Since A is totally bounded, (x_n) has a Cauchy subsequence, say (x_{n_k}). But $\mathbb{R}$ is complete, so $x_{n_k} \to x \in \mathbb{R}$. Thus, x is a limit point of A. □

We know that a metric space X is complete iff every Cauchy sequence in X has limit in X, and every closed set in X is complete. In fact, if X is complete, then $A \subseteq X$ is complete if and only if A is closed.

We can see that complete metric spaces have some common properties like $\mathbb{R}$:

Theorem 8.16. *Let (X, d) be a metric space. Then the following are equivalent:*

(1) (X, d) is complete.
(2) (Nested Set Theorem:) Let F_n be a decreasing sequence of closed sets in X with $\delta(F_n) \to 0$, then $\bigcap_{n=1}^{\infty} F_n \neq \phi$ (exactly one point).
(3) (Bolzano-Weierstrass Theorem:) Every infinite totally bounded subset of X has a limit point in X.

Proof. $(1) \implies (2)$:

Let $F_n \supset F_{n+1} \supset \cdots$ and $\delta(F_n) \to 0$. Choose $x_n \in F_n$, then $\delta\{x_k : k \geq n\} \leq \delta(F_n) \to 0$.

Hence, (x_n) is a Cauchy sequence in X, and by (1), $x_n \to x \in X$.

Since F_n's are closed, $x \in F_n$ for each n,

$$\implies x \in \bigcap_{n=1}^{\infty} F_n \implies \bigcap_{n=1}^{\infty} F_n \neq \varnothing.$$

(In fact, $\bigcap_{n=1}^{\infty} F_n = \{x\}$, exactly one point.)

$(2) \implies (3)$:

Let A be an infinite, totally bounded set in X. Notice that A contains a distinct Cauchy sequence x_n ($x_n \neq x_m$ for $n \neq m$), because A is totally bounded. Set

$$A_n = \{x_k : k \geq n\}$$

Then $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, and $\delta(A_n) \to 0$ ($\because x_n$ is a Cauchy sequence).

But then $\overline{A_n} \supseteq \overline{A_{n+1}} \cdots$ and

$$\delta(\overline{A_n}) = \delta(A_n) \to 0.$$

By (ii), there exists $x \in \bigcap_{n=1}^{\infty} \overline{A_n} \neq \varnothing$.

Now, $x_n \in A$, and $d(x_n, x) \leq \delta(\overline{A_n}) \to 0$.

Hence, $x_n \to x$. So x is a limit point of A.

(3) $\implies$ (1) :

Let x_n be a Cauchy sequence in X. We only need to show that (x_n) has a convergent subsequence.

Note that $A = \{x_n : n \in \mathbb{N}\}$ is totally bounded, because (x_n) is a Cauchy sequence. If A is finite, the result is trivial. Otherwise (iii) implies A has a limit point. That is, there exists a subsequence $x_{n_k} \to x \in X$.

Hence, $x_n \to x \in X$. □

Exercise 8.17. Suppose that every countable, closed subset in X is complete. Show that X is complete.

Exercise 8.18. Show that X is complete if and only if every closed ball in X is complete.

Remark 8.19. The total boundedness of a set is all about; an infinite set cannot be too scattered. That is, the substantial portion of the set can be put into (or lies in) a set of arbitrarily "small" size by a continuous dissection process by leaving finitely many.

9. Compact Metric Spaces

Definition 9.1. A metric space (X, d) is said to be compact if X is complete and totally bounded.

Theorem 9.2. *(X, d) is compact if and only if every sequence in X has a convergent subsequence.*

Proof. Suppose X is compact (complete and totally bounded). Let $x_n \in X$. Then $A = \{x_n : n \in \mathbb{N}\}$ is totally bounded and hence has Cauchy subsequence, say x_{n_k}. But X is complete, which implies $x_{n_k} \to x \in X$.

On the other hand, if every sequence (x_n) in X has a convergent subsequence (x_{n_k}), which is then Cauchy, and this implies that X is totally bounded.

Also, let x_n be a Cauchy sequence in X. Then again $A = \{x_n : n \in \mathbb{N}\}$ is totally bounded and has convergent subsequence, say $x_{n_k} \to x \in X$. Thus, $x_n \to x$.

$$\begin{bmatrix} \text{Totally Bounded} \\ + \text{Complete} \end{bmatrix} \iff \begin{bmatrix} \text{every sequence has a Cauchy subsequence} \\ + \text{Cauchy sequence is convergent} \end{bmatrix}$$

□

Corollary 9.3.

(1) *Let $A \subset X$. If A is compact, then A is closed.*
(2) *If X is compact and A is closed, then A is compact.*

i.e. compact subsets of a compact metric space are closed sets.

(Hint:(i) If A is compact, then for $x_n \in A$ and $x_n \to x \implies x_{n_k} \to y \in A \implies x_n \to x = y$.

(ii) If A is closed and $x_n \in A$, then $x_n \in X \implies x_{n_k} \to x \in X \implies x \in A$, since A is closed.)

Exercise 9.4. If K is a compact subset of $(\mathbb{R}, u)$, then $\inf K$ and $\sup K \in K$.

By definition of infimum, there exists $x_n \in K$ such that $x_n \to \inf K$. But, since K is compact, there exists $x_{n_k} \to x \in K$, which implies $\inf K = x$ etc.

Exercise 9.5. Let $E = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$. Show that E is closed and bounded in $(\mathbb{Q}, u)$, but not compact.

(*Hint:* $\mathbb{Q}$ is not complete.)

Exercise 9.6. Suppose $f : (X, d) \to (Y, \rho)$ is continuous, then for $K \subset X$ to be compact, $f(K)$ is compact in Y.

Let $y_n \in f(K)$, then $y_n = f(x_n)$ for some $x_n \in K$. Therefore, there exists a subsequence $x_{n_k} \to x \in K$ such that $f(x_{n_k}) \to f(x) \in f(K)$.

Exercise 9.7. If $A \subset X$ is compact, then show that $\delta(A) < \infty$. If $A \neq \emptyset$, then there exist $x, y \in A$ such that $\delta(A) = d(x, y)$.

Note that

$$\delta(A) = \sup\{d(x, y) : (x, y) \in A \times A\} \quad (\text{say} \quad S)$$

And $d : A \times A \to \mathbb{R}$ is (jointly) continuous.
As $A \times A$ is compact, the set S is compact in $\mathbb{R}$.
Hence, there exist $(x_0, y_0) \in A \times A$ such that $\delta(A) = d(x_0, y_0)$.

Exercise 9.8. Show that $S_1[0] = \{x \in \ell^2 : \|x\|_2 \leq 1\}$ is not compact.
(*Hint:* The set $\{e_n : n \in \mathbb{N}\}$ is not totally bounded.)

Exercise 9.9. Show that $A = \left\{x \in \ell^2 : |x_n| \leq \frac{1}{n},\ n = 1, 2, \ldots\right\}$ is compact.
(*Hint:* A is closed, hence complete. A is totally bounded, since for $\varepsilon = 1$, okay. For $\varepsilon < 1$, only finitely many coordinates are left unpatched (uncovered), hence for each $\varepsilon < 1$, $A = A_\varepsilon \cup B_\varepsilon$, $A_\varepsilon \in \mathbb{R}^n$ for some n.

Corollary 9.10. *Let (X, d) be compact. Suppose $f : X \to \mathbb{R}$ is continuous, then f is bounded. Moreover, f attains its maximum and minimum.*

Proof. $f(X)$ is compact in $\mathbb{R}$, which implies $f(X)$ is closed and bounded. Hence,
$$\sup_{x \in X} f(x) \in \mathbb{R}, \qquad \inf_{x \in X} f(x) \in \mathbb{R}.$$
i.e. there exist $x_0, y_0 \in X$ such that $f(y_0) = \sup_{x\in X} f(x)$, $f(x_0) = \inf_{x \in X} f(x)$. Hence,
$$f(x_0) \leq f(x) \leq f(y_0) \quad \forall x \in X.$$
□

Corollary 9.11. *If $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b])$ is compact and $f([a, b]) = [c, d]$ for some $c, d \in \mathbb{R}$.*

Corollary 9.12. *If (X, d) is a compact metric space and*
$$C(X) = \{f : X \to \mathbb{R} \quad \text{or} \quad \mathbb{C} \mid f \text{ is continuous}\}.$$
Define
$$\|f\|_\infty = \sup_{x \in X} |f(x)| < \infty.$$
Then $(C(X), \|\cdot\|_\infty)$ is complete normed linear space.

Lemma 9.13. *Let (X, d) be a metric space. Then the following are equivalent:*

(a) If $\mathcal{G}$ is a arbitrary collection of open sets in X with $\bigcup_{G \in \mathcal{G}} G \supseteq X$, then there exist $G_1, \ldots, G_n$ (finitely many) such that $\bigcup_{i=1}^n G_i \supseteq X$.
(In other words, every open cover has a finite subcover.)

