REPRESENTATIONS OF FINITE AND COMPACT GROUPS

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CERTIFICATE

This is to certify that the work contained in this report entitled **"REPRESENTATIONS OF FINITE AND COMPACT GROUPS"** submitted by **PRAMOD SINGH** (**Roll No: 172123026**) to Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him/her under my supervision.

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ABSTRACT

The aim of this project is to study the representation of compact groups and understand Peter-Weyl theorem and irreducible unitary representations of the special unitary group $SU(2)$.

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Chapter 1

Representation of Finite Group

In this chapter, we will discuss the representation theory of finite groups and their characters. In the first, section we will set up a few basic definitions, notations and some elementary results related to the representation of finite group. In the second section, we will setup definitions and some results related to character theory of finite group.

1.1 Representation of the finite groups

Let $\mathbb{V}$ be a vector space over the field $\mathbb{C}$ of complex numbers and let GL($\mathbb{V}$) be the group of all isomorphisms of $\mathbb{V}$ onto itself. Suppose $\mathbb{G}$ is a finite group with identity element e. Throughout this chapter by group, we mean a finite group.

Definition 1.1.1. A *representation* of a group $\mathbb{G}$ in $\mathbb{V}$ is a homomorphism $\pi : \mathbb{G} \to GL(\mathbb{V})$ i.e $\pi(gh) = \pi(g)\pi(h)\ \forall g, h \in \mathbb{G}$.

When π is given, we say that $\mathbb{V}$ is representation space of $\mathbb{G}$ and we denote group representation by $(\pi, \mathbb{V})$. If $\mathbb{G}$ is group and $\mathbb{V}$ is finite dimensional space, then the degree of representation is the dimension of $\mathbb{V}$.

Following are some properties.

(a) $\pi(e) = \mathbb{I}$, where is $\mathbb{I}$ is the identity map from $\mathbb{V}$ onto $\mathbb{V}$.

(b) $[\pi(g^{-1})] = [\pi(g)]^{-1}$, $\forall g \in \mathbb{G}$.

Definition 1.1.2. Suppose π and π' are two representation of the same group $\mathbb{G}$ in vector spaces $\mathbb{V}$ and $\mathbb{V}'$. Then we say that π and π' are similar (or isomorphic) if there exists an isomorphism $\mathbb{T} : \mathbb{V} \longmapsto \mathbb{V}'$ such that
$\mathbb{T} \circ \pi(g) = \pi'(g) \circ \mathbb{T}$, $\forall g \in \mathbb{G}$.

Definition 1.1.3. Suppose $\mathbb{W}$ is a subspace of vector space $\mathbb{V}$. Then we say that $\mathbb{W}$ is stable (or invariant) under π, if $\pi(\mathbb{G})\mathbb{W} \subseteq \mathbb{W}$. Suppose $(\pi_1, \mathbb{V}_1)$ and $(\pi_2, \mathbb{V}_2)$ are two representations of $\mathbb{G}$. Then we can make direct sum representation of $\mathbb{G}$ into $\mathbb{V}_1 \bigoplus \mathbb{V}_2$ by defining,
$(\pi_1 \bigoplus \pi_2)(g)(\mathbb{V}_1 \bigoplus \mathbb{V}_2) \;=\; (\pi_1(g)(\mathbb{V}_1), \pi_2(g)(\mathbb{V}_2))$ i.e.

$$(\pi_1 \bigoplus \pi_2)(g) \;=\; \begin{bmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{bmatrix}.$$

Definition 1.1.4. The representation $(\pi, \mathbb{V})$ is said to be *irreducible* if there is no non trivial π-invariant subspace. Suppose $\mathbb{W}$ is an invariant subspace of π. Then the restriction $\pi|_{\mathbb{W}}$ of π to $\mathbb{W}$ is an isomorphism of $\mathbb{W}$ onto itself. Thus $\pi|_{\mathbb{W}}$ is a representation of $\mathbb{G}$ in $\mathbb{W}$ and is called a *subrepresentation* of π. We denote it by $(\pi|_{\mathbb{W}}, \mathbb{W})$, or simply by $(\pi, \mathbb{W})$.

Theorem 1.1.5. *Let $\pi : \mathbb{G} \to GL(\mathbb{V})$ be a representation of $\mathbb{G}$ in $\mathbb{V}$ and let $\mathbb{W}$ be a π-invariant subspace of $\mathbb{V}$. Then there exists a complement $\mathbb{W}_0$ of $\mathbb{W}$ in $\mathbb{V}$ which is invariant under π.*

Proof. Suppose $\mathbb{W}'$ be an arbitrary complement of $\mathbb{W}$. Let $P : \mathbb{V} \longrightarrow \mathbb{W}$ be the projection map. Define, $P' = \frac{1}{n}\sum_{g\in\mathbb{G}} \pi(g)^{-1}P\pi(g)$, where n is order of $\mathbb{G}$. Since $P\pi(g)(\mathbb{V}) \subseteq \mathbb{W}$ and $\mathbb{W}$ is π-invariant, we get $P'(\mathbb{V}) \subseteq \mathbb{W}$ and $P'(w) = w,\ \forall w \in \mathbb{W}$. Thus P' is a projection map of $\mathbb{V}$ onto $\mathbb{W}$ corresponding to some complement $\mathbb{W}_0$ of $\mathbb{W}$.
Claim: $\pi(h)P' = P'\pi(h), \forall h \in \mathbb{G}$

$$\begin{aligned}
\pi(h)^{-1}P'\pi(h) &= \frac{1}{n}\sum_{g\in\mathbb{G}} \pi(h)^{-1}\pi(g)^{-1}P\pi(g)\pi(h)\\
&= \frac{1}{n}\sum_{g\in\mathbb{G}} \pi(gh)^{-1}P\pi(gh)\\
&= P'.
\end{aligned}$$

If $s \in \mathbb{W}_0$, we have $P'(s) = 0$, which gives $P'(\pi(g)(s)) = \pi(g)P'(s)) = 0$. Hence $\pi(g)(s) \in \mathbb{W}_0$. Therefore, $\mathbb{W}_0$ is stable under π. ☐

Remark 1.1.6. Suppose that in addition to above assumption $\mathbb{V}$ endowed with an inner product satisfying $\langle\pi(g)v, \pi(g)w\rangle = \langle v, w\rangle, \forall\ g \in \mathbb{G}$ and $\forall v, w \in \mathbb{V}$. Then the orthogonal complement $\mathbb{W}_0$ of $\mathbb{W}$ in $\mathbb{V}$ is π-invariant.

Example 1.1.7. Consider cyclic group $G = \{1, \omega, \omega^2\}$. Define $\pi : G \to GL(\mathbb{C})$ defined by $\pi(\omega) = z$, where $z \in \mathbb{C}^*$. For π to be a group homomorphism, $|\pi(1)| = 1$. That is $|z|^3 = 1$, this implies that ω can be mapped to any of the cube root of unity to give a representation of degree one.

Theorem 1.1.8. *Every finite degree representation of group $\mathbb{G}$ is a direct sum of irreducible representations.*

Proof. We will prove this theorem using the induction hypothesis on the degree of representation. Let $(\pi, \mathbb{V})$ be a finite degree representation of $\mathbb{G}$. If the degree of representation is one, then π is irreducible. Suppose degree of representation of π is n. Suppose the theorem hold for all representation of degree less than n. If π is irreducible, we are done. Otherwise by *Theorem 1.1.5*, $\mathbb{V}$ can be decompose into direct sums of $\mathbb{V}_1 \bigoplus \mathbb{V}_2$ with $\dim \mathbb{V}_i < \dim \mathbb{V}$, $i = 1, 2$. By the induction hypothesis, $\mathbb{V}_1$ and $\mathbb{V}_2$ are direct sum of irreducible representations, hence $\mathbb{V}$ is the direct sum of irreducible representations. $\square$

1.2 Character theory

Suppose $\mathbb{V}$ is a finite dimensional vector space having basis $(e_i)_{i=1}^n$. Let T be a linear map from $\mathbb{V}$ into itself with matrix (a_{ij}), then we denote trace of T by $Tr(T)$.

Definition 1.2.1. Let $(\pi, \mathbb{V})$ be a representation of a finite group $\mathbb{G}$. Then for each $g \in \mathbb{G}$, define $\chi_\pi(g) := \mathrm{Tr}(\pi(g))$. Then the complex-valued function χ_π on $\mathbb{G}$ is called the character of representation.

Proposition 1.2.2. *Suppose χ is the character of representation $(\pi, \mathbb{V})$ of degree m. Then,*

(a) $\chi(e) = m$*, where e is the identity element of the group G.*

(b) $\chi(g^{-1}) = \chi(g)^*$, $\forall\ g \in \mathbb{G}$.

(c) $\chi(ghg^{-1}) = \chi(h)$, $\forall\, g, h \in \mathbb{G}$.

