HARDY-LITTLEWOOD MAXIMAL FUNCTION AND LEBESGUE DIFFERENTIATION THEOREM

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CERTIFICATE

This is to certify that the work contained in this report entitled **"Hardy-Littelwood maximal function and Lebesgue Differentiation theorem"** submitted by **Prasanta Sardar** (**Roll No: 222123037.**) to Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him/her under my supervision.

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ABSTRACT

This project explores the connection between differentiation and integration, focusing on when one can recover a function from its integral or derivative. We begin by studying conditions under which integrals can be differentiated and when derivatives can be integrated back.

Key topics include the Vitali Covering Lemma, monotone functions, functions of bounded variation, and absolute continuity. We also discuss the differentiation of integrals in $\mathbb{R}^n$, the Hardy–Littlewood maximal function, and the Lebesgue Differentiation Theorem.

Contents

Chapter 1

1.1 Introduction

In this project , we consider two problems related to the reciprocity of differentiation and integration.

(1) Let f be integrable on $[a, b]$, and define

$$F(x) = \int_a^x f(y)\, dy.$$

Does this imply that f is differentiable (at least almost everywhere) and that $F' = f$ almost everywhere? We shall see that the answer to this question is connected to a broader idea, not limited to dimension one.

(2) What conditions on a function F on $[a, b]$ guarantee that $F'(x)$ exists for almost every x, F' is integrable, and

$$\int_a^b F'(x)\, dx = F(b) - F(a)?$$

The second problem is more difficult than the first one.

As an example, we shall see that if $F : [a, b] \to \mathbb{R}$ is monotone increasing, then F is differentiable almost everywhere in $[a, b]$, and

$$\int_a^b F'(x)\, dx \leq F(b) - F(a).$$

However, there exist continuous monotone increasing functions that fail to satisfy this inequality. For example, the Cantor-Lebesgue function F on $[0, 1]$, with $F(0) = 0$ and $F(1) = 1$, satisfies $F'(x) = 0$ almost everywhere. To prove this result, we need the Vitali covering lemma.

Chapter 2

2.1 Vitali Covering Lemma

Definition 2.1.1 : A collection $\mathcal{S}$ of intervals in $\mathbb{R}$ is said to be a Vitali cover of a set $E \subset \mathbb{R}$ if for each $x \in E$ and for all $\epsilon > 0$, there exists $I \in \mathcal{S}$ such that $x \in I$ and $\ell(I) < \epsilon$.

Lemma 2.1.2 : Let $m^*(E) < \infty$ and let $\mathcal{S}$ be a Vitali cover of E. Then for all $\epsilon > 0$, there exists a finite subcollection $\{I_1, I_2, \ldots, I_N\} \subset \mathcal{S}$ such that

$$m^*\left(E - \bigcup_{n=1}^{N} I_n\right) < \epsilon.$$

proof :

It is sufficient to prove the lemma assuming that the intervals in $\mathcal{S}$ are closed. Since $m^*(E) < \infty$, we can always find an open set $O \supset E$ such that

$$m^*(E) \leq m(O) < \infty.$$

Thus, without loss of generality, we assume that each $I \in \mathcal{S}$ is contained in O.

We construct a sequence $\{I_n\}$ of disjoint intervals in $\mathcal{S}$ by induction. Choose $I_1 \in \mathcal{S}$. Suppose $I_1, I_2, \ldots, I_n$ have already been chosen. Define

$$k_n = \sup\{\ell(I) \mid I \in \mathcal{S}, I \cap (\bigcup_{i=1}^{n} I_i) = \emptyset\}.$$

Since $I \subset O$, we have $k_n < m(O) < \infty$.

If $E \subset \bigcup_{i=1}^n I_i$, we stop. Otherwise, we choose $I_{n+1} \in \mathcal{S}$ such that

$$\ell(I_{n+1}) > \frac{1}{2}k_n, \quad I_{n+1} \cap \bigcup_{i=1}^{n} I_i = \emptyset.$$

This process constructs a sequence of disjoint intervals $\{I_n\}$ in $\mathcal{S}$ such that

$$\bigcup_{n=1}^{\infty} I_n \subset O.$$

Since $m(O) < \infty$, we conclude

$$\sum_{i=1}^{\infty} \ell(I_i) \leq m(O) < \infty.$$

For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} \ell(I_i) < \frac{\epsilon}{5}.$$

Define

$$R = E \setminus \bigcup_{n=1}^{N} I_n.$$

For every $x \in R$, since $\mathcal{S}$ is a Vitali cover, there exists an interval $J_x \in \mathcal{S}$ such that $x \in J_x$ and $\ell(J_x) < \frac{\epsilon}{5m(O)}$. The set R is covered by these intervals, forming an open cover. Extracting a finite subcover $\{J_1, J_2, \ldots, J_M\}$, we obtain

$$m^*(R) \leq \sum_{j=1}^{M} \ell(J_j) < \frac{\epsilon}{5}.$$

Now consider $x \in R$. Since $\bigcup_{i=1}^N I_i$ is closed, we can find a small interval