(b) If $\mathcal{F}$ is a collection of closed sets in X with $\bigcap_{i=1}^n F_i \neq \emptyset$ for every choice of finitely many F_i's in $\mathcal{F}$, then
$$\bigcap_{F \in \mathcal{F}} F \neq \emptyset.$$
(This is called the finite intersection property.)

Notice that (a) $\implies$ X is totally bounded, since
$$X \subseteq \bigcup_{x \in X} B_\varepsilon(x) \implies X \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i).$$
(b) $\implies$ X is complete, since every decreasing sequence of closed sets has non-empty intersection.

Proof.

(a) $\implies$ (b): Let $\mathcal{F}$ be a collection of closed sets in X such that $\bigcap_{i=1}^n F_i \neq \emptyset$ for every choice of finitely many F_i's in $\mathcal{F}$. On contrary, suppose $\bigcap_{F\in\mathcal{F}} F = \emptyset$.
Then $X = \bigcup_{F\in\mathcal{F}} F^c$ is an open cover of X. Hence $X = \bigcup_{i=1}^n \{F_i^c \colon F_i \in \mathcal{F}\}$.
This implies $\bigcap_{i=1}^n F_i = \emptyset$, a contradiction.

(b) $\implies$ (a): Suppose $X = \bigcup_{G\in\mathcal{G}} G$ but $X \neq \bigcup_{i=1}^n G_i$ for any choice of finitely many G_i's in $\mathcal{G}$. Then $X \setminus \bigcup_{i=1}^n G_i \neq \emptyset$ for every choice of finitely many G_i's in $\mathcal{G}$, which implies $\bigcap_{i=1}^n G_i^c \neq \emptyset$ for every choice of finitely many sets G_i's from $\mathcal{G}$.

$$\bigcap_{G\in\mathcal{G}} G^c \neq \emptyset \implies \bigcup\{G : G \in \mathcal{G}\} \neq X.$$

□

Theorem 9.14. *X is compact iff either (a) or (b) (hence both) of the previous lemma is satisfied.*

Proof. Notice that (a) and (b) imply that X is totally bounded and complete. Hence X is compact.

Now suppose X is compact, and $\mathcal{G}$ is an open cover that admits no finite subcover.

Since X is totally bounded, it can be covered by finitely many closed sets of diameter ≤ 1. But then it implies that one of these, say A_1, will not be covered by finitely many open sets in $\mathcal{G}$.

It follows that $A_1 \neq \emptyset$, and it must be an infinite set (else covered by finitely many G's).

Next, A_1 is totally bounded, so A_1 is covered by finitely many closed sets of diameter $\leq \frac{1}{2}$.

Choose one of them, say A_2, such that A_2 cannot be covered by finitely many G's from $\mathcal{G}$.

Thus,

$$A_1 \supset A_2 \supset \cdots A_n \supset \cdots$$

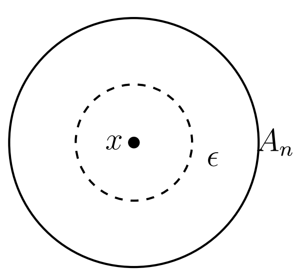
where A_n is closed, infinite, $\operatorname{diam}(A_n) \leq \frac{1}{n}$, and A_n cannot be covered by finitely many G's from $\mathcal{G}$.

Notice that $\bigcap_{n=1}^\infty A_n \neq \emptyset$ ($\because$ X is complete).

Let $x \in \bigcap_{n=1}^\infty A_n$, then $x \in A_n$. Then $x \in G$ for some $G \in \mathcal{G}$. But G is open, hence $x \in B_\epsilon(x) \subset G$ for some $\epsilon > 0$.

For $\frac{1}{n} < \epsilon$, we get $x \in A_n \subset B_\epsilon(x) \subset G$.

Hence, A_n is covered by a single $G \in \mathcal{G}$, which is a contradiction. □



Corollary 9.15. *X is compact if and only if every decreasing sequence of non-empty closed sets has non-empty intersection.*

$$i.e. \quad F_1 \supset F_2 \supset \cdots \supset F_n \supset F_{n+1} \cdots \implies \bigcap_{n=1}^{\infty} F_n \neq \emptyset$$

Proof. The forward implication is followed by the previous theorem.

Conversely, suppose every nested (decreasing) sequence of closed sets in X has non-empty intersection.

We prove compactness of X in the sense of the Bolzano–Weierstrass theorem. Let $x_n \in X$. Define

$$A_n = \{x_k : k \geq n\}.$$

Then $\bigcap_{n=1}^{\infty} \overline{A_n} \neq \emptyset$.

Let $x \in \bigcap_{n=1}^{\infty} \overline{A_n} = A$ (say). Then A is closed.

Hence, there exists a subsequence x_{n_k} such that $x_{n_k} \to x$.

(Notice that the sequence x_n has been taken distinct, i.e., an infinite set.) □

Remark 9.16. Note that, as long as compactness is concerned, we do not require the diameter of F_n tends to zero. Hence $\bigcap_{n=1}^{\infty} F_n$ can contain more than one point. This is in sharp contrast with the condition for completeness.

Corollary 9.17. *X is compact if and only if every countable open cover admits a finite subcover.*

Proof. ($\implies$:) Compact $\implies$ lemma (a) holds $\implies$ countable cover has finite subcover.

($\impliedby$:) Suppose every countable open cover has a finite subcover. This is equivalent to every countable family of closed sets having the finite intersection property (can be proved similar to the previous lemma).

Let $(x_n) \subset X$ be a sequence of distinct terms. Write

$$A_n = \overline{\{x_k : k \geq n\}}.$$

Then $x \in \bigcap_{n=1}^{\infty} A_n \neq \emptyset$, so there exists $x_{n_k} \in X$ such that $x_{n_k} \to x$.

Hence, X is compact. □

10. Separable Metric Spaces

If a space admits a countable dense set, we say that the space is **separable**. Eventually, it helps determine the size of the space, certainly not in terms of cardinality only, rather dimensions, or in a more general sense of size. Evidently, every totally bounded space is separable.

Definition 10.1. A metric space (X, d) is said to be **separable** if there exists a countable set $A \subset X$ such that $\overline{A} = X$.

For example, $\mathbb{Q}$ (the set of rationals) is a countable dense subset of $\mathbb{R}$.
Likewise, $\mathbb{Q}^n$ and $\mathbb{Q}^n + i\mathbb{Q}^n$ are countable dense subsets of $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively.

It is easy to see that $(\mathbb{R}^n, \|\cdot\|_p)$ is separable for $1 \leq p \leq \infty$. However, $(\ell^p, \|\cdot\|_p)$ is separable for $1 \leq p < \infty$ and *not* separable for $p = \infty$.

We know that $\overline{c_{00}} \subset \ell^p$, where c_{00} is the space of finite sequences. Let $x \in \ell^p$,

$$x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$$

Define $x_n = (x_1, \ldots, x_n, 0, 0, \ldots)$. Then

$$\|x - x_n\|_p \to 0 \quad \text{as } n \to \infty \tag{1}$$

Since $x_i \in \mathbb{C}$, there exists $x_i^k \in \mathbb{Q} + i\mathbb{Q}$ such that $|x_i^k - x_i|^p \to 0; \quad i = 1, 2, \ldots, n$. Thus,

$$\left(\sum_{i=1}^{n} |x_i^k - x_i|^p\right)^{1/p} \to 0$$

i.e.

$$\|x_n^k - x_n\|_p \to 0 \tag{2}$$

where $x_n^k = (x_1^k, \ldots, x_n^k) \in \mathbb{Q}^n + i\mathbb{Q}^n$.

From (1) and (2),

$$\|x - x_n^k\|_p \leq \|x_n^k - x_n\|_p + \|x_n - x\|_p \to 0$$

That is, $\overline{c_{00}(\mathbb{N}, \mathbb{Q} + i\mathbb{Q})} = \ell^p(\mathbb{N}, \mathbb{C})$.

Next, we shall show $\ell^\infty(\mathbb{N}, \mathbb{C})$ is not separable, by proving that ℓ^∞ cannot be the union of countably many balls of arbitrarily small radius.

Let $A = \{\tilde{x_1}, \tilde{x_2}, \ldots\}$ be any countable set in ℓ^∞. Consider

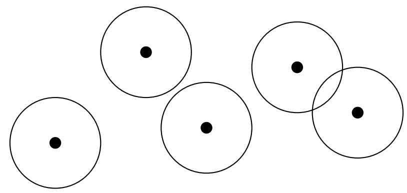
$$S = \{x = (x_1, x_2, ...) \in \ell^\infty : x_i \in \{0, 1\}\ \}$$

Then S is an uncountable set. For this, $x \in S \implies y = \frac{x_1}{2} + \frac{x_2}{2^2} + \cdots, x_i \in \{0, 1\}$. Then the map from S to $[0, 1]$ is surjective, and hence S is uncountable.

Let $x, y \in S$ be such that $x \neq y$. Then $\|x - y\|_\infty = 1$.

Hence, $\{B_{\frac{1}{2}}(x) : x \in S\}$ is an uncountable, disjoint collection of open balls in ℓ^∞.