Proof. Since $\chi_\pi(e) = Tr(\pi(e)) = Tr(\mathbb{I}) = m$, which proves (a).
Suppose order of $\mathbb{G}$ is n and let $\lambda_1,\ \lambda_2, \ldots, \lambda_m$ are eigenvalues of $\pi(g)$. Then $g^n =$ e, $\forall\, g \in \mathbb{G}$, which gives $\pi(g^n) = \mathbb{I}$, which in turn implies $(\pi(g))^n = \mathbb{I}$. Hence $\lambda^n = 1$ for each of its eigenvalue λ. Therefore,
$\chi(g^{-1}) = \sum_i \lambda_i^{-1} = \sum_i \overline{\lambda_i} = \overline{Tr(\pi(g))}$, which proves (b).
$\chi(ghg^{-1}) = Tr(\pi(ghg^{-1})) = Tr(\pi(g)\pi(h)\pi(g)^{-1}) = Tr(\pi(h)) = \chi(h)$. ☐

Remark 1.2.3. A function f on $\mathbb{G}$ satisfying identity (c) i.e, $f(ghg^{-1}) = f(h)$, $\forall\, g, h \in \mathbb{G}$ is called *class function* on $\mathbb{G}$.

Proposition 1.2.4. *Suppose $(\pi, \mathbb{V})$ and $(\pi', \mathbb{V}')$ are two representations of $\mathbb{G}$, and let χ and χ' be their characters respectively. Then the character χ_0 of direct sum representation $(\pi \bigoplus \pi', \mathbb{V} \bigoplus \mathbb{V}')$ is $\chi + \chi'$.*

Proof. Suppose $g \in \mathbb{G}$ and $\pi(g)$ and $\pi'(g)$ are in matrix forms. Then the representation of $\pi \bigoplus \pi'(g)$ in matrix form is given by, $\begin{bmatrix} \pi(g) & 0 \\ 0 & \pi'(g) \end{bmatrix}$.
Therefore $\chi_0(g) = Tr(\pi \bigoplus \pi'(g)) = Tr(\pi(g)) + Tr(\pi'(g))$ and hence $\chi_0(g) = \chi(g) + \chi'(g)$. ☐

Theorem 1.2.5. (**Schur's lemma**) *Suppose $(\pi_1, \mathbb{V}_1)$ and $(\pi_2, \mathbb{V}_2)$ are two irreducible representations of $\mathbb{G}$ and let T be a linear transformation from $\mathbb{V}_1$ to $\mathbb{V}_2$ such that, $\pi_2(g)T(v) = T(\pi_1(g)v)$ $\forall\, g \in \mathbb{G}$ and $\forall\, v \in \mathbb{V}_1$, Then*

(a) If π_1 and π_2 are not isomorphic, then T= 0.

(b) If $\mathbb{V}_1 = \mathbb{V}_2$ and $\pi_1 = \pi_2$, Then $T = \lambda\mathbb{I}$ for some $\lambda \in \mathbb{C}$.

Proof. Since $\pi_2(g)T(v) = T(\pi_1(g)v)$, $\forall\ g \in \mathbb{G}$ and $\forall\ v \in \mathbb{V}$, implies that ker(T) and Im(T) are invariant subspaces of $\mathbb{V}_1$ and $\mathbb{V}_2$. Since $(\pi_1, \mathbb{V}_1)$ is irreducible representation, therefore either ker(T) $= \mathbb{V}_1$ or ker(T) $= \{0\}$. If ker(T) $= \mathbb{V}_1$ then T $= 0$. If ker(T) $= \{0\}$ then T is one-one and maps $\mathbb{V}_1$ isomorphically to Im(T). Since $(\pi_2, \mathbb{V}_2)$ is irreducible representation, therefore, Im(T) $= \mathbb{V}_2$, it follows that T is an isomorphism, which proves (a).

Now, suppose $\mathbb{V}_1 = \mathbb{V}_2$ and $\pi_1 = \pi_2$, and suppose λ be an eigenvalue of T. Define $T' :=$ T - $\lambda\mathbb{I}$. Since λ is an eigenvalue of T, therefore ker$\{T'\} \neq \{0\}$. On the other hand $\pi_2 \circ T' = T' \circ \pi_1$. Therefore by part (a) $T' = 0$, which in turn implies that T $= \lambda\mathbb{I}$, that proves (b). □

Corollary 1.2.6. *Suppose $(\pi_1, \mathbb{V}_1)$ and $(\pi_2, \mathbb{V}_2)$ are two irreducible representations of group $\mathbb{G}$ of order n and let T: $\mathbb{V}_1 \longrightarrow \mathbb{V}_2$ be a linear transformation. Define,*

$$T' = \frac{1}{n} \sum_{g \in \mathbb{G}} \pi_2(g)^{-1} T \pi_1(g).$$

Then,

(a) If π_1 and π_2 are not isomorphic, then $T' = 0$.

(b) If $\mathbb{V}_1 = \mathbb{V}_2$ and $\pi_1 = \pi_2$, then $T = \frac{1}{m} Tr(T)\mathbb{I}$, where m is dimension of $\mathbb{V}_1$.

Proof. Let h $\in \mathbb{G}$, then

$$\begin{aligned}
\pi_2(h)T'\pi_1(h) &= \frac{1}{n}\sum_{g\in\mathbb{G}} \pi_2(h)^{-1}\pi_2(g)^{-1}T\pi_1(g)\pi_1(h) \\
&= \frac{1}{n}\sum_{g\in\mathbb{G}} \pi_2(gh)^{-1}T\pi(gh) \\
&= T'.
\end{aligned}$$

Applying schur's lemma on T', we see in case (a) $T' = 0$, and in case (b) $T' = \lambda\mathbb{I}$. Consider,

$$\begin{aligned}
Tr(T') &= \frac{1}{n}\sum_{g\in\mathbb{G}} Tr(\pi_2(g)^{-1}T\pi_1(g)) \\
&= Tr(T).
\end{aligned}$$

Hence $\mathrm{Tr}(\lambda\mathbb{I}) = m\lambda = Tr(T)$, thus $\lambda = \frac{1}{m}Tr(T)$, which proves (b). □

Suppose ϕ and ψ are two complex-valued functions on $\mathbb{G}$. Define, $\langle \phi, \psi\rangle := \frac{1}{n}\sum_{g\in\mathbb{G}} \phi(g^{-1})\psi(g)$, then, $\langle , \rangle$ is an inner product.

Suppose $\pi_1(g) = (a_{ij}(g))$ and $\pi_2(g) = (b_{lm}(g))$ are in matrix form. And suppose $T = (x_{li})$ and $T' = (y_{li})$ are in matrix form. Therefore,

$$y_{li} = \frac{1}{n}\sum_{g\in\mathbb{G},j,m} b_{lm}(g^{-1})x_{mj}a_{ji}(g). \tag{1.1}$$

The right hand side is linear form with respect of x_{mj}, and since it is true for arbitrary T, therefore in case (a), we have following result.

Corollary 1.2.7. *In case (a), we have*

$$\frac{1}{n}\sum_{g\in\mathbb{G}} b_{lm}(g^{-1})xa_{ji}(g) = 0$$
$$i.e\ \langle a_{lm}, a_{ji}\rangle = 0$$

for arbitrary choices of l, m, j, i.

In case (b) $T' = \lambda\mathbb{I}$

$$y_{li} = \lambda\delta_{li},\ where\ \delta\ is\ kronecker\ delta$$
$$\lambda = \frac{1}{m}Tr(T) = \frac{1}{m}\sum \delta_{mj}x_{mj}$$
$$\frac{1}{n}\sum_{g,j,m} b_{lm}(g^{-1})x_{mj}a_{ji}(g) = \frac{1}{m}\sum_{j,m}\delta_{li}\delta_{mj}x_{mj}.$$

by equating the coefficient of x_{mj} we get the following result.

Corollary 1.2.8. *In case (b) we have*

$$\frac{1}{n}\sum_{g\in\mathbb{G}} b_{lm}(g^{-1})a_{ji}(g) = \frac{1}{n}\delta_{li}\delta_{mj} \ = \begin{cases} \frac{1}{m} & if\ l=i,\ m=j \\ 0 & otherwise \end{cases}$$
$$and\ \langle b_{lm}, a_{ji}\rangle = \frac{1}{m}\delta_{li}\delta_{mj}$$

Theorem 1.2.9. *(a) If χ is the character of an irreducible representation, then $\langle\chi,\chi\rangle$= 1.*

(b) If χ and χ' are characters of two non isomorphic irreducible representations, then $\langle\chi,\chi'\rangle$= 0.

Proof. (a) Suppose (a_{ii}) be matrix form of irreducible representation. We know that $\langle \chi, \chi' \rangle = \sum_{i,j} \langle a_{ii}, a_{jj} \rangle$. By *Corollary 1.2.8*, we have
$\langle a_{ii}, a_{jj} \rangle = \frac{1}{n}\delta_{ij}$
$\langle \chi, \chi' \rangle = \frac{1}{m} \sum_{i,j} \delta_{ij} =$ 1. This proves (a).

(b) By *Corollary 1.2.7*, the result follows directly.

∎

Chapter 2

Haar measure on the topological groups

2.1 Topological group

Definition 2.1.1. A group $\mathbb{G}$ having a topology on it is said to be a *topological group* if the maps (g,h)$\longmapsto$ gh from $\mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{G}$ and g $\longmapsto$ g^{-1} from $\mathbb{G} \longrightarrow \mathbb{G}$ are continuous.