$I \in \mathcal{S}$ such that $x \in I$ and $I \cap \bigcup_{i=1}^{N} I_i = \emptyset$. If $I \cap I_i = \emptyset$ for all $i \leq n$, then $\ell(I) \leq k_n \leq 2\ell(I_{n+1})$. Since $\ell(I_{n+1}) \to 0$, I must intersect some I_{n_0} with $n_0 > N$. The smallest such n_0 satisfies

$$\ell(I) \leq k_{n_0-1} \leq 2\ell(I_{n_0}).$$

Since $x \in I$ and $I \cap I_{n_0} \neq \emptyset$, the distance from x to the midpoint of I_{n_0} is at most

$$\ell(I) + \frac{1}{2}\ell(I_{n_0}) \leq \frac{5}{2}\ell(I_{n_0}).$$

Thus, x belongs to an interval J_{n_0} centered at the midpoint of I_{n_0} with length $5\ell(I_{n_0})$. Hence,

$$R \subset \bigcup_{n_0=N+1}^{\infty} J_{n_0}.$$

This implies

$$m^*(R) \leq \sum_{n_0=N+1}^{\infty} \ell(J_{n_0}) = 5 \sum_{n_0=N+1}^{\infty} \ell(I_{n_0}) < \epsilon.$$

This completes the proof.

Consider $f : [a, b] \to \mathbb{R}$ and write

$$f'(x) = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

where$h \neq 0$.

Note that these limits exist if for all h_x tends to zero,

$$f'(x) = \lim_{h_x \to 0} \frac{f(x) - f(x - h_x)}{h_x} = \lim_{h_x \to 0} \frac{f(x + h_x) - f(x)}{h_x}.$$

In case f is not differentiable, the following four limits could be different:

$$D^+ f(x) = \limsup_{h_x \to 0} \frac{f(x + h_x) - f(x)}{h_x}$$

$$D^- f(x) = \liminf_{h_x \to 0} \frac{f(x) - f(x - h_x)}{h_x}.$$

$$D^+ f(x) = \limsup_{h_x \to 0} \frac{f(x + h_x) - f(x)}{h_x}$$

$$D^- f(x) = \liminf_{h_x \to 0} \frac{f(x) - f(x - h_x)}{h_x}.$$

Then

$$D^+ f(x) \geq D^+ f(x) \quad \textit{and} \quad D^- f(x) \geq D^- f(x).$$

If

$$D^+ f(x) = D^+ f(x) = D^- f(x) = D^- f(x) \neq \infty,$$

we say f is differentiable, and the common value is called the derivative of

f at x.

We denote it by $f'(x)$. If $D^+f(x) = D^-f(x)$, f is right differentiable at x and denote the right derivative by $f'(x+)$, similarly$f'(x-)$.

2.2 Monotone Function

If $f : [a,b] \to \mathbb{R}$ is monotone, then f is continuous a.e. x. Suppose f is monotone, then

$$f(x^-) = \lim_{y \to x^-} f(y) = \sup_{y<x} f(y) \leq f(x) \leq \inf_{z>x} f(x) \leq \lim_{z \to x^+} f(z) = f(x^+).$$

Thus, both $f(x^-)$ and $f(x^+)$ exist and

$$f(x^-) \leq f(x) \leq f(x^+).$$

2.2.1 Monotone Functions and Continuity

Let $f : [a,b] \to \mathbb{R}$ be a monotone increasing function. Then f is continuous almost everywhere (a.e.) on $[a,b]$.

If f is monotone increasing, then for any $x \in [a,b]$, the left and right limits exist:

$$f(x^-) = \lim_{y \to x^-} f(y) = \sup_{y<x} f(y), \quad f(x^+) = \lim_{z \to x^+} f(z) = \inf_{z>x} f(z).$$

Hence,

$$f(x^-) \leq f(x) \leq f(x^+).$$

If $f(x^-) < f(x) < f(x^+)$, then such points x are countable since the intervals $(f(x^-), f(x^+))$ corresponding to different such x's are disjoint, and each such interval contains a unique rational number. Therefore, the set of discontinuities is countable, and f is continuous a.e.

Theorem 2.2.2

Let $f : [a, b] \to \mathbb{R}$ be a monotone increasing function. Then:

- f' exists a.e. $x \in [a, b]$,
- f' is measurable, and
- $\int_a^b f'(x)\,dx \leq f(b) - f(a)$.

Proof:

To show that f is differentiable a.e., it suffices to show that the set

$$E = \{x : D^+ f(x) > D^- f(x)\}$$

has measure zero, where $D^+ f(x)$ and $D^- f(x)$ denote the right and left Dini derivatives, respectively.

Let

$$E_{u,v} = \{x : D^+ f(x) > u > v > D^- f(x)\}, \quad u, v \in \mathbb{Q}.$$

Then $E = \bigcup_{u,v \in \mathbb{Q}} E_{u,v}$, so it is enough to show that $m^*(E_{u,v}) = 0$ for each $u > v \in \mathbb{Q}$.