Since A is countable, A can intersect only countably many balls $B_{\frac{1}{2}}$'s. Hence A cannot be dense.



Exercise 10.2. Show that $\overline{c_{00}} = c_0$ and hence deduce c_0 is separable.

Exercise 10.3. Let $B([0,1])$ be the space of all bounded functions on $[0,1]$. Show that $(B([0,1]), \|\cdot\|_\infty)$ is not separable. For $t \in (0,1)$, define $f_t = \chi_{[0,t)}$. Then for $s \neq t$, $s, t \in (0,1)$, we get $\|f_s - f_t\|_\infty = 1$.

Then $S = \{B_{1/2}(f_t) : t \in (0,1)\}$ is an uncountable collection of disjoint open balls in $B([0,1])$. If A is any countable set, say $A = \{g_1, g_2, \ldots\} \in B([0,1])$,then there exists $t_0 \in (0,t)$ such that $B_{1/2}(f_{t_0}) \cap A = \varnothing$.

That is, except countably many, all the balls in S are left un-intersected by A

Exercise 10.4. The space $(C([0,1]), \|\cdot\|_\infty)$ is separable.
(*Hint:* proof of this will be done by Weierstrass approximation theorem, which we do later.)

Exercise 10.5. Every totally bounded metric space is separable.

Let (X,d) be totally bounded. For $\epsilon = \frac{1}{n}$, there exist $x_{n_1}, \ldots, x_{n_k}$ such that

$$X = \bigcup_{j=1}^{n_k} B_{\frac{1}{n}}(x_{n_j}).$$

Let $D_{n_k} = \{x_{n_1}, \ldots, x_{n_k}\}$. Then

$$\mathcal{D} = \bigcup D_{n_k}$$

is a countable dense set in X.

Next, we consider the compact subsets of the space of continuous functions $C(X)$, then X is a compact metric space.

Notice that $\dim C(X) < \infty$ if and only if X is a finite set. Hence, closed and bounded subset of $C(X)$ are compact if X is finite.

But the question of compact subsets of $C(X)$, X is compact, is same as when a subset of $C(X)$ is totally bounded?

In terms of the Bolzano–Weierstrass theorem, we can rephrase, when (uniformly) bounded sequence in $C(X)$ have a uniformly convergent subsequence?

We will see later that this question is related to the earlier question of asking, When does a pointwise convergent sequence imply uniform convergence?

That is, pointwise convergence + [something] $\implies$ uniform convergence.

Example 10.6. If $f_n \in C(X)$, X compact, $f_n \xrightarrow{\text{unif}} f$, then $\{f\} \cup \{f_n : n \in \mathbb{N}\}$ is compact. (i.e., every Cauchy sequence is totally bounded).

Definition 10.7. A collection $\mathcal{F} \subset C(X)$ is said to be uniformly bounded if

$$\sup_{f\in\mathcal{F}} \sup_{x\in X} |f(x)| = \sup_{f\in\mathcal{F}} \|f\|_\infty < \infty.$$

Example 10.8. Any uniformly convergent sequence f_n in $B(X)$ (or $C(X)$) is uniformly bounded. (*Hint:* $\|f_n\|_\infty \leq \|f\|_\infty + 1$ (for $\epsilon = 1$) for all $n \geq N$, $n \in \mathbb{N}$.)

Definition 10.9. A collection $\mathcal{F} \subset C(X)$ is said to be pointwise bounded if for each $x \in X$,

$$\sup_{f \in \mathcal{F}} |f(x)| < \infty.$$

Example 10.10. If $f_n \to f$ pointwise, then f_n is pointwise bounded.

Theorem 10.11. *Let (X, d) be a compact metric space and $f : X \to \mathbb{R}$ (or $\mathbb{C}$) be continuous. Then f is uniformly continuous.*

Proof. Let $x \in X$ (compact), and $\epsilon > 0$. Then there exists $\delta_x > 0$ such that $d(x, y) < \delta_x \implies |f(x) - f(y)| < \epsilon$. i.e. $y \in B_{\delta_x}(x) \implies |f(x) - f(y)| < \epsilon$

Notice that

$$X = \bigcup_{x \in X} B_{\delta_x}(x).$$

Since X is compact,

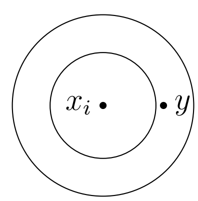
$$X = \bigcup_{i=1}^{n} B_{\delta_{x_i}}(x_i).$$

Let

$$\delta = \frac{1}{2} \min_{1 \leq i \leq r} \{\delta_{x_i}\}.$$

Then $\delta > 0$.

Let $x, y \in X$ and they close enough.There exist x_i such that $x \in B_{\delta_{x_i}}(x_i)$.Choose $\delta' > 0$ such that $\delta' < \delta$ and $d(x, y) < \delta'$ with $y \in B_{\delta_{x_i}}(x_i)$.



Then $d(x, y) < \delta' \implies |f(x) - f(y)| \leq 2\epsilon$.

Thus, for $\epsilon > 0$, there exists $\delta' > 0$ such that whenever $d(x, y) < \delta' \implies |f(x) - f(y)| < 2\epsilon$. ☐

Next, we shall discuss the missing ingredient of pointwise convergence to the uniform convergence.

11. The Space of Continuous Functions

11.1. **Equicontinuity.** A collection $\mathcal{F} \subset C(X)$ is said to be (uniformly) equicontinuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon, \quad \forall f \in \mathcal{F}.$$

Example 11.1.
- (i) Finite subset of $C(X)$ is (uniformly) equicontinuous and every sub-collection of a (uniformly) equicontinuous collection is equicontinuous.
- (ii) Let $0 < \alpha \leq 1$ and $k > 0$. Define
$$\mathrm{Lip}_K^\alpha = \{f \in C([0,1]) : |f(x) - f(y)| \leq k|x-y|^\alpha\}$$
This collection is equicontinuous, but not totally bounded, since all constant functions are satisfying this condition.

Lemma 11.2. *If $\mathcal{F} \subset C(X)$ is totally bounded, then $\mathcal{F}$ is uniformly bounded and (uniformly) equicontinuous.*

Proof. Since a totally bounded set is (uniformly) bounded, we only need to show that $\mathcal{F}$ is equicontinuous.

Since $\mathcal{F}$ is totally bounded, for $\varepsilon > 0$, there exists $f_1, \ldots, f_n \in \mathcal{F}$ such that for $f \in \mathcal{F}$, there exists f_i with
$$\|f - f_i\|_\infty < \varepsilon.$$
But $\{f_1, \cdots, f_n\}$ is equicontinuous, so for $\varepsilon > 0$, there exists $\delta > 0$ such that
$$d(x,y) < \delta \implies |f_i(x) - f_i(y)| < \varepsilon, \quad \forall i = 1, \ldots, n.$$
Now, for any $f \in \mathcal{F}$,
$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$
□

Corollary 11.3. *If $f_n \xrightarrow{unif} f$ in $C(X)$, then $\{f_n\}$ is uniformly bounded and (uniformly) equicontinuous.*

Proof. Notice that $\{f\} \cup \{f_n : n \in \mathbb{N}\}$ is compact, hence $\{f_n\}$ is totally bounded, so (uniformly) equicontinuous. □

11.2. **Arzelà-Ascoli Theorem.** Let X be a compact metric space, and $\mathcal{F} \subset C(X)$. Then $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is closed, uniformly bounded, and uniformly equicontinuous.

Proof. The forward implication follows from the previous lemma.

Conversely, let $(f_n) \subset \mathcal{F}$ be a sequence.

Claim: (f_n) has a (uniformly) convergent subsequence.

Note that (f_n) is equicontinuous. For $\varepsilon > 0$, there exists $\delta > 0$ such that
$$d(x,y) < \delta \implies |f_n(x) - f_n(y)| < \varepsilon, \quad \forall n \geq 1.$$
Since X is totally bounded, there exists a finite set $x_1, \ldots, x_k \in X$ such that
$$X = \bigcup_{i=1}^k B_\delta(x_i).$$
Let $x \in X$, then there exist x_i such that $d(x, x_i) < \delta$.

Also, (f_n) is uniformly bounded, so for each i,

$$\{f_n(x_i)\}_{n=1}^{\infty} \text{ is bounded in } \mathbb{R}.$$

So WLOG, we may assume that $\{f_n(x_i)\}_{n=1}^{\infty}$ is convergent for each $i = 1, \dots, k$.

In particular, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_m(x_i) - f_n(x_i)| < \varepsilon$$

for all $m, n \geq N$, for each $i = 1, 2, \dots, k$.