Remark 2.1.2. Continuity of the above two maps is equivalent to the continuity of map (g,h)$\longmapsto$ gh^{-1}, from $\mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{G}$.

Suppose $U, V \subset \mathbb{G}$, we will write $UV = \{uv : \ u \in U, v \in V\}$ and $U^{-1} = \{u^{-1} : \ u \in U\}$. A neighbourhood U of e $\in \mathbb{G}$ is called *symmetric* if $U = U^{-1}$. In this chapter, we will denote e to be the identity of group $\mathbb{G}$.

Definition 2.1.3. A topological space is said to be *locally compact* if every point possesses a compact neighbourhood. A topological group is called a *locally compact group* if it is Hausdorff and locally compact.

Proposition 2.1.4. *Suppose $\mathbb{G}$ be a topological group,*

(a) if $z = gh$, O is a neighbourhood of z, then there exist neighbourhoods U of g and V of h such that $UV \subset O$.

(b) For each $g \in \mathbb{G}$, the maps $t \longmapsto gt$, $t \longmapsto tg$, and $g \longmapsto g^{-1}$ are homeomorphisms of $\mathbb{G}$. A set $U \subset \mathbb{G}$ is a neighbourhood of $g \in \mathbb{G}$ if $g^{-1}U$ is a neighbourhood of $e \in \mathbb{G}$.

(c) If U is a neighbourhood of $e \in \mathbb{G}$, then U^{-1} is also a neighbourhood of e. Every neighbourhood U of $e \in \mathbb{G}$ contains a symmetric neighbourhood of e.

(d) For every neighbourhood U of $e \in \mathbb{G}$, there exists a neighbourhood V of e such that $V^2 \subset U$.

(e) If $A, B \subset \mathbb{G}$ such that, if any of them is open in $\mathbb{G}$, then AB is open in $\mathbb{G}$.

(f) A, B are compact subsets of $\mathbb{G}$, then AB is a compact subset of $\mathbb{G}$.

(g) For $A \subset \mathbb{G}$, $\overline{A} = \cap_V AV$, where the intersection is taken over all neighbourhoods V of e.

Proof. (a) Suppose $\phi : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ be such that $\phi(g, h) = gh$. Let O be a neighbourhood of z, then $\phi^{-1}(O)$ is open and contains (g,h). Hence it contains a set of the form $O_1 \times O_2$, where O_1 is a neighbourhood of g and O_2 is a neighbourhood of h such that $\phi(O_1 \times O_2) = O_1O_2 \subset O$. Take U to be O_1 and V to be O_2.

(b) Suppose ϕ be the map from $\mathbb{G}$ to $\mathbb{G}$ defined by t $\longmapsto$ gt. Then ϕ is a bijective function from $\mathbb{G}$ to $\mathbb{G}$ and by the continuity of maps t $\longmapsto$ (t,g) and (t,g) $\longmapsto$ gt, we get continuity of $t \longmapsto gt$. The inverse function ϕ^{-1} maps t $\longmapsto g^{-1}t$ and its continuity follows similarly. g $\longmapsto$ g^{-1} is bijective and self inverse function, hence continuity of inverse function follows by definition. Suppose U is any neighbourhood of g then define $V = g^{-1}U$, e $\in g^{-1}U$ and $g^{-1}U$ is open. Hence V is a neighbourhood of e.

(c) Since g $\longmapsto$ g^{-1} is homeomorphism, which in turn implies that if U is any neighbourhood of e then U^{-1} is a neighbourhood of $e^{-1} = e$. For the second part, take $V = U \cap U^{-1}$, $e \in V$ and $V = V^{-1}$.

(d) Consider the map $f : U \times U \to \mathbb{G}$ defined by $(g, g) \longmapsto gh$. Suppose U is a neighbourhood of e. Since f is continuous, hence f$^{-1}(U)$ is open and contains (e,e). Therefore, by (a) there exist neighbourhoods V_1, V_2 of e such that (e,e) $\in V_1 \times V_2$ and $V_1V_2 \subset U$. Take $V_3 = V_1 \cap V_2$, clearly e $\in V_3$, and take $V = V_3 \cap V_3^{-1}$. Therefore $V = V^{-1}$ and V is open and $V^2 \subset U$.

(e) Since A and B are compact subsets of $\mathbb{G}$, therefore, $A \times B$ is a compact subset of $\mathbb{G} \times \mathbb{G}$ and the map (g,h) $\longmapsto$ gh is continuous map. We know that the continuous image of compact set is compact. Hence AB is a compact subset of $\mathbb{G}$.

(f) Assume A is open. Then $AB = \cup_{b \in B} Ab$ is the union of open sets, hence AB is open.

(g) Suppose $g \in \overline{A}$, and let V be any neighbourhood of e. Then gV^{-1} is a neighbourhood of g, so $gV^{-1} \cap A \neq \phi$. Suppose $a \in gV^{-1} \cap A$, then $a = gv^{-1}$ for some v in V. So $g = av$. Since V is an arbitrary neighbourhood of e, $\overline{A} \subset \cap_V AV$.

Conversely, suppose $x \in \cap_V AV$, and let W be a neighbourhood of x. Then $V = x^{-1}W$ is a neighbourhood of e. So V^{-1} is a neighbourhood of e. Therefore, $x = gv^{-1}$ for some $g \in A$ and $v \in V$, which implies $g = xv \in xV = W$. This, in turn, implies $W \cap A \neq \phi$. Since W is an arbitrary neighbourhood of x. Hence $x \in \overline{A}$.

□

Lemma 2.1.5. *Suppose $\mathbb{G}$ is a topological group and $\mathbb{H}$ is a subgroup of $\mathbb{G}$. Then $\overline{\mathbb{H}}$ is a subgroup of $\mathbb{G}$. Moreover, if $\mathbb{H}$ is normal subgroup of $\mathbb{G}$, then $\overline{\mathbb{H}}$ is a normal subgroup of $\mathbb{G}$.*

Proof. Suppose $g, h \in \overline{\mathbb{H}}$ and let U be any neighbourhood of gh. Suppose $\mu : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ be defined by $(g, h) \longmapsto gh$ is continuous. So $\mu^{-1}(U)$ is open in $\mathbb{G} \times \mathbb{G}$ and contains (g,h), so there are neighbourhoods V of g and W of h such that $V \times W \subset \mu^{-1}(U)$. Since$g, h \in \overline{\mathbb{H}}$, there exist $x \in V \cap \mathbb{H} (\neq \phi)$ and $y \in W \cap \mathbb{H} (\neq \phi)$. Therefore, $xy \in \mathbb{H}$ and $(x, y) \in \ \mu^{-1}(U)$, which implies $xy \in U$. This, in turn, implies that $xy \in U \cap \mathbb{H}$ and since U is arbitrary neighbourhood of gh, it implies that $gh \in \overline{\mathbb{H}}$.

For the second part, suppose $a \in \overline{\mathbb{H}}$ and $g \in \mathbb{G}$. Let V be a neighbourhood of e in $\mathbb{G}$, and put $W = g^{-1}Vg$, then W is a neighbourhood of e in $\mathbb{G}$ and $Wa \cap \mathbb{H} \neq \phi$. Let $h \in \mathbb{H}$, be such that $h \in Wa$. Since $\mathbb{H}$ is normal subgroup, $ghg^{-1} \in \mathbb{H}$ and $ghg^{-1} \in gWag^{-1} = gg^{-1}Vgag^{-1} = Vgag^{-1}$, thus, $Vgag^{-1} \cap \mathbb{H} \neq \phi$. This, in turn, implies that $gag^{-1} \in \mathbb{H}$ and hence the result.

$\square$

2.2 Haar measure

Suppose X is locally compact space. Then the Borel σ algebra $\mathcal{B}$ on X is the sigma algebra generated by its open subsets. A measure μ on this sigma algebra is said to be a regular Borel measure if it satisfies:

1. μ is finite for every compact subset.
2. If $B \in \mathcal{B}$ then $\mu(B) = inf\{\mu(O) : \ Ois\ open,\ B \subseteq O\}$
3. If $B \in \mathcal{B}$ and $\mu(B) < \infty$, then
   $\mu(B) = sup\{\mu(K) : \ K\ is\ compact,\ K \subseteq B\}$

Members of $\mathcal{B}$ are called Borel sets.

Definition 2.2.1. A left Haar measure on a locally compact group $\mathbb{G}$ is a regular Borel measure μ such that $\mu(xE) = \mu(E)$, $E \in \mathcal{B}$ and $x \in \mathbb{G}$. Analogously we can define right Haar measure.

Suppose f is a continuous function on a topological group $\mathbb{G}$ and $y \in \mathbb{G}$, then left translate of f through y is defined as

$$L_y f(x) = f(y^{-1}x)$$

and its right translate is defined as

$$R_y f(x) = f(xy).$$

f is called *left uniformly continuous*, if for every $\epsilon > 0$, there is a neighbourhood V of e such that $\|L_y f - f\|_u < \epsilon$, for $y \in V$. Analogously we can define right uniformly continuous. we will write $C_c(G)$ for compactly supported continuous function on $\mathbb{G}$ and $C_c^+(G)$ denotes the positively compactly supported continuous function.