Let $s = m^*(E_{u,v})$, and fix $\epsilon > 0$. By outer regularity, there exists an open set $O \supset E_{u,v}$ such that $m(O) < s + \epsilon$. For each $x \in E_{u,v}$, there exists $h > 0$ such that:

$$\frac{f(x) - f(x-h)}{h} < v \Rightarrow f(x) - f(x-h) < hv.$$

Thus, the collection $\{[x-h, x] : x \in E_{u,v}, h > 0\}$ is a Vitali cover of $E_{u,v}$. By Vitali's Covering Lemma, there exists a finite disjoint subcollection $\{I_n = [x_n - h_n, x_n]\}_{n=1}^N$ such that:

$$\bigcup_{n=1}^{N}(x_n - h_n, x_n) \supset A = E_{u,v} \cap \bigcup_{n=1}^{N}(x_n - h_n, x_n),$$

and

$$m^*(A) > s - \epsilon.$$

Since f is monotone increasing,

$$\sum_{n=1}^{N}(f(x_n) - f(x_n - h_n)) < v\sum_{n=1}^{N} h_n = vm\left(\bigcup_{n=1}^{N} I_n\right) < v(s + \epsilon).$$

Similarly, for $y \in A \subset E_{u,v}$, there exists $k > 0$ such that:

$$\frac{f(y+k) - f(y)}{k} > u \Rightarrow f(y+k) - f(y) > uk.$$

Again, $\{[y, y+k]\}$ forms a Vitali cover of A, and a disjoint subcollection $\{J_i = [y_i, y_i + k_i]\}_{i=1}^M$ exists such that:

$$m^*(B) > s - 2\epsilon, \quad B = A \cap \bigcup_{i=1}^{M} J_i,$$

and

$$\sum_{i=1}^{M}(f(y_i+k_i)-f(y_i)) > u(s-2\epsilon).$$

However, since $J_i \subset I_n$ for some n, and the intervals are disjoint:

$$\sum_{i=1}^{M}(f(y_i+k_i)-f(y_i)) \leq \sum_{n=1}^{N}(f(x_n)-f(x_n-h_n)) < v(s+\epsilon).$$

So,

$$u(s-2\epsilon) < v(s+\epsilon),$$

and since $u > v$, this implies $s = 0$. Thus, $m^*(E_{u,v}) = 0$, and f is differentiable a.e.

Let

$$g(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

and define

$$g_n(x) = n(f(x+1/n)-f(x)), \quad f(x) = f(b) for x \geq b.$$

Then $g_n(x) \to g(x)$ a.e., so g is measurable. By Fatou's Lemma:

$$\int_a^b g(x)\,dx \leq \liminf_{n\to\infty}\int_a^b g_n(x)\,dx.$$

Now,

$$\int_a^b g_n(x)\,dx = n\int_a^b (f(x+1/n)-f(x))\,dx = n\left[\int_{a+1/n}^{b+1/n} f(x)\,dx - \int_a^b f(x)\,dx\right].$$

So,

$$\int_a^b g(x)\,dx \le \liminf_{n\to\infty} \left[\int_b^{b+1/n} f(x)\,dx - \int_a^{a+1/n} f(x)\,dx \right] \le f(b) - f(a).$$

Therefore, $g \in L^1([a,b])$ and $f'(x) = g(x)$ a.e. on $[a,b]$.

□

2.3 Functions of Bounded Variation

Let $f : [a,b] \to \mathbb{R}$, and consider a partition

$$Q = \{a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_k = b\}.$$

Define:

$$p = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+, \quad n = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^-$$

where, for any real-valued function g,

$$g^+(x) = \{ g\ (x) if g(x) \ge 0,$$

0 otherwise , $\mathrm{g}^-(x) = g^+(x) - g(x), \quad |g(x)| = g^+(x) + g^-(x).$

Let

$$t = n + p = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|\,.$$

Define

$$P = \sup p, \quad N = \sup n, \quad T = \sup t,$$

where the suprema are taken over all partitions Q of $[a, b]$.

If $T_a^b(f) < \infty$, then we say that f is of **bounded variation** on $[a, b]$, abbreviated as $BV[a, b]$.

Lemma 2.3.1

If $f \in BV[a, b]$, then

$$T_a^b(f) = P_a^b + N_a^b, \quad and \quad f(b) - f(a) = P_a^b - N_a^b.$$

Proof:

For any partition Q of $[a, b]$, we have:

$$p = n + f(b) - f(a) \Rightarrow P = N + f(b) - f(a).$$

Also,

$$t = p+n = p+(p-\{f(b)-f(a)\}) = 2p-\{f(b)-f(a)\} \Rightarrow T = 2P-\{f(b)-f(a)\} = P+N. \quad \square$$

Theorem 2.3.2

A function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if f can be written as the difference of two monotone real-valued functions.

Proof:

Suppose $f \in BV[a,b]$. Define:

$$g(x) = P_a^x, \quad h(x) = N_a^x$$

Then both g and h are increasing real-valued functions since

$$0 \leq P_a^x \leq T_a^x \leq T_a^b < \infty, \quad 0 \leq N_a^x \leq T_a^x \leq T_a^b < \infty.$$

By the lemma above,

$$f(x) = g(x) - h(x) + f(a).$$

Conversely, suppose $f = g - h$, where g and h are monotone increasing. Then, for any partition Q, we have:

$$\sum |f(x_i) - f(x_{i-1})| \leq \sum (g(x_i) - g(x_{i-1})) + \sum (h(x_i) - h(x_{i-1})).$$

Hence,

$$T_a^b(f) \leq g(b) - g(a) + h(b) - h(a) < \infty.$$

Corollary 2.3.3

If $f \in BV[a,b]$, then the derivative $f'(x)$ exists almost everywhere on $[a,b]$.