Now, for $x \in X$, there exists x_i such that $d(x, x_i) < \delta$. Hence,

$$\begin{aligned}|f_m(x) - f_n(x)| &\leq |f_m(x) - f_m(x_i)| + |f_m(x_i) - f_n(x_i)| + |f_n(x_i) - f_n(x)| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.\end{aligned}$$

So $\|f_m - f_n\|_\infty \leq 3\varepsilon$ for all $m, n \geq N$. Therefore, (f_n) is a (uniformly) Cauchy sequence, hence convergent (because $C(X)$ is complete). □

Corollary 11.4. *Let X be compact. If (f_n) is uniformly bounded and (uniformly) equicontinuous in $C(X)$, then (f_n) has convergent subsequence.*
(Hint: $A = \overline{\{f_n : n \in \mathbb{N}\}}$ is closed.)

Example 11.5. Let $X = (0, 1)$ and define

$$f_n(t) = \begin{cases} 1 - nt & \text{if } t < \frac{1}{n} \\ 0 & \text{if } t \geq \frac{1}{n} \end{cases}$$

Show that $(f_n)_{n=1}^{\infty}$ is pointwise equicontinuous but not uniformly equicontinuous on $(0, 1)$.

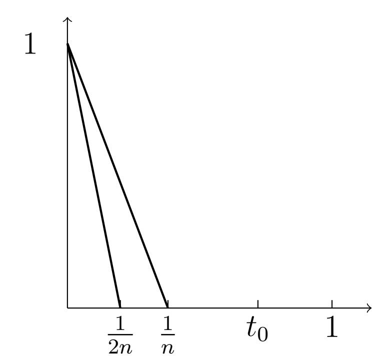
Notice that for any point $t \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n - 0$,

$$f_n(t) = 0 \quad \text{in a small neighborhood of } t.$$

Hence, $(f_n)_{n\geq 1}$ is pointwise equicontinuous on $(0, 1)$. However,

$$\left| f_n\left(\frac{1}{2n}\right) - f_n\left(\frac{1}{n}\right) \right| = |1 - n \cdot \frac{1}{2n} - 0| = \frac{1}{2}$$

where $|\frac{1}{2n} - \frac{1}{n}| = \frac{1}{2n} \to 0$. Hence, $(f_n)_{n\geq 1}$ is not uniformly equicontinuous on $(0, 1)$.



Example 11.6. For $X = [0, 1]$, define

$$f_n(t) = \max\left\{1 - 2(n+1)^2 \left| t - \frac{1}{n} \right|, 0\right\}.$$

Then $(f_n)_{n\geq 1}$ is equicontinuous at each point $t > 0$, but not at $t = 0$.

For $t_0 > 0$, it follows from the fact that
$$1 - 2(n+1)^2\left|t_0 - \frac{1}{n}\right| \le 0 \quad \text{iff} \quad \frac{1}{n} + \frac{1}{2(n+1)^2} \le t_0.$$
And hence, $f_n(t) = 0$ for $n \ge n_0$ in a small neighborhood of $t_0 > 0$. Notice that the above means that $(f_n)_{n=n_0}^{\infty}$ is pointwise equicontinuous at $t_0 > 0$. Since $\{f_1, \cdots, f_{n_0-1}\}$ (finitely many) is always equicontinuous. Thus $(f_n)_{n\ge1}$ is pointwise equicontinuous for $t > 0$.

However, for $t = 0$, $f_n(0) = 0$, $f_n\left(\frac{1}{n}\right) = 1$, but $|0 - \frac{1}{n}| \to 0$ and $|f_n(0) - f_n(\frac{1}{n})| = 1$. Thus, $(f_n)_{n\ge1}$ is not pointwise equicontinuous at $t = 0$.

Remark 11.7. We end this section with a remark on structural property of sets in real line. Any set can be inscribed into countably many disjoint open intervals, however, a bounded (totally bounded) set can be covered by finitely many almost disjoint intervals of arbitrarily small length.

Remark 11.8. A closed observation of totally bounded sets reveals that most of the properties which are true for finitely many points (centers) in a totally bounded metric space, can easily be percolated to the full space, since any point of the space is in a small (arbitrarily small) ball.

11.3. **Dini's Theorem.**

Theorem 11.9. *Let X be a compact metric space, and $f, f_n \in C(X)$ such that $f_n \downarrow f$ pointwise on X. Then $f_n \downarrow f$ uniformly on X.*

Proof. Let $g_n = f_n - f$. Then $g_n \downarrow 0$ pointwise on X. Notice that for each $\epsilon > 0$, $|g_n(x)| < \epsilon$ for large n (depending upon x).

Let
$$E_n = \{x \in X : g_n(x) < \epsilon\}.$$
Then $E_n = g_n^{-1}(-\infty, \epsilon)$, hence open. Also, $E_n \subset E_{n+1} \subset \cdots$. Since $g_n \downarrow 0$ at each point, it follows that $X = \bigcup_{n=1}^{\infty} E_n$.

(If $x \in X$ and $x \notin E_n$ for all $n \in \mathbb{N}$, then $g_n(x) \ge \epsilon$ for all $n \in \mathbb{N}$, which is a contradiction.)

But X is compact, hence there exist $N \in \mathbb{N}$ such that $X = \bigcup_{n=1}^{N} E_n = E_N$. Thus, for $x \in X$ and $n \ge N$, $g_n(x) \le f_N(x) < \epsilon$, i.e., $|g_n(x)| < \epsilon$ for all $n \ge N$, for all $x \in X$.

Hence $g_n \downarrow 0$ uniformly on X. □

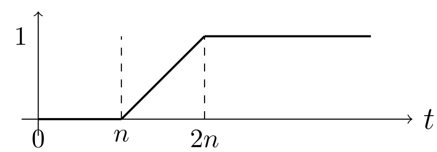
Corollary 11.10. *Suppose $f_n, f \in C(X)$ and $f_n \uparrow f$ pointwise, then $f_n \uparrow f$ uniformly. (Hint: $g_n = f - f_n \downarrow 0$ pointwise, so use the above argument.)*

1. Notice that the limit function f must be continuous, else $f_n(x) = x^n$ will contradict the above theorem.
2. If X is not compact, then the conclusion of the theorem might not be true.
   For $X = \mathbb{R}$,
$$f_n(t) = \begin{cases} 0 & \text{if } -\infty < t \le n \\ \frac{t}{n} - 1 & \text{if } n < t \le 2n \\ 1 & \text{if } t > 2n \end{cases}$$

$$f_n \downarrow 0 \text{ pointwise,}$$

but

$$\|f_n\|_\infty = 1.$$



Remark 11.11. However, a pointwise convergent sequence can differ with uniform convergence on an arbitrarily small set (*Egoroff's Theorem*).

11.4. **Upper Semi-Continuity.** Let $f : (X, d) \to \mathbb{R}$. Then f is said to be **upper semi-continuous** on X if for each $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) < \alpha\}$ is open.

Result: $f : X \to \mathbb{R}$ is upper semi-continuous iff for any $x \in X$, and each sequence $x_n \to x$ implies

$$\limsup_{n\to\infty} f(x_n) \leq f(x).$$

Proof. Let $x_0 \in X$ and $\epsilon > 0$. Then $x_0 \in \{x : f(x) < f(x_0) + \epsilon\}$ is open.
$\implies$ There exists a neighbourhood $B_\delta(x_0)$ such that $f(x) < f(x_0) + \epsilon$ for all $x \in B_\delta(x_0)$. Let $\frac{1}{n} < \delta$ and $x_n \to x$ then $x_n \in B_{\frac{1}{n}}(x_0)$ such that $f(x_n) < f(x_0) + \epsilon$.

Hence,

$$x_n \to x_0 \implies \limsup_{n\to\infty} f(x_n) \leq f(x_0) + \epsilon \quad \text{for all} \quad \epsilon > 0$$

So $\limsup_{n\to\infty} f(x_n) \leq f(x_0)$.

Conversely, suppose (on contrary) that f is not upper semi-continuous on X.

Then there exists $\alpha \in \mathbb{R}$ such that

$A_\alpha = \{x \in X : f(x) < \alpha\}$ is not open. That is there exists $x_0 \in A_\alpha$ such that for any neighbourhood $B_\delta(x_0)$, there exist $x_\delta \in B_\delta(x_0)$ with $x_\delta \notin A_\alpha \implies f(x_\delta) > \alpha$.

For $\delta = \frac{1}{n}$, choose $x_n \in B_{\frac{1}{n}}(x_0) \implies x_n \to x_0$, but $f(x_n) > \alpha > f(x_0)$. Thus,

$$\limsup_{n\to\infty} f(x_n) \geq \alpha > f(x_0)$$

which is a contradiction. □

Example 11.12. If X is compact and $f : X \to \mathbb{R}$ is upper semi-continuous, then f attains its maximum.

Note that $X = \bigcup_{\alpha\in\mathbb{R}}\{x \in X : f(x) < \alpha\}$, but X is compact, hence $X = \bigcup_{i=1}^{k}\{x \in X : f(x) < \alpha_i\}$. For any $x \in X$, $f(x) < \alpha_i < \max\{\alpha_i\} = \alpha < \infty$. Hence f is bounded above. Next, f attains its supremum on X.