Example 2.2.2. Suppose G is a finite group then for any set M, let $|M|$ denotes the number of element in M. Suppose order of the group G is n. Then we define the measure ν on G by $\nu(M) = \frac{1}{n}|M|$, $M \subseteq G$. Then ν is normalised Haar measure on G.

Proposition 2.2.3. *If $f \in C_c(G)$, then f is right uniformly continuous.*

Proof. Suppose $K = \text{supp}(f)$ and let $\epsilon > 0$. For each $x \in K$ there exists a neighbourhood U_x of e such that $|f(xy) - f(x)| < \frac{\epsilon}{2}$ for every $y \in ; U_x$. By *Proposition 2.1.4*, there exists a symmetric neighbourhood V_x of e such that$V_x V_x \subset U_x$. Then $\{xV_x\}_{x \in K}$ will form a cover of K. Since K is compact, there exist finitely many points, say $x_1, x_2, x_3, \ldots, x_n \in K$ such that $K \subset \cup_{i=1}^{n} x_i V_{x_i}$. Suppose $V = \cap_{i=1}^{n} V_{x_i}$.
Claim: $|f(xy) - f(x)| < \epsilon$ if for all $y \in V$.
If $x \in K$ then $X \in x_j V_{x_j}$ for some j and $x_j^{-1}x \in V_{x_j}$ and hence $xy \in x_j U_{x_j}$, therefore

$$|f(xy) - f(x)| < |f(xy) - f(x_j)| + |f(x_j) - f(x)| < \epsilon.$$

And if $x \notin K$ then $f(x) = 0$, and hence either $f(xy) = 0$ (if $xy \notin K$) or $xy \in x_j V_{x_j}$ for some j and $x_j^{-1}x = x_j^{-1}xyy^{-1} \in U_{x_j}$ so that $|f(x_j| < \frac{\epsilon}{2}$ and hence the results. ∎

Suppose $f, \phi \in C_c^+(G)$, then we define Haar covering number of f with respect to ϕ by

$$(f : \phi) = inf\{\sum_{i=1}^{n} c_i : \ f \leq \sum_{i=1}^{n} c_i L_{x_i}\phi \ for \ some \ x_1, x_2, \ldots, x_n \in \mathbb{G}\}$$

The above definition make sense as the set $\{x : \phi(x) \geq \frac{1}{2}\|\phi\|_u\}$ is open, non empty and as supp(f) is compact, so finitely many left translates of it cover supp(f) and therefore there are $x_1, x_2, \ldots, x_n$ such that

$$f \leq \frac{2\|f\|_u}{\|\phi\|_u} \sum_{j=1}^{n} Lx_j\phi$$

and hence $(f : \phi) > 0$.

Lemma 2.2.4. *Suppose that $f, \phi, g \in C_c^+$, then*

(a) $(f : \phi) = (L_x f : \phi)$ *for any* $x \in \mathbb{G}$.

(b) $(cf : \phi) = c(f : \phi)$ *for any positive* c.

(c) $(f + g : \phi) \leq (f : \phi) + (g : \phi)$.

(d) $(f : \phi) \leq (f : g)(g : \phi)$.

Proof. Since $f \leq \sum c_i L_{x_i}\phi$ if and only if $L_x f \leq \sum c_i L_{xx_i}\phi$ which proves (a) part. Similarly part (b) is easy to see. For part (c), suppose $f \leq \sum_i c_i L_{x_i}\phi$ and $g \leq \sum_j d_j L_{x_j}\phi$, thus if $(f : \phi) \leq \sum c_i$ and $(g : \phi) \leq \sum d_j$ then $(f + g : \phi) \leq \sum c_i + \sum d_j$ and hence prove (c). □

Now, we shall make a normalization by choosing $f_0 \in C_c^+$ and define

$$I_\phi(f) = \frac{(f:\phi)}{(f_0:\phi)} \; for \; f, \phi \in C_c^+.$$

Hence by the previous lemma

$$(f_0, \phi)^{-1} \leq I_\phi(f) \leq (f : f_0).$$

Lemma 2.2.5. *If $f_1, f_2 \in C_c^+(\mathbb{G})$ and $\epsilon > 0$, then there is a neighbourhood V of e such that $I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + \epsilon$, whenever supp$(\phi) \subset V$.*

Proof. Fix $g \in C_c^+(\mathbb{G})$ such that $g = 1$ on supp$(f_1 + f_2)$ and let $\delta > 0$. Suppose $h = f_1 + f_2 + \delta g$ and $h_i = \frac{f_i}{h}$ for $i = 1, 2$, and $h_i = 0$ outside of supp(f_i). Then $h_i \in C_c^+(\mathbb{G})$, therefore by *Proposition 2.2.3*, there is a neighbourhood V of e such that $|h_i(x) - h_i(y)| < \delta$ and $y^{-1}x \in V$. Suppose $\phi \in C_c^+(\mathbb{G})$ be such that supp$(\phi) \subset V$, and $h \leq \sum_{i=1}^n c_j L_{x_j}\phi$, then if $x_j^{-1}x \in$ supp(ϕ) we get $|h_i(x) - h_i(x_j)| < \delta$, so

$$f_i(x) = h(x)h_i(x) \leq \sum_{j=1}^n c_j \phi(x_j^{-1}x)h_i(x) \leq \sum_{j=1}^n c_j \phi(x_j^{-1}x)[h_i(x_j) + \delta].$$

But then $(f_i : \phi) \leq \sum c_j[h_i(x_j) + \delta]$ and since $h_1 + h_2 < 1$

$$(f_1 : \phi) + (f_2 : \phi) \leq \sum_j c_j[1 + 2\delta].$$

Now $\sum c_j$ can be made arbitrary close to $(h : \phi)$, so by *Lemma 2.2.4*,

$$I_\phi(f_1) + I_\phi(f_2) \leq (1 + 2\delta)I_\phi(h) \leq (1 + 2\delta)[I_\phi(f_1 + f_2) + \delta I_\phi(g)].$$

Therefore, by the inequality which we get just earlier of this lemma, it suffices to choose δ to be as small so that

$2\delta(f_1 + f_2 : f_0) + \delta()1 + 2\delta(g : f_0) < \epsilon.$ $\square$

Theorem 2.2.6. *Every locally compact group $\mathbb{G}$ possesses a left Haar measure.*

Proof. Suppose X_f be the interval $[(f_0 : f)^{-1}, (f : f_0)]$ for every $f \in C_c^+\mathbb{G}$, suppose $X = \prod_{f \in C_c^+} X_f$. Then by Tychonoff's theorem, X is compact Hausdorff space and by *lemma 2.2.5*, every I_ϕ is an element of X. Suppose for every compact neighbourhood V of e, $K(V)$ denotes the closure of $\{I_\phi : \operatorname{supp}(\phi) \subset V\}$ in X. It is easy to see $\cap_{i=1}^n K(V_i) \supset K(\cap_{i=1}^n V_i)$, so by finite intersection property of compact space, there is an element I in the intersection of all the $K(V)$, such that for every neighbourhood V of e and any $g_1, g_2, \ldots, g_n$ and $\epsilon > 0$, there exists $\phi \in C_c^+(\mathbb{G})$ with $\operatorname{supp}(\phi) \subset V$ such that $|I(g_j) - I_\phi(g_j)| < \epsilon$ for $j = 1, 2, \ldots, n$. Therefore in view of *Lemma 2.2.3* and *Lemma 2.2.4*, I is left invariant and satisfies $I(af + bg) = aI(f) + bI(g)$ $\forall\ f, g \in C_c^+(\mathbb{G})$ where $a, b > 0$. Now, if we extend I to $C_c(\mathbb{G})$ by defining $I(f) = I(f^+) - I(f^-)$, then I is left invariant positive linear functional on $C_c(\mathbb{G})$. Therefore, by invoking the Riesz representation theorem, we get the result. $\square$

Chapter 3

Representation of Compact Group

Throughout this chapter, by a group, we mean a topological group.

3.1 Representation of the topological groups

Suppose H is a Hilbert space over $\mathbb{C}$. Let $\mathcal{B}(H)$ be the set of all bounded linear operator on H and $GL(H)$ denotes the group of all invertible members of $\mathcal{B}(H)$.

Definition 3.1.1. A representation of a group $\mathbb{G}$ on H is a group homomorphism $\pi : \mathbb{G} \to GL(H)$ such that for every $v \in H$, the map $g \longmapsto \pi(g)v$ from $\mathbb{G}$ into H is continuous.

For given π, we called H to be representation space and denotes group representation by (π, H). The dimension of H is called the dimension of the representation.

Definition 3.1.2. A representation (π, H) of $\mathbb{G}$ is said to be *unitary representation* if $\pi(g)$ is unitary operator on H for every $g \in \mathbb{G}$, that is, if $\langle \cdot, \cdot \rangle$ is an inner product on H, then $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$, $\forall\ v, w \in H$ and $\forall\ g \in \mathbb{G}$.