2.4 Differentiation of Integrals

Let f be an integrable function on $[a, b]$. Define

$$F(x) = \int_a^x f(y)\,dy.$$

Does this imply that $F'(x) = f(x)$ almost everywhere?

Lemma 2.4.1

If f is an integrable function on $[a, b]$, then the function

$$F(x) = \int_a^x f(y)\,dy$$

is continuous and of bounded variation.

Proof:

For $h \in \mathbb{R}$,

$$F(x+h) - F(x) = \int_x^{x+h} f(y)\,dy = \int_a^b \chi_{[x,x+h]}(y) f(y)\,dy,$$

where $\chi_{[x,x+h]}$ is the indicator function on $[x, x+h]$.

Since $f \in L^1([a, b])$ and $\chi_{[x,x+h]} \to 0$ pointwise as $h \to 0$, we have $\chi_{[x,x+h]} f \to 0$ almost everywhere. By the Dominated Convergence Theorem (DCT), it follows that

$$\lim_{h\to 0}(F(x+h) - F(x)) = 0,$$

i.e., F is continuous.

Now, for any partition $\{x_0, x_1, \ldots, x_k\}$ of $[a, b]$, we have

$$\sum_{i=1}^{k} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{k} \left| \int_{x_{i-1}}^{x_i} f(y)\, dy \right| \le \int_a^b |f(y)|\, dy.$$

Hence,

$$T_a^b(F) \le \int_a^b |f(y)|\, dy < \infty.$$

Therefore, $F \in BV[a, b]$. □

Lemma 2.4.2

Let $f \in L^1([a, b])$, and suppose

$$\int_a^x f(t)\, dt = 0, \quad forall x \in [a, b].$$

Then $f = 0$ almost everywhere on $[a, b]$.

Proof:

Define the set

$$E = \{x \in [a, b] : f(x) > 0\}.$$

Assume $m(E) > 0$. Then there exists a closed set $F \subset E$ with $m(F) > 0$. Let

$$O = (a, b) \setminus F.$$

Then O is open, and can be written as a countable disjoint union:

$$O = \bigcup_{n=1}^{\infty} I_n, \quad I_n = (a_n, b_n).$$

Now, since $\int_a^b f = 0$, we write:

$$0 = \int_a^b f = \int_F f + \int_O f.$$

But since $f > 0$ on F, we have $\int_F f > 0$, implying $\int_O f < 0$, which is a contradiction. So $\int_{I_n} f \neq 0$ for some n. Then

$$\int_a^{a_n} f - \int_a^{b_n} f \neq 0,$$

implying one of them is non-zero, again contradicting the assumption that $\int_a^x f = 0$ for all $x \in [a, b]$.

Hence, $m(E) = 0$. Similarly, the set $\{x : f(x) < 0\}$ also has measure zero. Thus, $f = 0$ almost everywhere. □

Lemma 2.4.3

Let f be a bounded measurable function on $[a, b]$. Define

$$F(x) = \int_a^x f(y)\, dy + F(a).$$

Then $F'(x) = f(x)$ almost everywhere on $[a, b]$.

Proof:

By Lemma 2.4.1, F is of bounded variation, which implies that $F'(x)$ exists almost everywhere.

Assume $|f(x)| \leq M$ for all $x \in [a, b]$, so $|F'(x)| \leq M$ almost everywhere.

Define

$$f_n(x) = n\left(F(x + 1n) - F(x)\right).$$

Then

$$f_n(x) = n \int_x^{x+\frac{1}{n}} f(t)\,dt.$$

Since $|f_n(x)| \le M$, and $f_n(x) \to F'(x)$ pointwise almost everywhere, we may apply the Bounded Convergence Theorem (BCT). Therefore,

$$\int_a^c F'(x)\,dx = \lim_{n\to\infty} \int_a^c f_n(x)\,dx = \lim_{n\to\infty} n\left[\int_a^c (F(x + 1n) - F(x))\,dx\right].$$

Changing variables:

$$\int_a^c (F(x + 1n) - F(x))\,dx = \int_c^{c+\frac{1}{n}} F(x)\,dx - \int_a^{a+\frac{1}{n}} F(x)\,dx.$$

Since F is continuous, we have:

$$\int_a^c F'(x)\,dx = F(c) - F(a) = \int_a^c f(x)\,dx.$$

Thus,

$$\int_a^c (F'(x) - f(x))\,dx = 0 \quad forallc \in [a, b].$$

By Lemma 2.4.2, it follows that $F'(x) = f(x)$ almost everywhere on $[a, b]$.

□

Theorem 2.4.4

Let f be an integrable function on $[a, b]$, and define

$$F(x) = F(a) + \int_a^x f(t)\,dt.$$

Then $F'(x) = f(x)$ almost everywhere on $[a, b]$.