If not, then $f(x) < \sup f$ for all $x \in X$. For $n \in \mathbb{N}$, there exists $x_n \in X$ such that

$$\sup f - \frac{1}{n} < f(x_n)$$

Now, $x_n \in X$, X is compact, hence $\exists$ subsequence $x_{n_k} \to x \in X$. But, then

$$\sup f \leq \limsup_{k\to\infty} f(x_{n_k}) \leq f(x)$$

Thus $\sup f \leq f(x)$, which is not possible, as it contradicts our assumption.

Note that, in similar way, we can define **lower semi-continuity**, i.e., $f : X \to \mathbb{R}$, $\{x \in X : f(x) > \alpha\}$ is open for each $\alpha \in \mathbb{R}$. Also, it follows that f is lower semi-continuous if and only if for all $x_n \to x$,

$$f(x) \leq \liminf_{n\to\infty} f(x_n)$$

Thus, f is continuous if and only if f is both lower and upper semi-continuous.

(*Hint:* $\lim_{n\to\infty} sup f(x_n) \leq f(x) \leq \lim_{n\to\infty} inf f(x_n)$ whenever $x_n \to \infty$.)

Remark 11.13. Note that if $f : X \to \mathbb{R}$ is **upper semi-continuous** (USC), then $f^{-1}\{(-\infty, \alpha)\}$ is open, and hence $f^{-1}\{[\beta, \alpha)\}$ is open if $\beta < \alpha$, but it does *not* imply that $f^{-1}\{(\beta, \alpha)\}$ is open for each $\alpha, B \in \mathbb{R}$, else f is continuous.
(However, f is **Lebesgue measurable**!)

But if f is both lower semi-continuous (LSC) and upper semi-continuous (USC), then

$$f^{-1}\{(\alpha, \beta)\} = f^{-1}\{(-\infty, \beta) \cap (\alpha, \infty)\}$$

is open, hence f is continuous.

Remark 11.14. There is no relation between lower semi-continuity and upper semi-continuity with left limit and right limit.

Example 11.15.

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

is upper semi-continuous, but none of left limit and right limit exists at $x = 0$.

Exercise 11.16. Check for lower semi-continuity and upper semi-continuity for $f(x) = \lfloor x \rfloor$, the greatest integer function.

11.5. **Weierstrass Approximation Theorem.** We shall see that polynomials are dense in $(C[a, b], \| \cdot \|_\infty)$ if $b - a < \infty$. As a consequence, $C[a, b]$ is a separable space.

The question of density of polynomials in $C[a, b]$ can be transferred to $C[0, 1]$ with the help of the map:

$$f(t) = \frac{t - a}{b - a}$$

For $f \in C[0, 1]$ and $n = 0, 1, 2, \ldots$, define (Bernstein polynomial):

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}$$

Then $B_n(f_n)$ is a polynomial of degree at most n. Here, $B_n(f_n)$ is known as **Bernstein polynomial**.

In fact, we have

$$B_n(f)(0) = f(0), \qquad B_n(f)(1) = f(1)$$

Let us denote $f_n(x) = x^n$ for $n = 0, 1, 2, \ldots$.

The following lemma, which is involved with combinatorics, is crucial in proving the density of $B_n(f)$ in $C[0,1]$.

Lemma 11.17.

(1) $B_n(f_0) = f_0$ *and* $B_n(f_1) = f_1$
(2) $B_n(f_2) = \left(1 - \frac{1}{n}\right) f_2 + \frac{1}{n} f_1$, *hence* $B_n(f_2) \to f_2$ *uniformly.*
(3) $\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \frac{x(1-x)}{n} \le \frac{1}{4n}$
(4) *Given* $\delta > 0$, $0 \le x \le 1$, *let* F *denote the set of* $F = \left\{k \in \{0,1,\ldots,n\} : \left|\frac{k}{n} - x\right| \ge \delta\right\}$. *Then*
$$\sum_{k\in F} \binom{n}{k} x^k (1-x)^{n-k} \le \frac{1}{4n\delta^2}$$

Proof. (i) is trivial, as it follows from simple binomial expansions.
Hint:
$$\sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} x^k(1-x)^{n-k} = x\sum_{k=1}^{n}\binom{n-1}{k-1} x^{k-1}(1-x)^{n-k}$$
$$= x\sum_{j=0}^{n-1}\binom{n-1}{j} x^j (1-x)^{(n-1)-j} = x[x + (1-x)]^{n-1} = x$$
So $B_n(f_1) = f_1$.

(ii) To compute $B_n(f_2)$, we break the sum into two parts:
$$\left(\frac{k}{n}\right)^2 \binom{n}{k} = \frac{k}{n}\binom{n-1}{k-1} = (1-\frac{1}{n})\binom{n-2}{k-2} + \frac{1}{n}\binom{n-1}{k-1} \quad \text{for } k \ge 2$$
Thus,
$$B_n(f_2) = \left(1-\frac{1}{n}\right)\sum_{k=2}^{n}\binom{n-2}{k-2} x^k (1-x)^{n-k} + \frac{1}{n}\sum_{k=1}^{n}\binom{n-1}{k-1} x^k(1-x)^{n-k}$$
$$= \left(1-\frac{1}{n}\right)x^2 + \frac{1}{n}x \to f_2 \text{ uniformly}$$

(iii) Note that
$$\left(\frac{k}{n} - x\right)^2 = \left(\frac{k}{n}\right)^2 - 2x\frac{k}{n} + x^2$$
hence
$$\sum_{k=0}^{n}\left(\frac{k}{n} - x\right)^2\binom{n}{k}x^k(1-x)^{n-k} = (1-\frac{1}{n})x^2 + \frac{1}{n}x - 2x^2 + x^2 = \frac{x(1-x)}{n} \le \frac{1}{4n} \quad (\text{ by (ii)})$$

(iv) For $k \in F$, $1 \le \frac{(\frac{k}{n}-x)^2}{\delta^2}$.
Hence,
$$\sum_{k\in F}\binom{n}{k}x^k(1-x)^{n-k} \le \frac{1}{\delta^2}\sum_{k\in F}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k} \le \frac{1}{\delta^2}\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k} \le \frac{1}{4n\delta^2}$$
□

Theorem 11.18. ***Bernstein:*** *Let* $f \in C[0,1]$, *then* $B_n(f) \to f$ *uniformly.*

Proof. Since f is uniformly continuous, for $\varepsilon > 0$ there exists $\delta > 0$ such that $|x-y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$.

Now,

$$\begin{aligned} |f(x) - B_n(f)(x)| &= \left| \sum_{k=0}^{n} \left(f(x) - f\left(\frac{k}{n}\right) \right) \binom{n}{k} x^k (1-x)^{n-k} \right| (\because \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1) \\ &\leq \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k} \end{aligned}$$

Let us fix a n (to be specified soon). Let F denote the set of $k \in \{0, 1, \dots, n\}$ such that $\left|\frac{k}{n} - x\right| \geq \delta$. Then

$$|f(x) - f\left(\frac{k}{n}\right)| < \frac{\varepsilon}{2} \quad \text{for } k \notin F,$$

and

$$|f(x) - f\left(\frac{k}{n}\right)| \leq 2\|f\|_\infty \quad \text{for } k \in F.$$

Thus,

$$\begin{aligned} |f(x) - B_n(f)(x)| &\leq \frac{\varepsilon}{2} \sum_{k \notin F} \binom{n}{k} x^k (1-x)^{n-k} + 2\|f\|_\infty \sum_{k \in F} \binom{n}{k} x^k (1-x)^{n-k} \\ &\leq \frac{\varepsilon}{2} \cdot 1 + 2\|f\|_\infty \left(\frac{1}{4n\delta^2} \right) \\ &< \varepsilon \end{aligned}$$

if $n > \frac{\|f\|_\infty}{\varepsilon\delta^2}$.

Therefore,

$$\|B_n(f) - f\|_\infty < \varepsilon \quad \text{if} \quad n > \frac{\|f\|_\infty}{\varepsilon\delta^2}.$$

□

Exercise 11.19. If $f \in C[0,1]$ and $\int_0^1 x^n f(x)\,dx = 0$ for all $n \geq 0$, then $f = 0$.

12. Connected Sets

The structure of the real line has been invaded in several ways to know the peculiar hidden properties. We have already seen that any open set $O \subset \mathbb{R}$ can be expressed as the disjoint union of countably many open intervals. That is,

$$O = \bigsqcup_{n=1}^{\infty} I_n, \text{ where } I_n = (a_n, b_n).$$

Hence, for any set $A \subset \mathbb{R}$, we get an open set $O \supset A$, and thus

$$A \subset O \subset \bigcup I_n.$$

Hence, any set can be embedded into countably many open intervals. The "connected set" has its natural meaning, and we can extract its definition from the intervals.

We know that an interval cannot be broken into two relatively open parts.

On the contrary, suppose that

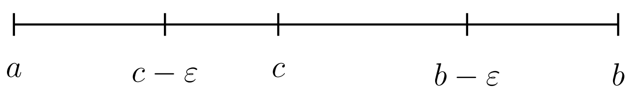
$$[a, b] = A \sqcup B,$$

where A and B are non-empty, disjoint, and relatively open sets in $[a, b]$. This implies that A and B are disjoint closed sets too, as

$$A = [a, b] \setminus B, \qquad B = [a, b] \setminus A.$$

Thus A and B are disjoint, non-empty open and closed sets (called clopen sets).