Definition 3.1.3. Suppose (π, H) and (ρ, M) are two representations of the group $\mathbb{G}$. Then π and ρ are called *equivalent* if there exists a continuous linear isomorphism $T : H \to M$ such that $T\pi(g) = \rho(g)T$, for all $g \in \mathbb{G}$.

Definition 3.1.4. Suppose (π, H) be a representation of group $\mathbb{G}$ on finite dimensional Hilbert space H. Suppose H^* denotes the dual of H. We define *contragredient representation* by $\pi^c : \mathbb{G} \to GL(H^*)$ by $\pi^c(g)\gamma(v) = \gamma(\pi(g^{-1})v)$, $\forall\ \gamma \in H^*$ and $\forall\ v \in H$.

The above definition make sense because,

1. $\pi^c(e)\gamma(v) = \gamma(v)$, $\forall\ \gamma \in H^*$ and $\forall\ v \in H$, which gives $\pi^c(e) = I$ on H^*.

2. $\pi^c(gh)\gamma(v) = \gamma(\pi((gh)^{-1})v)\pi^c(h)(\gamma(\pi(g^{-1})v)) = \pi^c(g)\pi^c(h)(\gamma(v))$, $\forall\ g \in \mathbb{G}$ and $\forall\ v \in H$.

Definition 3.1.5. Suppose (π, H) be a representation of $\mathbb{G}$. A closed subspace M of H is said to be *invariant subspace* of π if $\pi(g)v \in M$ for all $v \in M$ and $\forall\ g \in \mathbb{G}$. If M is an invariant subspace of π, then $g \longmapsto \pi|_M(g)$ is group representation and called *subrepresentation* of π and denoted by $(\pi|_M, M)$.

Remark 3.1.6. If M is a closed subspace of H and $T \in \mathcal{B}(H)$, then M is invariant under T if and only if $TP = PTP$, where P is orthogonal projection on M. This is because, if M is invariant under T, then $TP(x) \in M$ for all $x \in H$, which in turn implies that $TP(X) = PTP(x)$. Conversely, if $TP = PTP$ and $x \in M$, then $T(x) = TP(x) = PTP(x) \in M$, which implies that $T(x) \in M$. Hence M is invariant under T.

Example 3.1.7. Suppose $\mathbb{G}$ be a compact group and let μ be a Haar measure on $\mathbb{G}$. Let $H = L^2(\mathbb{G})$. Define,

$$(R(g)f)(s) = f(sg),\ g \in \mathbb{G}, f \in H.$$

Then $g \longmapsto R(g)$ is a unitary representation and is called *right regular representation.* Analogously we define *left regular representation* as $(L(g)f)(s) = f(g^{-1}s),\ g \in \mathbb{G}, f \in H.$

Lemma 3.1.8. *Suppose (π, H) be a unitary representation of a group $\mathbb{G}$.*

(a) If M is an invariant subspace of π, then the subspace $M^{\perp}$ is also invariant subspace of π.

(b) Let M be a closed subspace of H and let P be the orthogonal projection of H on M. Then M is invariant for π if and only if $P\pi(g) = \pi(g)P$ for all $g \in \mathbb{G}$.

Proof. (a) Suppose $m \in M$ and $u \in M^{\perp}$ and $g \in \mathbb{G}$.

$$\begin{aligned}\langle \pi(g)u, m\rangle &= \langle u, \pi^*(g)m\rangle \\ &= \langle u, \pi(g^{-1})m\rangle,\ \textit{since } \pi \textit{ is unitary} \\ &= 0,\ \textit{since } M \textit{ is invariant under } \pi\end{aligned}$$

and hence $\pi(g)u \in M^{\perp}$.

(b) Suppose $P\pi(g) = \pi(g)P$, and let $m \in M$ then $\pi(g)(m) = \pi(g)P(m) = P\pi(g)(m) \in M$, and hence M is invariant for π. Conversely, suppose M is invariant for π, then by *Remark 3.1.6*, $\pi(g)P = P\pi(g)P$, which in turn implies that $\pi(g)^*P = P\pi(g)^*P$ (as P is self adjoint). Which gives $(\pi(g)^*P)^* = (P\pi(g)^*P)^*$, which in turn implies $P\pi(g) = P\pi(g)P$.

□

Direct Sum

Suppose $\{H_i\}_{i\in\Lambda}$ be a collection of Hilbert spaces. Consider the set $H = \{u = (u_i) : u_i \in H_i \textit{ and } \sum_i \|u_i\|^2 < \infty\}$. Then H is vector space with coordinate-wise addition and scalar multiplication. Suppose $v = (v_i)$ and $u = (u_i)$, define $\langle u, v\rangle = \sum_i \langle u_i, v_i\rangle$, with this inner product H become Hilbert space. This Hilbert space H is called the direct sum of the collection $\{H_i\}_{i\in\Lambda}$ and is denoted by $\bigoplus_{i\in\Lambda} H_i$. Suppose $\{(\pi_i, H_i)\}_{i\in\Lambda}$ be a family of unitary representations of a group $\mathbb{G}$. Then for each $g \in \mathbb{G}$, consider the map $\pi(g)$ defined on H by

$$\pi(g)v = (\pi_i(g)v_i) \; for \; all \; v \in H.$$

Then π is a unitary representation of $\mathbb{G}$ on H. This representation is called the *direct sum* of family $\{\pi_i\}_{i\in\Lambda}$ of unitary representations of $\mathbb{G}$ and denoted by $\bigoplus \pi_i$.

Definition 3.1.9. Suppose (π, H) be a group representation of group $\mathbb{G}$. Then (π, H) is said to be *irreducible* if there is no non trivial π-invariant closed subspace of H.

Definition 3.1.10. Suppose (π, H) be a unitary representation of $\mathbb{G}$, then it is said to be *completely reducible* if there exists a family $\{H_i\}$ of closed mutually orthogonal invariant subspaces of π such that each (π, H_i) is irreducible and $H = \bigoplus H_i$.

Notation: Suppose (π, H) be unitary representation of $\mathbb{G}$. We will denote $\mathcal{I}_\pi = \{T \in \mathcal{B}(H) : T\pi(g) = \pi(g)T \ \forall \ g \in \mathbb{G}\}$, the space of *intertwining operators.*

Theorem 3.1.11. ***(Schur's Lemma)*** *Suppose (π, H) is a unitary representation of a group $\mathbb{G}$, then π is irreducible if and only if $\mathcal{I}_\pi = \{\lambda I : \lambda \in \mathbb{C}\}$, where I is identity operator.*

Proof. Suppose $\mathcal{I}_\pi = \{\lambda I : \lambda \in \mathbb{C}\}$. If π is reducible, then there exists a proper closed subspace M of H, which is invariant under π. Suppose P is the orthogonal projection on M. Clearly $P \neq cI$ for any scalar c. By *Lemma 3.1.8*, $P\pi(g) = \pi(g)P$ for all $g \in \mathbb{G}$, which is a contradiction. Hence π is irreducible.
Converse part can be proved using the spectral theorem for the self-adjoint compact operators. We omit the proof here. □

Corollary 3.1.12. *Suppose (π, H) is an irreducible unitary representation of an abelian group $\mathbb{G}$, then the dimension of H is one.*

Proof. Since the group G is abelian, $\pi(gh) = \pi(g)\pi(h) = \pi(h)\pi(g) \ \forall \ g, h \in \mathbb{G}$. Hence $\pi(h) \in \mathcal{I}_\pi \ \forall \ h \in \mathbb{G}$. Thus, by Schur's lemma, each $\pi(g)$ is a scalar multiple of identity operator. Therefore every subspace of H is invariant under π. This proves the result. $\square$

Cyclic representation

Suppose (π, H) is a representation of $\mathbb{G}$. Let $v(\neq 0) \in H$, define $M_v = \overline{span}\{\pi(g)v : g \in \mathbb{G}\}$. Then M_v is a closed invariant subspace of π, and is called *cyclic space* generated by v. If there exists $v \neq 0$ such that $M_v = H$, then v is called *cyclic vector* for π.

Proposition 3.1.13. *(a) A representation (π, H) is irreducible if and only if every non zero $v \in H$ is a cyclic vector for π.*

(b) Any unitary representation of a group is a direct sum of cyclic representations.

Proof. (a) Suppose π is irreducible. Let $v(\neq 0) \in H$, then M_v is non trivial invariant subspace for π. Hence $M_v = H$.

Conversely, suppose π is not irreducible. Then there exists a non trivial invariant closed subspace M for π. Let $v(\neq 0) \in M$, then $M_v \subset M \subsetneq H$, which is a contradiction.

(b) This proposition can be proved with the help of Zorn's lemma on the family of mutually orthogonal cyclic subspaces of H, where partial order is the set inclusion.