Proof:

Without loss of generality, assume $f \geq 0$.

For each $n \in \mathbb{N}$, define

$$\mathrm{f}_n(x) = \Big[\begin{array}{lll} f(x), & if & f(x) \leq n, \\ n, & if & f(x) > n. \end{array}$$

Then $f - f_n \geq 0$, and define

$$G_n(x) = \int_a^x (f(t) - f_n(t))\,dt.$$

By Lemma 2.4.3, G_n is differentiable almost everywhere and $G_n'(x) = f(x) - f_n(x) \geq 0$.

We write:

$$F(x) = F(a) + \int_a^x (f(t) - f_n(t))\,dt + \int_a^x f_n(t)\,dt = G_n(x) + \int_a^x f_n(t)\,dt.$$

Differentiating both sides, we get:

$$F'(x) = G_n'(x) + \frac{d}{dx}\left(\int_a^x f_n(t)\,dt\right) \geq f_n(x) \quad a.e.$$

Since $f_n(x) \to f(x)$ as $n \to \infty$, we obtain:

$$F'(x) \geq f(x) \quad a.e.$$

Now integrate both sides over $[a, b]$:

$$\int_a^b F'(x)\,dx \geq \int_a^b f(x)\,dx = F(b) - F(a). \tag{1}$$

Since $f \geq 0$, F is monotone increasing, and hence:

$$\int_a^b F'(x)\,dx \leq F(b) - F(a). \tag{2}$$

Combining (1) and (2), we get:

$$\int_a^b F'(x)\,dx = \int_a^b f(x)\,dx,$$

which implies

$$\int_a^b \left(F'(x) - f(x)\right)\,dx = 0.$$

Since $F'(x) - f(x) \geq 0$, it must be that $F'(x) = f(x)$ almost everywhere on $[a, b]$. □

2.5 Absolute Continuity

A function $f : [a, b] \to \mathbb{R}$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \varepsilon,$$

whenever $\{(x_i, x_i')\}_{i=1}^{n}$ is a finite collection of disjoint intervals in $[a, b]$ satisfying

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta.$$

Lemma 2.5.1

Every absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Proof:

Given that f is absolutely continuous. Take $\varepsilon = 1$, then there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < 1,$$

whenever $\{(x_i, y_i)\}_{i=1}^{n}$ is a disjoint family of intervals with

$$\sum_{i=1}^{n} |y_i - x_i| < \delta.$$

Let N be the smallest positive integer such that

$$N > \frac{b-a}{\delta}.$$

Define points $a_j = a + \frac{j(b-a)}{N}$ for $j = 0, 1, \ldots, N$, so that each subinterval $[a_{j-1}, a_j]$ has length $< \delta$.
Then for each subinterval $[a_{j-1}, a_j]$, any partition Q of it satisfies $\|Q\| < \delta$, so

$$T_{a_{j-1}}^{a_j}(f) < 1.$$

Therefore,

$$T_a^b(f) \leq \sum_{j=1}^{N} T_{a_{j-1}}^{a_j}(f) < N < \infty.$$

Hence, $f \in BV[a,b]$. □

Corollary 2.5.2

If f is absolutely continuous on $[a,b]$, then f is differentiable almost everywhere on $[a,b]$.

Lemma 2.5.3

If f is absolutely continuous on $[a,b]$ and $f'(x) = 0$ almost everywhere, then f is constant.

Proof:

We prove that $f(a) = f(c)$ for any $c \in [a,b]$. Let $E = \{x \in (a,c) : f'(x) = 0\}$. Since $f' = 0$ a.e., we have $m(E) = c - a$.

Let $\varepsilon > 0$ and $\eta > 0$ be arbitrary. For each $x \in E$, there exists a small interval $[x, x+h]$ such that

$$|f(x+h) - f(x)| < \eta h.$$

By the Vitali Covering Lemma, there exists a disjoint finite subcollection $\{[x_k, y_k]\}_{k=1}^{n}$ that covers all of E except for a set A of measure less than some $\delta > 0$, where δ corresponds to ε in the definition of absolute continuity.

Assume $x_k < x_{k+1}$, and define:

$$y_0 = a \leq x_1 < y_1 < x_2 < \cdots < y_n \leq c = x_{n+1},$$

so that

$$\sum_{k=0}^{n} |x_{k+1} - y_k| < \delta.$$

Now,

$$\sum_{k=1}^{n} |f(y_k) - f(x_k)| \leq \eta \sum_{k=1}^{n} (y_k - x_k) < \eta(c-a).$$

Also, by absolute continuity of f,

$$\sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| < \varepsilon.$$

Hence,

$$|f(c) - f(a)| = \left| \sum_{k=0}^{n} (f(x_{k+1}) - f(y_k)) + \sum_{k=1}^{n} (f(y_k) - f(x_k)) \right| < \varepsilon + \eta(c-a).$$

Since ε and η are arbitrary, it follows that $f(c) = f(a)$. $\square$

Theorem 2.5.4

A function F is an indefinite integral if and only if it is absolutely continuous.