To start with, let $b \in B$. Since B is open, $(b - \varepsilon, b] \subset B$ for some $\varepsilon > 0$.



Now, let $c = \sup A$. Then $a < c < b$.
(if $a = c$, then $A = \{a\}$ (not open), and if $c = b$, then $A \cap B \neq \emptyset$.)

By definition of supremum, $(c - \varepsilon, c) \cap A \neq \emptyset$ and $(c, c + \varepsilon) \cap B \neq \emptyset$ (since c is the dead end of A).

That is, $c \in \overline{A} = A$ and $c \in \overline{B} = B$, which is a contradiction that $A \cap B = \emptyset$.

Hence, based on the above observation, we can define connected/disconnected sets.

Definition 12.1. A metric space X is said to be *disconnected* (not connected) if there exist two non-empty open sets A and B such that $X = A \sqcup B$. The sets A and B are called a *disconnection* of X.

We say that X is *connected* if X cannot be expressed as a disjoint union of two non-empty open sets in X.

Thus, the interval $[a, b]$ is connected.

Note that, when $X = A \sqcup B$ where A and B are disjoint, non-empty open sets, it follows that A and B are closed sets too (as $A = B^c$, $B = A^c$). Thus, A and B are disjoint, non-empty clopen sets.

Thus, X is connected if and only if X has no nontrivial clopen sets.
(*Hint:* if A is clopen, then $X = A \sqcup A^c$, and A^c is also open.)

Definition 12.2. A subset E of a metric space X is called *disconnected in E* if there exist non-empty disjoint open sets U and V in E such that $E = U \sqcup V$.

Note that there exist open sets A and B in X such that

$$U = A \cap E, \qquad V = B \cap E$$

$\implies$

$$E = (A \cap E) \cup (B \cap E) = (A \cup B) \cap E \implies E \subset A \cup B.$$

It is clear that A and B need not be disjoint. However, we can filter them further to make them disjoint and still cover E.

Lemma 12.3. *Let $E \subset X$. If U and V are disjoint open sets in E, then there exist disjoint open sets A and B in X such that*

$$U = A \cap E \quad \text{and} \quad V = B \cap E.$$

Proof. For $x \in U$, there exists $\varepsilon_x > 0$ such that

$$E \cap B_{\varepsilon_x}(x) \subset U \qquad (\because U \text{ is open in } E).$$

Similarly, for $y \in V$, there exists $\varepsilon_y > 0$ such that

$$E \cap B_{\varepsilon_y}(y) \subset V.$$

Now, $U \cap V = \emptyset \implies E \cap (B_{\varepsilon_x}(x) \cap B_{\varepsilon_y}(y)) = \emptyset$.

Claim: $B_{\frac{\varepsilon_x}{2}}(x) \cap B_{\frac{\varepsilon_y}{2}}(y) = \emptyset$.

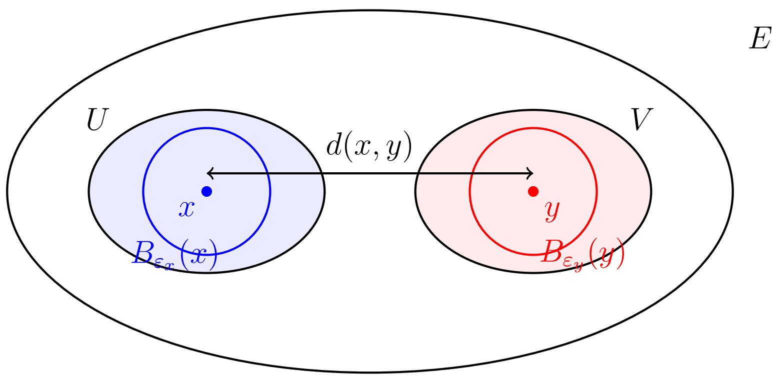
Note: If $d(z, x) < \frac{\varepsilon_x}{2}$ and $d(z, y) < \frac{\varepsilon_y}{2}$, then

$$d(x, y) < \frac{\varepsilon_x}{2} + \frac{\varepsilon_y}{2}.$$

Choose ε_x and ε_y so that

$$d(x, y) > \frac{\varepsilon_x}{2} + \frac{\varepsilon_y}{2}.$$

Then the claim will be satisfied.



Write:

$$A = \bigcup_{x \in U} B_{\frac{\varepsilon_x}{2}}(x), \qquad B = \bigcup_{y \in V} B_{\frac{\varepsilon_y}{2}}(y)$$

$\Rightarrow A \cap B = \emptyset$ and A, B are open in X, and $E \subset A \sqcup B$.

Thus, we say $E \subset X$ is *disconnected* if there exist disjoint open sets A, B in X with $A \cap E \neq \emptyset$, $B \cap E \neq \emptyset$, and $E \subset A \sqcup B$. $\square$

Next, we see that connected subsets of $\mathbb{R}$ are precisely singletons or intervals.

Theorem 12.4. *A subset E of $\mathbb{R}$ (containing more than one point) is connected iff for every $x, y \in E$ with $x < y$ it follows that $[x, y] \subset E$.*

Proof. If for some $x, y \in E$, with $x < z < y \implies z \notin E$, then
$$E \subset (-\infty, z) \sqcup (z, \infty)$$
is a disconnection of E. This proves the assertion.

On the other hand, suppose for each pair $x, y \in E$ we have $[x, y] \subset E$, but E is disconnected. Then there exist non-empty open sets $A, B \subset \mathbb{R}$ such that $A \cap E \neq \emptyset$, $B \cap E \neq \emptyset$ and $E \subset A \sqcup B$.
Let $a \in A \cap E$, $b \in B \cap E$ and assume $a < b$. Then $[a, b] \subset E \subset A \sqcup B$.

So, $[a, b]$ is disconnected ($\because [a, b] = (A \cap [a, b]) \sqcup (B \cap [a, b])$), which is a contradiction.

Finally, suppose for each $x, y \in E$, $[x, y] \subset E$. We claim E is an interval.

Note that $(\inf E, \sup E) \subset E$,where we include the possibilities that $\inf E = -\infty$, $\sup E = +\infty$.

(*Hint:* For any $\epsilon > 0$, there exist $x, y \in E$ with $\inf E + \epsilon > x$ and $\sup E - \epsilon < y$ so that $[x, y] \subset E \quad \forall \epsilon > 0$).

Thus, E must be an interval, and it depends upon disposition of $\inf E, \sup E$ as finite or infinite as element or not of E. □

Exercise 12.5. Show that the connected subsets of Cantor's set are only singletons (i.e., Cantor set is totally disconnected).

Now, we simplify our study of connected sets with the help of continuous functions.

Notice that a discrete metric space (containing more that one point) is always disconnected. We use this fact to identify disconnected sets through comparison via continuous maps.

Theorem 12.6. *A space X is disconnected if and only if there exists a continuous surjective map $f : X \to \{0, 1\}$ (two-point discrete space).*

Proof. If $f : X \to \{0, 1\}$ is continuous and surjective, then
$$A = f^{-1}(\{0\}), \quad B = f^{-1}(\{1\}).$$
are non-empty, disjoint open sets and $A \sqcup B = X$. Since f is continuous, A, B are closed. Thus, X has a disconnection.

Conversely, if $X = A \sqcup B$ where A and B are non-empty, disjoint open sets in X. Define
$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$
which is a continuous and surjective map. □

This result gives a perfect replacement of definition of connected sets.

Thus, we conclude that X is connected if and only if every continuous map from X into a discrete space is constant.

Theorem 12.7. *Let $f : (X, d) \to (Y, \rho)$ be continuous, and let $E \subseteq X$. If E is connected, then $f(E)$ is connected.*

Proof. Suppose $f(E)$ is not connected. Then there exists a continuous surjective map $g : f(E) \to \{0, 1\}$. Thus, $g \circ f : E \to \{0, 1\}$ is continuous and surjective, so E is disconnected, a contradiction. □

Remark 12.8. A non-constant continuous image of an interval is again an interval. This is nothing but the intermediate value theorem.

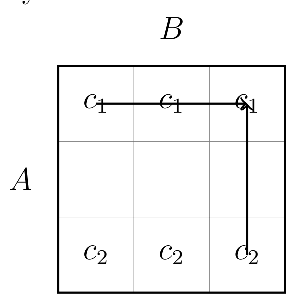
Corollary 12.9. *Let I be an interval in $\mathbb{R}$, and $f : I \to \mathbb{R}$ be a non-constant continuous function, then $f(I)$ is an interval.*

In particular, if $a, b \in I$ and $f(a) \neq f(b)$, then f assumes all values between $f(a)$ and $f(b)$.