$\square$

Lemma 3.1.14. *Suppose H and K are two Hilbert spaces and let $T : H \to K$*

be a bounded linear map. Suppose $f : \mathbb{G} \to H$ is a continuous map, then

$$T\Bigg(\int_{\mathbb{G}} f(y) d\mu(y) \Bigg) = \int_{\mathbb{G}} T f(y) d\mu(y)$$

Proof. Suppose $u \in K$,

$$\begin{aligned} \langle \int_{\mathbb{G}} f(y) d\mu(y), T^* u \rangle &= \int_{\mathbb{G}} \langle T f(y), u \rangle d\mu(y) \\ &= \langle \int_{\mathbb{G}} T f(y) d\mu(y), u \rangle. \end{aligned}$$

Since U is unitary, it follows that $T\Bigg(\int_{\mathbb{G}} f(y) d\mu(y) \Bigg) = \int_{\mathbb{G}} T f(y) d\mu(y)$ ☐

Lemma 3.1.15. *Suppose (π, H) is a cyclic unitary representation of $\mathbb{G}$ with $x \in H, x \neq 0$ a cyclic vector. Then the operator defined by*

$$Ky = \int_{\mathbb{G}} \langle y, \pi(g)x \rangle \pi(g) x d\mu(g) \tag{3.1}$$

is bounded, strictly positive compact on H such that $K\pi(g) = \pi(g)K$ for all $g \in \mathbb{G}$.

Proof. Suppose (h_α) is a net in $\mathbb{G}$ converging to $h \in \mathbb{G}$. Since the inner product is continuous, we get $\langle y, \pi(h_\alpha)x \rangle \pi(h_\alpha)x \longrightarrow \langle y, \pi(h)x \rangle \pi(h)x$. Now to show K is bounded, consider

$\|Ky\|^2 = |\langle Ky, Ky \rangle| = | \int_{\mathbb{G}} \langle y, \pi(g)x \rangle \langle \pi(g)x, Ky \rangle d\mu(g)|$

$\leq \int_{\mathbb{G}} |\langle y, \pi(g)x \rangle||\langle \pi(g)x, Ky \rangle| d\mu(g) \leq \|y\| \|x\|^2 \|Ky\|$.

That gives $\|Ky\| \leq \|y\| \|x\|^2$ and hence K is bounded. Now consider,

$\langle y, y \rangle = \langle \int_{\mathbb{G}} \langle y, \pi(g)x \rangle \pi(g) x d\mu(g), y \rangle = \int_{\mathbb{G}} |\langle y, \pi(g)x \rangle|^2 d\mu(g) \geq 0$. If possible suppose $\langle Ky, y \rangle = 0$, then $< y, \pi(g)x >= 0$ a.e. But $g \longmapsto \langle y, \pi(g)x \rangle$ is a continuous map, and support of μ is $\mathbb{G}$, it follows $\langle y, \pi(g)x\rangle = 0$ for all $g \in \mathbb{G}$. That is, $y = 0$, since x was cyclic vector. Thus K is positive.

To prove K is compact, suppose $z_n \xrightarrow{w} z$, then

$\langle Kz_n, Kz\rangle = \langle z_n, K^*Kz\rangle \to \|Kz\|^2$ and

$\|Kz_n - Kz\|^2 = \|Kz_n\|^2 + \|Kz\|^2 - 2Re\langle Kz_n, Kz\rangle$.

$\|Kz_n\|^2 = \int_{\mathbb{G}} \langle z_n, \pi(g)x\rangle\langle \pi(g)x, Kz_n\rangle d\mu(g)$

$= \int_{\mathbb{G}} \langle z_n, \pi(g)x\rangle\langle K\pi(g)x, z_n\rangle d\mu(g) \to \int_{\mathbb{G}} \langle z, \pi(g)x\rangle\langle k\pi(g)x, z\rangle d\mu(g)$

$= \int_{\mathbb{G}} \langle z, \pi(g)x\rangle\langle \pi(g)x, Kz\rangle d\mu(g) = \|Kz\|^2$. Hence K is compact.

Suppose $g \in \mathbb{G}$ and $u, v \in H$, then $\langle K\pi(g)u, v\rangle = \int_{\mathbb{G}} \langle \pi(g)u, \pi(h)x\rangle\langle \pi(h)x, v\rangle d\mu(h)$

$= \int_{\mathbb{G}} \langle u, \pi(g^{-1}h)x\rangle\langle \pi(g^{-1}h)x, \pi(g^{-1})v\rangle d\mu(h) = \int_{\mathbb{G}} \langle u, \pi(h)x\rangle\langle \pi(h)x, \pi(g^{-1}v)d\mu(h)\rangle$

$= \langle Ku, \pi(g^{-1})v\rangle = \langle \pi(g)u, v\rangle$. Hence $K\pi(g) = \pi(g)K\ \forall\ g \in \mathbb{G}$. ☐

Corollary 3.1.16. *Suppose (π, H) is a unitary representation of group $\mathbb{G}$. Then there exists a non-trivial finite dimensional subspace of H which is invariant under π.*

Proof. By *Proposition 3.1.13*, every unitary representation is nothing but the direct sum of cyclic representations, therefore, without loss of generality, we may assume (π, H) is a cyclic representation. By the spectral theorem for the self-adjoint compact operators, there exists $\gamma \neq 0$ such that $H_\gamma = ker(K - \gamma I) \neq \{0\}$ and finite dimensional, where K is defined by (3.1). Suppose $v \in H_\gamma$, then $K\pi(g)v = \pi(g)Kv = \pi(g)\gamma v = \gamma\pi(g)v$ and hence H_γ is invariant subspace. ☐

Suppose (π, H) is an irreducible unitary representation of a compact group, then by the previous corollary, there exists a finite dimensional non trivial invariant subspace, and hence we have the following results.

Corollary 3.1.17. *Every irreducible unitary representation of a compact group is finite dimensional.*

Lemma 3.1.18. *Any finite dimensional unitary representation of a group is completely reducible.*

Proof. We shall prove this theorem by using the induction hypothesis on the degree of representation. Suppose $(\pi, \mathbb{H})$ be a finite degree representation of $\mathbb{G}$. If the degree of representation is one, then π is irreducible. Suppose degree of representation of π, i.e $\deg(\pi, \mathbb{H})$ = n, n$\in \mathbb{N}$. Now, suppose the theorem holds for all representations of degree less than n. If π is irreducible, we are done. Otherwise, there exists proper nontrivial invariant closed subspace M of H. Since π is unitary, which gives $M^{\perp}$ is also invariant under π and dimension of M and $M^{\perp}$ is less than the dimension of H. Hence by induction hypothesis we get the results. $\square$

Theorem 3.1.19. *Every unitary representation of a compact group is a direct sum of irreducible finite dimensional unitary representations.*

Proof. Suppose (π, H) is a unitary representation of compact group $\mathbb{G}$ and let $\mathcal{F} = \{H_i : \ i \in I\}$, where H_i's are pairwise disjoint orthogonal finite dimensional irreducible subrepresentations of π. By the *Corollary 3.1.16* and *Lemma 3.1.18*, $\mathcal{F}$ is non empty, and set inclusion is partial order on $\mathcal{F}$. Suppose $\mathcal{O} = \{\{H_i\}_{i\in I_j} : \ j \in \Lambda\}$ be a chain in $\mathcal{F}$, and let $K = \cup_{j\in\Lambda}\{H_i\}_{i\in I_j}$. Since $\mathcal{O}$ is chain it follows that K is an upper bound of $\mathcal{O}$, so by the Zorn's lemma, there exists a maximal element $\{H_j\}_{j\in J}$.

Claim: $H = \bigoplus_{j\in J} H_j$

Suppose $H \neq \bigoplus_{j\in J} H_j$ and let $H_\alpha = \bigoplus_{j\in J} H_j$, then by *Corollary 3.1.16* and

Lemma 3.1.18, there exists non trivial finite dimensional irreducible subspace H'. Then $H' \perp H_j \ j \in J$, which contradicts the maximality of $H_j, \ j \in J$. $\square$

Notation

1. Let $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of $\mathbb{G}$.

2. If (π, H) is a finite dimensional unitary representation of $\mathbb{G}$ and $\{e_1, \ldots, \ e_n\}$ is an orthonormal basis of H, we define $\phi_{ij} = \langle \pi(g)e_j, e_i \rangle, 1 \leq i, j \leq n$. Then, ϕ_{ij}'s are continuous functions on $\mathbb{G}$ and are called *matrix coefficient* of the representation of π.

Proposition 3.1.20. *Suppose (π, H) and (ρ, K) are two finite dimensional representations of $\mathbb{G}$, and let $T : K \to H$ be a linear map such that $\pi(g)T = T\pi(g)$, $\forall \ g \in \mathbb{G}$. Then $ker(T)$ and $Im(T)$ are invariant subspaces of K and H respectively. Also if π and ρ are irreducible and inequivalent, then $T = 0$.*

Proof. Proof of this theorem is easy. $\square$

Schur's orthogonality relations

Let π and ϱ be two irreducible representations of $\mathbb{G}$ and $(\phi_{ij}(g))$ and $(\psi_{kl}(g))$ are the corresponding matrix coefficients of π and ϱ respectively with respect to some fixed orthonormal bases in the respective Hilbert spaces. Then,

(a) $\langle \phi_{ij}, \psi_{kl} \rangle_{L^2(\mathbb{G})} = 0$, if π and ϱ are not equivalent.

(b) $\langle \phi_{ij}, \psi_{kl} \rangle_{L^2(\mathbb{G})} = \frac{1}{dim(H_\pi)} \delta_{ik} \delta_{jl}$.