Proof:

($\Rightarrow$) Suppose $F(x) = \int_a^x f(t)\,dt$. Then F is absolutely continuous.
($\Leftarrow$) Suppose F is absolutely continuous. Then $F \in BV[a,b]$, so $F(x) = F_1(x) - F_2(x)$, where F_1, F_2 are increasing functions. Therefore, $F'(x)$ exists almost everywhere and

$$|F'(x)| \leq F_1'(x) + F_2'(x).$$

This implies

$$\int_a^b |F'(x)|\,dx \leq F_1(b) + F_2(b) - F_1(a) - F_2(a) < \infty,$$

so $F' \in L^1[a,b]$.
Define $G(x) = \int_a^x F'(t)\,dt$. Then G is absolutely continuous, and so is $f = F - G$. Thus,

$$f'(x) = F'(x) - G'(x) = 0 \quad a.e.$$

By Lemma 2.5.3, f is constant. Hence,

$$F(x) = \int_a^x F'(t)\,dt + F(a).$$

□

Corollary 2.5.5

Every absolutely continuous function is the indefinite integral of its derivative.

Remark

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

2.6 Differentiation of Integrals in $\mathbb{R}^n$

Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Define

$$F(x) = \int_a^x f(y)\,dy \quad for \quad x \in [a, b].$$

Then

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(y)\,dy.$$

Let $I = (x, x+h)$ and $|I|$ be the length of the interval I. Letting $|I| \to 0$, the question is whether

$$\lim_{|I|\to 0}\left(\frac{1}{|I|}\int_I f(y)\,dy\right) = f(x) \quad a.e. \quad x.$$

We can reformulate this problem by considering intervals containing x.

That is, does

$$\lim_{|I|\to 0, x\in I} \left(\frac{1}{|I|} \int_I f(y)\, dy \right) = f(x) \quad a.e.?$$

In an analogous way, a similar question can be posed in higher dimensions. For instance, in $\mathbb{R}^n$ $(n \geq 1)$: Let

$$r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\},$$

where $\|x-y\|$ is the Euclidean distance. Then the Lebesgue measure of the ball $B_r(x)$ is given by

$$m(B_r(x)) = r^n \cdot m(B_1(0)),$$

since the Lebesgue outer measure m^* is translation and dilation invariant. If we denote

$$v_h = m(B_1(0)),$$

then

$$m(B_r(x)) = v_h r^n.$$

Suppose f is integrable on $\mathbb{R}^n$, and B denotes a ball containing x. The question is whether

$$\lim_{m(B)\to 0, x\in B} \left(\frac{1}{m(B)} \int_B f(y)\, dy \right) = f(x).$$

As an example, if f is continuous at $x \in \mathbb{R}^n$, then the above limit converges to $f(x)$. We have:

$$\left|\frac{1}{m(B)}\int_B |f(y)-f(x)|\,dy\right| \leq \frac{1}{m(B)}\int_B |f(y)-f(x)|\,dy.$$

For $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x-y\| < \delta|f(x)-f(y)| < \varepsilon.$$

If B is a ball of radius less than $\frac{\delta}{2}$ and containing x, then

$$\left|\frac{1}{m(B)}\int_B |f(y)-f(x)|\,dy\right| < \varepsilon,$$

which is the desired result.

From the above, we can make the observation that the limit is the result of taking the supremum of a sequence of shrinking balls. This leads to a way to define the maximum function for $|f|$.

Chapter 3

3.1 Hardy-Littlewood Maximum Function

For $f \in L^1(\mathbb{R}^n)$, define the Hardy-Littlewood maximal function $f^*(x)$ by

$$f^*(x) = \sup_{x \in B} \left(\frac{1}{m(B)} \int_B |f(y)|\, dy \right),$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. The function f^* is known as the Hardy-Littlewood maximal function of $|f|$.

Theorem 3.1.1

Let $f \in L^1(\mathbb{R}^n)$. Then the following hold:

1. f^* is measurable,
2. $f^*(x) < \infty$ for almost every $x \in \mathbb{R}^n$,
3. $m\left(\{x \in \mathbb{R}^n : f^*(x) > \alpha\}\right) \leq \frac{A}{\alpha}\|f\|_1$, where $A = 3^n$.

Before proving this result, the focus will be on conclusion (iii). It can be shown later that $f^*(x) \geq |f(x)|$ for almost every x. However, conclusion (iii) suggests that f^* is not much larger than $|f|$. This observation leads to the expectation that if f is integrable, then f^* should be integrable as

well. But this is not necessarily the case, as f^* of a non-zero function $f \in L^1(\mathbb{R}^n)$ may decay too slowly at infinity.
For this, let $\alpha > 0$ and $r = |x| > \alpha$. Then $B_r(0) \subset B_{2r}(x)$, and we have

$$f^*(x) \geq \frac{1}{m(B_{2r}(x))} \int_{B_{2r}(x)} |f(y)|\, dy = \frac{C}{|x|^n} \int_{B_{2r}(x)} |f(y)|\, dy \geq \frac{C}{|x|^n} \int_{B_\alpha(0)} |f(y)|\, dy,$$

where C is some constant.
However, $\frac{1}{|x|^n}$ is not integrable on $\mathbb{R}^n \setminus B_\alpha(0)$. Hence, if $f^* \in L^1(\mathbb{R}^n)$, then

$$\int_{B_\alpha(0)} |f(y)|\, dy = 0 \quad forall \quad \alpha > 0.$$

This implies that $|f| = 0$ almost everywhere.