Example 12.10. If A, B are connected subsets of a metric space X, then $A \times B$ is connected in $(X \times X, d \times d)$, where

$$(d \times d)\{(x_1, y_1), (x_2, y_2))\} = d(x_1, x_2) + d(y_1, y_2).$$

Suppose $f : A \times B \to \{0, 1\}$ is continuous. We claim f is constant. For $a \in A$ and $b \in B$, $f(a, \cdot)$ and $f(\cdot, b)$ are continuous function on A and B respectively . Since A and B are connected implies $f(a, \cdot)$ and $f(\cdot, b)$ both are constant. That is, f is constant on every vertical and horizontal lines. Hence, f is constant.



Exercise 12.11. Show that $(0, 1) \times (0, 1)$ cannot be written as disjoint union of countably many open balls.

(*Hint:* $(0, 1) \times (0, 1)$ is connected)

Exercise 12.12. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ continuous. Show that D is connected if and only if the graph of f, $G_f = \{(x, f(x)) : x \in D\}$ is connected in $\mathbb{R}^2$.

(*Hint:* $g : X \to X \times X$, $g(x) = (x, f(x))$ is continuous $\implies G_f$ is connected ($\because X$ is connected)

On the other hand, projection $\rho_1 : G_f \to X \implies \rho_1(x, f(x)) = x$, is continuous $\implies X$ is connected)

Exercise 12.13. If $A \subset X$ is connected, then for $A \subseteq B \subseteq \overline{A}$, it implies that B is connected. In particular, $\overline{A}$ is connected.

Suppose $f : B \to \{0, 1\}$ is continuous and surjective, then $f|_A : A \to \{0, 1\}$ is continuous $\implies f$ is constant on B.

Exercise 12.14. Let $A \subset B \subset X$. If A and X are connected, does it imply B is connected?

($(0, 1) \subset (0, 1) \sqcup (1, 2) \subset \mathbb{R}$)

Exercise 12.15. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(Topologist's sine curve)

Show that f is not continuous, but G_f is connected.

(*Hint:* Consider $g : (0,1] \to [-1,1]$ by $g(x) = \sin\left(\frac{\pi}{x}\right)$. Then g is continuous, and hence $g\{(0,1]\}$ is connected $\implies$ g is onto.

Also, G_g is connected. Since

$$G_g \subset G_f \subset \overline{G_g} \implies G_f \text{ is connected.}$$

Exercise 12.16. If $f : X \to Y$ is continuous and onto, Y not connected, then X is not connected.

(*Hint:* $Y = C \sqcup D \implies X = f^{-1}(C) \sqcup f^{-1}(D)$)

Exercise 12.17. $\text{Ln}(\mathbb{R}) = \{\text{space of all } n \times n \text{ real matrices}\}$ and $\text{GL}_n(\mathbb{R}) = \{A = (x_{ij}) \in \text{Ln}(\mathbb{R}) : \det A \neq 0\}$.

Then, $\text{GL}_n(\mathbb{R})$ is disconnected in the usual metric on $\text{Ln}(\mathbb{R})$.

(*Hint:* $\det(A) = \sum_{i=1}^{n} x_{ii} \implies$ det is continuous

$$\implies \text{GL}_n(\mathbb{R}) = (\det)^{-1}(\mathbb{R} \setminus \{0\}) \text{ is open.}$$

Now,

$$\det : \text{GL}_n(\mathbb{R}) \xrightarrow{\text{continuous, onto}} \mathbb{R} \setminus \{0\}$$

$\implies \text{GL}_n(\mathbb{R}) = \text{GL}_n^+(\mathbb{R}) \cup \text{GL}_n^-(\mathbb{R})$

is disconnected, where

$$(\det)^{-1}\{(-\infty, 0))\} = \text{GL}_n^-(\mathbb{R}), \quad (\det)^{-1}\{(0,\infty))\} = \text{GL}_n^+(\mathbb{R}).$$

(*Hint:* An easiest metric on $\text{Ln}(\mathbb{R})$ is $d(A,B) = \max_{ij} |a_{ij} - b_{ij}|$)

12.1. **Path Connectedness.** A set $E \subset X$ is said to be *path connected* if for every $x, y \in E$, there exists a continuous function $\gamma : [0,1] \to E$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Example 12.18. Show that the continuous image of a path connected set is path connected.

Let $E \subset X$ be path connected and $f : E \to \mathbb{C}$ be continuous. Then, for $f(x), f(y) \in f(E)$, there exists a path $\gamma : [0,1] \to E$ ($\because x, y \in E$) such that $\gamma(0) = x$ and $\gamma(1) = y$. Therefore, $f \circ \gamma(0) = f(x)$ and $f \circ \gamma(1) = f(y)$. So $f \circ \gamma$ is the required path connecting $f(x)$ and $f(y)$.

Example 12.19. Let P be a polynomial in $\mathbb{C}^n$. Then $\mathbb{C}^n \setminus P^{-1}(0)$ is path connected.

Let $z, w \in \mathbb{C}^n \setminus P^{-1}(0)$. Define $\gamma : \mathbb{C} \to \mathbb{C}^n$ by $\gamma(t) = (1-t)z + tw$, $t \in \mathbb{C}$.

Then $\{t \in \mathbb{C} : \gamma(t) \in P^{-1}(0)\} = (P \circ \gamma)^{-1}(0)$. Since $(P \circ \gamma)$ is a polynomial on $\mathbb{C}$, it implies that $(P \circ \gamma)^{-1}(0)$ is a finite set. Hence, $\mathbb{C} \setminus (P \circ \gamma)^{-1}(0)$ is path connected in $\mathbb{C}$.

Hence, $f(\mathbb{C} \setminus (P \circ \gamma)^{-1}(0))$ is path connected in $\mathbb{C}^n \setminus P^{-1}(0)$ (since $\gamma(\mathbb{C} \setminus P^{-1}(0))$ is contained in $\mathbb{C}^n \setminus P^{-1}(0)$) containing z and w. Hence, $\mathbb{C}^n \setminus P^{-1}(0)$ is path connected.

(Note that γ is not onto unless $n = 1$, hence $\gamma(\mathbb{C} \setminus (P \circ \gamma)^{-1}(0)) \subsetneq \mathbb{C}^n \setminus P^{-1}(0)$.)

Once again Topologist's Sine Curve:

Let $f : [0,1] \to [-1,1]$ by

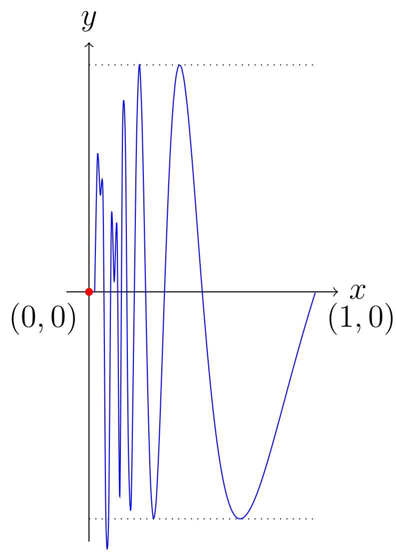
$$f(x) = \begin{cases} \sin \frac{\pi}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then

$$G_f = \{(x, \sin \frac{\pi}{x}) : x \in (0,1]\} \sqcup \{(0,0)\}$$

is not open. G_f is not path connected.
(The hope comes from the fact that f is not continuous at 0.)

Diagram: Sine Curve



On contrary, suppose there is a continuous path

$$\gamma : [0,1] \to G_f = \{(x, \sin \frac{1}{x}) : x \neq 0\} \cup \{(0,0)\}$$

where $\gamma(0) = (0,0)$ and $\gamma(1) = (1,0)$; write $\gamma = (\gamma_1, \gamma_2)$. Since γ is continuous, γ becomes uniformly continuous. For $\varepsilon = 1 > 0$, there exists $\delta > 0$ such that

$$|s - t| < \delta \implies |\gamma_2(s) - \gamma_2(t)| < 1$$

Since $0 \in \gamma^{-1}\{(0,0)\}$, let

$$t^* = \sup\ \gamma^{-1}\{(0,0)\} < 1 \qquad (\because \gamma(1) = (1,0))$$

Choose $\delta_1 > 0$ such that $0 \leq t^* < t^* + \delta_1 < 1$ and $\delta_1 < \delta$.

Note that

$$t^* = \sup\{\, t : \gamma(t) = (\gamma_1(t), \gamma_2(t)) = (0,0)\,\}$$

So, there exists $t_n \to t^*$, with $\gamma_1(t_n) = 0 \implies \gamma_1(t^*) = 0$, but $\gamma_1(t^* + \delta_1) > 0$.

For large N, there exists $s, t \in (t^*, t^* + \delta_1)$ such that

$$\gamma_1(t) = \frac{2}{N+1}, \quad \gamma_1(s) = \frac{2}{N}$$

Therefore,

$$\gamma_2(t) = \sin\left(\frac{N+1}{2}\right)\pi, \quad \gamma_2(s) = \delta_1 \sin\left(\frac{N\pi}{2}\right)$$

So,

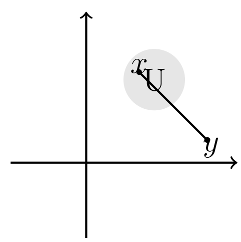
$$|\gamma_2(t) - \gamma_2(s)| = 1$$

This is a contradiction.