3.2 Character of representation

Definition 3.2.1. Suppose (π, H_π) is a finite dimensional representation of a group $\mathbb{G}$. The character of π is the function $\chi_\pi(g) = Tr(\pi(g))$, where Tr denotes trace.

Proposition 3.2.2. *(a) If π and ϱ are equivalent, then $\chi_\pi = \chi_\varrho$.*

(b) $\chi_\pi(g) = \chi_\pi(hgh^{-1})$, $g, h \in \mathbb{G}$.

(c) $\chi_{\pi \bigoplus \varrho} = \chi_\pi + \chi_\varrho$.

(d) $\chi_\pi(e) = d_\pi$, where d_π is dimension of H_π

(e) χ_π is continuous function on $\mathbb{G}$.

(f) If π is unitary, then for all $g \in \mathbb{G}$, $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$.

Proof. We will prove (e) only, others are easy to see. Suppose (h_α) is a net in $\mathbb{G}$ converging to $h \in \mathbb{G}$. Then, $\phi_{ii}(h_\alpha) \to \phi_{ii}(h)$, where $1 \leq i \leq dim(\pi)$. This, in turn, implies $\sum_{i=1}^{dim(\pi)} \phi_{ii}(h_\alpha) \to \sum_{i=1}^{dim(\pi)} \phi_{ii}(h)$. Hence $\chi_\pi(h_\alpha) \to \chi_\pi(h)$. Thus χ_π is continuous. □

Proposition 3.2.3. *Suppose (π, H_π) and (ϱ, H_ϱ) are two irreducible unitary representations of a compact group $\mathbb{G}$.*

(a) If π and ϱ are equivalent, then $\langle \chi_\pi, \chi_\varrho \rangle_{L^2(G)} = 1$.

(b) If π and ϱ are inequivalent, then $\langle \chi_\pi, \chi_\varrho \rangle_{L^2(G)} = 0$.

Proof. Suppose ϕ_{ij} and ν_{ij} are matrix coefficient of π and ϱ respectively. Then $\langle \chi_\pi, \chi_\varrho \rangle_{L^2(G)} = \int_{\mathbb{G}} \chi_\pi(g)\overline{\chi_\varrho(g)} d\mu(g) = \int_{\mathbb{G}} Tr(\pi(g))\overline{Tr(\pi(g))} d\mu(g)$. By

Schur's orthogonality relation (b), we get $\sum_{i,j} \langle \phi_{ii}, \phi_j \rangle_{L^2(\mathbb{G})} = 1$. Similarly using Schur's orthogonality relation (a) we get (b). $\square$

3.3 Peter-Weyl Theorem

we shall state and prove this theorem in five related assertions, PW1-PW5.

PW1

Every irreducible unitary representation of a compact group is equivalent to a subrepresentation of the right regular representation.

Proof. Suppose (π, H) is an irreducible unitary representation of group $\mathbb{G}$. Then by *Corollary 3.1.17*, the dimension of H is finite, say n. Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of H and let $\phi_{lj}(g) = \langle \pi(g)e_j, e_l \rangle$. Fix l (say l=1), and suppose $\psi_j(g) = \sqrt{n}\phi_{1j}(g)$. Then, by Schur's orthogonality relations, $\{\psi_1, \psi_2, \dots, \psi_n\}$ is an orthonormal set in $L^2(G)$. Suppose $E_1 = span\{\psi_1, \psi_2, \dots, \psi_n\}$. Then,

$$\begin{aligned}
R(h)\psi_j(g) = \psi_j(gh) = \sqrt{n}\phi_{1j}(gh) &= \sqrt{n}\langle \pi(gh)e_j.e_1 \rangle \\
&= \sqrt{n}\langle \pi(g)(\sum_{l=1}^{n} \langle \pi(h)e_j, e_l \rangle_l), e_1 \rangle \\
&= \sqrt{n}\sum_{l=1}^{n} \langle \pi(h)e_j, e_l \rangle \langle \pi(g)e_l, e_1 \rangle \\
&= \sqrt{n}\sum_{l=1}^{n} \phi_{lj}(h)\phi_{1l}(g) \\
&= \sum_{l=1}^{n} \phi_{lj}(h)\psi_l(g).
\end{aligned}$$

Thus subspace E_1 is invariant with respect to right regular representation and $R(g)|_{E_1}$ has matrix entries $(\phi_{lj}(g))$ with respect to orthonormal basis $\{\psi_1, \psi_2, \ldots, \psi_n\}$. Thus, $(R(h)|_{E_1}, E_1)$ is equivalent to (π, H). $\square$

Remark 3.3.1. Let $\psi^i_j(g) = \sqrt{n}\phi_{ij}(g)$, and $E_i = span\{\psi^i_1, \ldots, \psi^i_n\} \subseteq L^2(\mathbb{G})$. Thus, by Schur's orthogonality relations, E_i's are mutually orthogonal. Suppose $E_\pi = \bigoplus_{i=1}^{d_\pi} E_i$, where d_π is the dimension of π. Thus $\dim(E_\pi) = d_\pi^2$.

Proposition 3.3.2. *(a) Define, $\pi_{y,x}(g) = \langle \pi(g)x, y\rangle$ for all $g \in \mathbb{G}$, then $\pi_{y,x} \in L^2(G)$ and called matrix coefficient functions. Then, $E_\pi = span(\{\pi_{y,x} : y, x \in H\})$.*

(b) If π and ϱ are equivalent representations, then E_π and E_ϱ are identical with $dim(E_\pi) = d_\pi^2$.

Proof. (a) Obviously, $E_\pi \subset span\{\pi_{y,x} : y, x \in H\}$. Suppose $y, x \in H$ and $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis for H and $g \in \mathbb{G}$, then

$$\begin{aligned}\pi_{y,x}(g)\langle\pi(g)\left(\sum_{i=1}^{n}\langle x, e_i\rangle e_i\right), \sum_{j=1}^{n}\langle y, e_j\rangle e_j\rangle\\ = \sum_{i=1}^{n}\sum_{j=1}^{n}\langle x, e_i\rangle\overline{\langle y, e_j\rangle}\langle\pi(g)e_i, e_j\rangle\\ = \sum_{i=1}^{n}\sum_{j=1}^{n}\langle x, e_i\rangle\overline{\langle y, e_j\rangle}\phi_{ij}(g).\end{aligned}$$

Hence $\pi_{y,x} \in E_\pi$.

(b) Since π and ϱ are equivalent, there exists an isomorphisms U of π and ϱ such that $U\pi(g) = \rho(g)\varrho$.
$\pi_{y,x} = \langle\pi(g)x, y\rangle = \langle U^{-1}\varrho(g)Ux, \rangle = \langle\varrho(g)Ux, (U^{-1})^* y\rangle$.

This implies $\pi_{y,x} = \varrho_{(U^{-1})^* y, Ux}$. Since U is an isomorphism, it follows that E_π and E_ϱ are identical.

$\square$

For $\lambda \in \mathbb{G}$, suppose E_λ is a finite dimensional space that spanned by the matrix coefficient functions of the representation π in the equivalence class λ. Then, by the previous lemma, $E_\alpha = E_\beta$ for $\alpha, \beta \in \lambda$.

PW2

(a) Each E_λ is invariant under the right regular representation of $\mathbb{G}$.

(b) If π is an irreducible unitary representation in the equivalence class λ, then $R|_{E_\lambda}$ is equivalent to the direct sum of d_π copies of π and consequently, $\dim(E_\lambda) = d_\pi^2$.

(c) $L^2(\mathbb{G}) = \bigoplus_{\lambda \in \hat{G}} E_\lambda$.

Proof. (a) It is directly followed from PW1 and remark after PW1.

(b) Since each E_i is invariant under R and by remark after PW1 implies that $R|_{E_i}$ is equivalent to π. Hence proves (b).

(c) Suppose $M = \bigoplus_{\lambda \in \hat{G}}$. Then M and $M^\perp$ are invariant under right regular representation R. Suppose $U \subset M^\perp$ be an invariant subspace such that $R|_K$ is irreducible and it is in the equivalence class $\lambda \in \hat{G}$. For $f \in K$, suppose $F(h) = \langle R(h)f, f \rangle$, where $h \in \mathbb{G}$. Clearly F is continuous function, $F \in E_\lambda \subset M$ and $F(e) = \|f\|^2$.