Exercise

Let $f(x) = \frac{1}{x(\log x)^2}\chi_{(0,\frac{1}{2})}(x)$, where $\chi_{(0,\frac{1}{2})}(x)$ is the characteristic function of the interval $(0, \frac{1}{2})$. Then $f \in L^1(\mathbb{R})$ but $f^* \notin L^1_{loc}(\mathbb{R})$.
For $0 < x < \frac{1}{2}$, we have:

$$f^*(x) \geq \frac{1}{2x} \int_0^{2x} |f(y)|\, dy \geq \frac{1}{2x} \int_0^x \frac{1}{y(\log y)^2}\, dy.$$

We then estimate this as follows:

$$f^*(x) \geq \frac{1}{2x} \int_0^x \frac{1}{y(\log y)^2}\, dy > \frac{1}{2x|\log x|}.$$

The term $\frac{1}{2x|\log x|}$ is not integrable near $x = 0$, which implies that $f^* \notin L^1_{loc}(\mathbb{R})$.

Considering $F(x) = f(|x|)$, it is possible to construct $F \in L^1(\mathbb{R}^n)$ with the above properties via polar decomposition.

The inequality in (iii) is called a *weak inequality* because it is weaker than the corresponding inequality in L^1-norm (due to the Chebyshev inequality):

$$m\{x : |f(x)| > \alpha\} \leq \frac{1}{\alpha} \int_{\mathbb{R}^n} |f(x)|\, dx = \frac{1}{\alpha} \|f\|_1.$$

Proof (of theorem 3.1.1) :

(i) f^* is a measurable function

To show this, it is enough to show that the set $E_\alpha = \{x \in \mathbb{R}^n : f^*(x) > \alpha\}$ is open. For any $x \in E_\alpha$, there exists an open ball B such that $x \in B$ and

$$\frac{1}{m(B)} \int_B |f(y)|\, dy > \alpha.$$

If x' is any point close enough to x, then $x' \in B$, and we have:

$$\sup_{x' \in B'} \frac{1}{m(B')} \int_{B'} |f(y)|\, dy \geq \frac{1}{m(B)} \int_B |f(y)|\, dy > \alpha,$$

where B' is a ball containing x'. Since B is part of the family over which the supremum is taken, it means that $x' \in E_\alpha$. Hence, there exists a small ball around x that is contained within E_α. This shows that E_α is open.

(ii) Proof of $\{x : f^*(x) = \infty\} \subset \{x : f^*(x) > \alpha\}$

If we assume (iii) for the time being, then

$$\{x : f^*(x) = \infty\} \subset \{x : f^*(x) > \alpha\}, \quad \forall \quad \alpha > 0.$$

Hence, by the result in (iii),

$$m\{x : f^*(x) = \infty\} \leq \frac{1}{\alpha}\|f\|_1 \to 0 \quad as \quad \alpha \to \infty.$$

Finally, the proof of (iii) will be followed by the following lemma.

Covering Lemma

Let $B = \{B_1, B_2, \ldots, B_N\}$ be a finite collection of balls in $\mathbb{R}^n$. Then there exists a disjoint subcollection $\{B_{i_1}, B_{i_2}, \ldots, B_{i_k}\}$ of B such that

$$m\left(\bigcup_{l=1}^{N} B_l\right) \leq 3^n \sum_{j=1}^{k} m(B_{i_j}).$$

Proof of Covering Lemma:

If all balls in B are disjoint, then the result holds trivially. If not, let B and B' be two balls in B that intersect, with $radius(B) \geq radius(B')$. Then, $B' \subset 3B = \tilde{B}$, where $\tilde{B}$ is the ball with the same center as B and radius 3 times that of B.

First, pick a ball B_{i_1} in B with the largest radius. Then, delete B_{i_1} from B and any other ball that intersects B_{i_1}. All the deleted balls are contained in $\tilde{B}_{i_1}$. The remaining balls yield a new collection, say B', for which the procedure is repeated. Pick a ball B_{i_2} in B' with the largest radius, and delete B_{i_2} and any other ball intersecting B_{i_2}.