Example 12.20. $\mathbb{R}^n \setminus \{0\}, (\ n \geq 2)$ is connected.

Suppose not, let U be an open and closed set in $\mathbb{R}^n \setminus \{0\}$. For $x \in U$ and $y \in \mathbb{R}^n \setminus \{0\} \setminus U$, we get a line segment path connecting x and y, say L.

Then $L \cap U$ is the finite union of open and closed sets in $\mathbb{R}$, but $\mathbb{R}$ is connected. Hence, our assumption is wrong, and $\mathbb{R}^n \setminus \{0\}$ is connected. In fact, path connected.



Example 12.21. Let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Then S^{n-1} is connected.

Define $\varphi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ by

$$\varphi(x) = \frac{x}{\|x\|}$$

Then φ is continuous and onto, hence S^{n-1} is connected. In fact, S^{n-1} is continuous image of a path connected set $\mathbb{R}^n \setminus \{0\}$, hence path connected.

Example 12.22. Alternative: If $I \subset \mathbb{R}$ is connected, then I is an interval.

Suppose there exist $x, y \in I$, $x < z < y$, but $z \notin I$. Define

$$f(s) = \begin{cases} 1 & \text{if } s < z \\ -1 & \text{if } s > z \end{cases}$$

So,

$$f : I \setminus \{z\} = I \to \{1, -1\}$$

is continuous and onto, so I is not connected!

Example 12.23. Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$G_f = \{(x, f(x)) : x \in \mathbb{R}\}$$

is closed and connected in $\mathbb{R}^2$. Then f is continuous.

Let $x_n \to x$. Assume $f(x_n) \to y$. Then $(x_n, f(x_n))$ is a Cauchy sequence in $\mathbb{R}^2$ and hence

$$(x_n, f(x_n)) \to (x, y)$$

But G_f is closed, so $(x, y) \in G_f$, which implies $y = f(x)$. Hence, f is continuous.

Note that: $f(x_n) \to y$ can be achieved by considering the boundedness of f where $x_n \to x$.

If f is bounded, then $f(x_n)$ is bounded in $\mathbb{R}$, and by Bolzano-Weierstrass Theorem, there exists a subsequence $f(x_{n_k}) \to y \in \mathbb{R}$. Thus,

$$(x_{n_k}, f(x_{n_k}))$$

is a Cauchy sequence in $\mathbb{R}^2$ and hence convergent, say

$$(x_{n_k}, f(x_{n_k})) \to (x, y)$$

But G_f is closed, implies $y = f(x)$.

Notice that there is no other limit point for $(x_n, f(x_n))$ than $(x, f(x))$, else f will not be well-defined. Thus,

$$(x_n, f(x_n)) \to (x, f(x))$$

Hence, f is continuous.

Notice that so far we have not used the fact that G_f is connected.

Next case is when $|f(x_n)| \to \infty$, where $x_n \to x$. In this case, we reach to a contradiction that G_f is disconnected in a neighborhood of x.

We claim that there exists $\delta > 0$ such that for $|x - y| < \delta \implies$ either $|f(x) - f(y)| < 1$ or $|f(x) - f(y)| > 2$.

(Bounded below and above in a neighbourhood of x)

If it is false, then there is a sequence u_n with $|u_n - x| < \frac{1}{n}$ such that $1 \leq |f(x) - f(u_n)| \leq 2$.

$$\Rightarrow \text{There is a subsequence } f(u_{n_k}) \text{ of } f(u_n) \text{ such that } f(u_{n_k}) \to w.$$

Then

$$(u_{n_k}, f(u_{n_k})) \to (x, w),$$

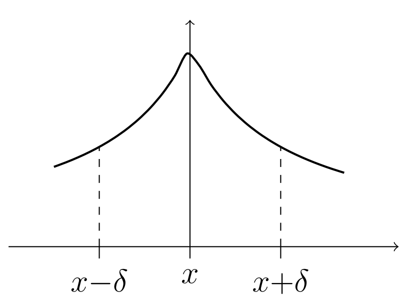
and the graph G_f is closed (by hypothesis), which implies $f(x) = w$.

But $1 \leq |f(x) - w| \leq 2$

Thus, our claim is true.

Let $[a, b] = [x - \delta, x + \delta]$.

We claim that $G_f \cap \{[a, b] \times \mathbb{R}\}$ is connected.



On the other hand,

$$G_f \cap ([a,b] \times \mathbb{R})$$
$$= (G_f \cap \{[a,b] \times \mathbb{R}\}) \cap \{(t,s) : |f(x) - s| < 1\} \cup (G_f \cap ([a,b] \times \mathbb{R})) \cap \{(t,s) : |f(x) - s| > 1\}$$

i.e. $G_f \cap ([a,b] \times \mathbb{R}) = A \sqcup B$. (*)

Thus, $G_f \cap ([a,b] \times \mathbb{R})$ is disconnected as $(x, f(x)) \in A$ and $(x_n, f(x_n)) \in B$ for large n.

This implies $x_n \to x \implies f(x_n)$ is bounded.

Hence, from the previous case, it follows that $f(x_n) \to f(x)$.

To show G_f is connected, let

$$g : G_f \cap ([a,b] \times \mathbb{R}) \to \{0,1\}$$

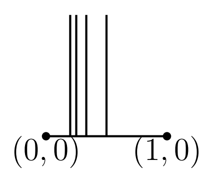
be continuous. Then g can be extended continuously outside $G_f \cap ([a,b] \times \mathbb{R})$ by constant.

Hence $g : G \to \{0,1\}$ is continuous. But G is connected, hence g is constant.

Thus, $G_f \cap ([a,b] \times \mathbb{R})$ is connected.

Example 12.24. Let $K = \{\frac{1}{n} : n \geq 1\}$ and $E = ([0,1] \times \{0\}) \cup (K \times [0,1])$.

Then E is path connected (Why?)



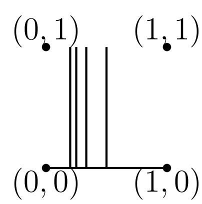
Let $C = E \times (\{0\} \times [0,1])$, known as the *comb space*, which is path connected.

The deleted comb space

$$C_0 = E \cup \{(0,1)\}$$

is connected, since $E \subset C_0 \subset \overline{E}$ and E is connected.

But C_0 is **not** path-connected,



because there is no path connecting $(0,1)$ and $(1,0)$.

On the contrary, suppose

$$\gamma : [0,1] \to C_0$$

be a continuous path such that $\gamma(0) = (0,1)$ and $\gamma(1) = (1,0)$.

Then $\gamma^{-1}((0,1))$ is a closed set, and

let $t_0 = \sup \gamma^{-1}((0,1)) = \sup\{t \in [0,1] : \gamma(t) = (0,1)\}$.

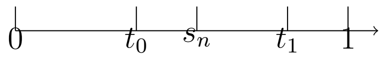
We claim that there exists $t_1 \in (t_0, 1]$ such that

$$(P_1 \circ \gamma)\{(t_0, t_1)\} \subseteq K,$$

where $P_1 : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the x-axis.

Suppose the claim is false. Then $\exists t_n \in (t_0, 1]$ with $t_n \to t_0$. By assumption, $\exists s_n \in (t_0, t_n)$ such that $\gamma(s_n) = (x_n, 0)$ for some $x_n \in [0,1] \setminus K$.

Note that $s_n \to t_0$. By continuity, $(x_n, 0) = \gamma(s_n) \to \gamma(t_0) = (0, 1)$, which is absurd.



Thus, there exists $t_1 \in (t_0, 1]$ such that $(P_1 \circ \gamma)\{(t_0, t_1)\} \subseteq K$.
$\implies 1 \in (P_1 \circ \gamma)(t_0, t_1)$ is a connected subset of K.
Hence $(P_1 \circ \gamma)(t_0, t_1) = \{1\}$ (by contunuity),
but $(P_1 \circ \gamma)(t_0) = 0$, an absurd.

Example 12.25. Let U be an open set in $\mathbb{R}^n$ (or $\mathbb{C}^n$). Then U is path connected if and only if U is connected.

Let $\mathcal{A}$ be the collection of all path connecting a point $p \in U$. Then $\mathcal{A}$ is open.

Let $q \in \mathcal{A}$, then $q \in U \implies B_r(q) \subset U$ for some $r > 0$.

Let $S \in B_r(z)$, then S is connected by a path to q by straight line, and q is connected to p. Hence, $B_r(q) \subset U$.

Let $B = \mathcal{A} \setminus U$. Then B is also open. Since for $t \in B$, there does not exist a path connecting p, then we can draw a path for both surrounding t, which is not connected to p. Thus,

$$U = \mathcal{A} \sqcup B.$$

Since U is connected, this implies $B = \emptyset$, thus U is path connected.