Continuing this way, after at most N steps, we obtain a collection of disjoint balls $B_1, B_2, \ldots, B_k$. Let $\tilde{B}_j = 3B_j$. Since any ball $B \in B$ must intersect some of the B_j's, and hence B has equal or smaller radius than B_j, we must have $B \subset \tilde{B}_j$. That is,

$$\bigcup_{l=1}^{N} B_l \subset \bigcup_{j=1}^{k} \tilde{B}_j.$$

Therefore,

$$m\left(\bigcup_{l=1}^{N} B_l\right) \leq m\left(\bigcup_{j=1}^{k} \tilde{B}_j\right) \leq 3^n \sum_{j=1}^{k} m(B_j).$$

$$= 3^n m\left(\bigcup_{j=1}^{k} B_j\right).$$

Proof of (iii):

Let $x \in E_\alpha$. Then there exists a ball B_x containing x such that

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)|\, dy > \alpha.$$

This implies

$$m(B_x) < \frac{1}{\alpha} \int_{B_x} |f(y)|\, dy.$$

Since m is inner regular, we have

$$m(E_\alpha) = \sup_{K \subset E_\alpha} m(K).$$

Let $K \subset E_\alpha$. Then $K \subset \bigcup_{x \in E_\alpha} B_x$. Hence, by the covering lemma, there exist disjoint balls $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ such that

$$m(K) \leq m\left(\bigcup_{l=1}^{N} B_l\right) \leq 3^n \sum_{j=1}^{k} m(B_{i_j}).$$

Now, using the inequality for each ball B_{i_j}, we obtain

$$m(K) \leq \frac{3^n}{\alpha} \sum_{j=1}^{k} \int_{B_{i_j}} |f(y)|\, dy = \frac{3^n}{\alpha} \int_{\bigcup B_{i_j}} |f(y)|\, dy.$$

Thus, we have

$$m(K) \leq \frac{3^n}{\alpha} \|f\|_1.$$

3.2 Lebesgue Differentiation Theorem

Theorem 3.2.1 : (Lebesgue Differentiation)

If $f \in L^1(\mathbb{R}^n)$, then

$$\lim_{m(B) \to 0, x \in B} \frac{1}{m(B)} \int_B f(y)\, dy = f(x) \quad a.e. \quad x.$$

Proof:

It is enough to show that for each $\alpha > 0$, the set

$$N_\alpha = \left\{ x \mid \liminf_{m(B) \to 0, x \in B} \left| \frac{1}{m(B)} \int_B f(y)\, dy - f(x) \right| > \alpha \right\}$$

has measure zero. Since $f \in L^1(\mathbb{R}^n)$, for each $\epsilon > 0$, there exists $g \in C_c(\mathbb{R}^n)$ such that

$$\|f - g\|_1 < \epsilon.$$

Since g is continuous, for each $x \in \mathbb{R}^n$, we have

$$\lim_{m(B)\to 0, x\in B} \frac{1}{m(B)} \int_B g(y)\, dy = g(x).$$

Now, we can decompose the difference:

$$\frac{1}{m(B)} \int_B f(y)\, dy - f(x) = \frac{1}{m(B)} \int_B (f(y) - g(y))\, dy + \frac{1}{m(B)} \int_B g(y)\, dy - g(x) + g(x) - f(x).$$

Hence,

$$\liminf_{m(B)\to 0, x\in B} \left| \frac{1}{m(B)} \int_B f(y)\, dy - f(x) \right| \leq (f-g)^*(x) + |f(x) - g(x)|. \quad (*)$$

Let $G_\alpha = \{x : |f(x) - g(x)| > \alpha\}$ and $F_\alpha = \{x : (f-g)^*(x) > \alpha\}$. Then,

$$N_\alpha \subset F_\alpha \cup G_\alpha \quad [from(*)].$$

By Chebyshev's inequality, we have

$$m(G_\alpha) \leq \frac{1}{\alpha} \|f - g\|_1.$$

And by the weak inequality, we have

$$m(F_\alpha) \leq \frac{A}{\alpha} \|f - g\|_1.$$

Thus,

$$m(E_\alpha) \leq \frac{A}{\alpha}\epsilon + \frac{1}{\alpha}\epsilon \quad \forall \epsilon > 0.$$

This implies that

$$m(E_\alpha) = 0.$$

Applying the above result to $|f|$, we get

$$f^*(x) = \sup_{x\in B} \frac{1}{m(B)} \int_B |f(y)|\, dy \geq \lim_{x\in B, m(B)\to 0} \frac{1}{m(B)} \int |f(y)|\, dy = |f(x)|.$$

Hence, $f^*(x) \geq |f(x)|$ for almost every x.

Since differentiation is a local notion, and the behavior of the function is considered on balls which shrink to a point x, it is enough for the function to be locally integrable.

$f \in L^1_{loc}(\mathbb{R}^n)$ if f is integrable over each compact subset of $\mathbb{R}^n$. That is, $\int f\chi_K \in L^1(\mathbb{R}^n)$ for any compact set K $\subset \mathbb{R}^n$.

Corollary 3.2.2 : If $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\lim_{m(B)\to 0, x\in B} \frac{1}{m(B)} \int_B f(y)\, dy = f(x) \quad a.e. \quad x.$$

Let E be a measurable set. By the corollary, for $\chi_E \in L^1_{loc}(\mathbb{R}^n)$,

$$\lim_{m(B)\to 0, x\in B} \frac{m(B\cap E)}{m(B)} = 1 \quad a.e. \quad x.$$

This means that small balls around x are almost entirely covered by E.

Moreover, if $0 < \epsilon < 1$, then there exists a ball B containing x such that

$$m(B \cap E) > (1 - \epsilon)m(B).$$

This implies that E covers at least $(1 - \epsilon)$ part of B.

Bibliography

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