Suppose $\{\phi_{ij}^\lambda\}$ is matrix coefficient functions of some representation

$(\pi_\lambda, H_\lambda) \in \lambda$, i.e. $\phi_{ij}^\lambda(g) = \langle \pi_\lambda(ge_j), e_i \rangle$ with respect to some orthonormal basis $\{e_1, e_2, \ldots, e_{d_\lambda}\}$, where d_λ is dimension of H_λ.Then,

$$\begin{aligned}
\langle F, \phi_{ij}^\lambda \rangle_{L^2(\mathbb{G})} &= \int_{\mathbb{G}} F(h)\overline{\phi_{ij}^\lambda(h)} d\mu h \\
&= \int_{\mathbb{G}} \int_{\mathbb{G}} f(gh)\overline{f(g)\phi_{ij}^\lambda(h)} d\mu(g) d\mu(h) \\
&= \int_{\mathbb{G}} \int_{\mathbb{G}} f(h)\overline{f(g)\phi_{ij}^\lambda(g^{-1})} d\mu(g) d\mu(h).
\end{aligned}$$

Since μ is left invariant and by the Fubini's theorem,

$$\begin{aligned}
\langle F, \phi_{ij}^\lambda \rangle_{L^2(\mathbb{G})} &= \int_{\mathbb{G}} \int_{\mathbb{G}} f(gh)\overline{f(g)} \sum_{k=1}^{d_\lambda} \overline{\phi_{ik}^\lambda(g^{-1})\phi_{kj}^\lambda(h)} d\mu(g) d\mu(h), \\
&= \sum_{k=1}^{d_\lambda} \int_{\mathbb{G}} \overline{f(g)\phi_{ik}^\lambda(g^{-1})} \langle f, \phi_{kj}^\lambda \rangle d\mu(g), \\
&= 0, since\ f \ \in \ M^\perp \ and\ \phi_{ij}^\lambda \ \in \ E_\lambda.
\end{aligned}$$

Therefore, $F \in E_\lambda^\perp \cap E_\lambda = \{0\}$. Which gives $f = 0$. So $M = L^2(\mathbb{G})$.

□

Remark 3.3.3. 1. By PW2, $(R, L^2(\mathbb{G}))$ is the direct sum of irreducible representations of $\mathbb{G}$ and each $\lambda \in \hat{G}$ occurs in the decomposition d_λ times.

2. For any $f \in L^2(\mathbb{G})$ we have, $f = \sum_{\lambda \in \hat{G}} d_\lambda \sum_{i=1}^{d_\lambda} \sum_{j=1}^{d_\lambda} \langle f, \phi_{ij}^\lambda \rangle \phi_{ij}^\lambda$ and

$\|f\|_{L^2}^2 = \sum_{\lambda \in \hat{G}} d_\lambda \sum_{i=1}^{d_\lambda} \sum_{j=1}^{d_\lambda} |\langle f, \phi_{ij}^\lambda \rangle|^2.$

Definition 3.3.4. Suppose $f, g \in L^1(\mathbb{G})$, then we define *convolution* of *f and g* as $f * g(x) = \int_{\mathbb{G}} f(y)g(y^{-1}x)d\mu(x)$.

Lemma 3.3.5. *Suppose $f \in L^2(\mathbb{G})$, then $f * \chi_\lambda = \sum_{i=1}^{d_\lambda}\sum_{j=1}^{d_\lambda}\langle f, \phi_{ij}^\lambda\rangle\phi_{ij}^\lambda$. Consequently, we have*

$$f = \sum_{\lambda \in \hat{G}} d_\lambda f * \chi_\lambda,$$

where χ_λ is the character of the representation π_λ. If $f, g \in L^2(\mathbb{G})$, then the above series for the function $f * g$ converges to $f * g$ uniformly.

Proof. Suppose dimension of π_λ is d_λ and let $g \in \mathbb{G}$, then

$$\begin{aligned}
f * \chi_\lambda &= \int_{\mathbb{G}} f(x)\chi_\lambda(x^{-1}g)d\mu(x) \\
&= \int_{\mathbb{G}} f(x)Tr(\pi_\lambda(x^{-1})\pi_\lambda(g))d\mu(x) \\
&= \int_{\mathbb{G}} f(x)\sum_{i,j=1}^{d_\lambda}\overline{\phi_{ij}^\lambda(x)}\phi_{ij}^\lambda(g)d\mu(x), \text{ since } \pi_\lambda(x^{-1}) = \pi_\lambda(h)^* \\
&= \sum_{i,j=1}^{d_\lambda}\langle f, \phi_{ij}^\lambda\rangle\phi_{ij}^\lambda.
\end{aligned}$$

□

PW3

For $f \in L^2(\mathbb{G})$ and $\lambda \in \hat{G}$, choose $\pi_\lambda \in \lambda$. Define, $f(\hat{\lambda}) = \int_{\mathbb{G}} f(g)\pi_\lambda(g^{-1})d\mu(g)$. Then $f(\hat{\lambda}) \in \mathcal{B}(H_\lambda)$ and $f(g) = \sum_{\lambda \in \hat{G}} d_\lambda Tr(f(\hat{\lambda})\pi_\lambda(g))$ and $\|f\|_{L^2(\mathbb{G})}^2 = \sum_{\lambda \in \hat{G}} d_\lambda \|\hat{f}\|_{HS}^2$, where $\|A\|_{HS} = Tr(AA^*)$.

Proof. It is easy to see that $f(\hat{\lambda})_{ij} = \langle f, \phi_{ji}^\lambda\rangle$. Now consider

$$f(g) = \sum_{\lambda \in \hat{G}} d_\lambda \sum_{i=1}^{d_\lambda} \sum_{j=1}^{d_\lambda} \langle f, \phi_{ij}^\lambda \rangle \phi_{ij}^\lambda(g) = \sum_{\lambda \in \hat{G}} d_\lambda \sum_{i=1}^{d_\lambda} \sum_{j=1}^{d_\lambda} f(\hat{\lambda})_{ji} \phi_{ij}^\lambda(g) = \sum_{\lambda \in \hat{G}} d_\lambda Tr(f(\hat{\lambda}) \pi_\lambda(g)).$$

Then $\|f\|_{L^2(\mathbb{G})}^2 = \sum_{\lambda \in \hat{G}} d_\lambda \sum_{i=1}^{d_\lambda} \sum_{j=1}^{d_\lambda} |f(\hat{\lambda})_{ji}|^2 = \sum_{\lambda \in \hat{G}} d_\lambda \|f(\hat{\lambda})\|_{HS}^2$. Hence we get the result. □

Define $L^2(\hat{G})$ as the space of function Φ from $\hat{G}$ to the disjoint union $\bigcup_{i=1}^{\infty} GL(n, \mathbb{C})$ that satisfies;

1. $\Phi(\lambda) \in GL(d_\lambda, \mathbb{C}) \ \forall \ \lambda \in \hat{G}$.
2. $\sum_{\lambda \in \hat{G}} d_\lambda \|\Phi(\lambda)\|_{HS}^2 < \infty$.

Then $L^2(\hat{G})$ is a Hilbert space with inner product

$$\langle \Phi, \Psi \rangle = \sum_{\lambda \in \hat{G}} d_\lambda Tr(\Phi(\lambda) \Psi(\lambda)^*).$$

PW4

The fourier transform $\mathcal{F} : f \to \hat{f}$ is an isometry from $L^2(\mathbb{G})$ onto $L^2(\mathbb{G})$.

Proof. We have already proved that $\mathcal{F}$ is an isometry in PW3.
Suppose $\Phi \in L^2(\hat{G})$. Then,

$$\sum_{\lambda \in \hat{G}} d_\lambda Tr(\Phi(\lambda) \pi_\lambda(g)) = \sum_{\lambda \in \hat{G}} d_\lambda \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi(\lambda)_{ij} \phi_{ji}^\lambda(g)$$

converges in $L^2(\mathbb{G})$. By definition, $\sum_{\lambda \in \hat{G}} d_\lambda \sum_{i=1}^{n} \sum_{j=1}^{n} |\Phi(\lambda)_{ij}|^2 < \infty$ and suppose it converges to $f \in L^2(\mathbb{G})$. Then by PW3 and Schur's orthogonality relations, we get $f(\hat{\lambda}) = \Phi(\lambda)$. □

Suppose $\mathcal{R}(G)$ be the linear space spanned by $\{\pi_{x,y} : dim(\pi) < \infty,\ and\ x, y \in H\}$. Elements belonging to $\mathcal{R}(G)$ are called *representative function*, and in view of *Proposition 3.3.2*, $\mathcal{R}(G)$ is spanned by the orthogonal family of function $\{\phi_{ij}^{\lambda} :\ \lambda \in \hat{G}, 1 \leq i, j \leq d_{\lambda}$.

PW5

$\mathcal{R}(G)$ is dense in $C(G)$, equipped with sup norm.

We skip the proof of PW5.

3.4 Irreducible Unitary Representation of $SU(2)$

Suppose $G = SU(2)$, the special unitary group consists of all matrices A of degree 2 satisfying $A^*A = I$ and $\det(A) = 1$. Suppose $C_2 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}\}$ is the vector space of all row vector, then $SU(2)$ act on C_2 from right, i.e if $A \in SU(2)$ and $z \in C_2$ then $z \longmapsto zA$ is group action.

Suppose S^3 is the unit sphere in $\mathbb{R}^4$. Then the map $\phi : S^3 \to SU(2)$ defined by,

$$\phi(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{bmatrix}, (x_1, x_2, x_3, x_4) \in S^3$$

is a homeomorphism of S^3 onto $SU(2)$. Let $(x_1, x_2, x_3, x_4) \in S^3$. Then we can write

$$x_1 = \cos\theta$$

$$x_2 = \sin\theta\cos\phi$$

$$x_3 = \sin\theta\sin\phi\cos\psi$$

$$x_4 = \sin\theta\sin\phi\sin\psi$$

where $0 \leq \theta, \phi \leq \pi$ and $0 \leq \psi \leq 2\pi$.

The normalised Haar measure of element dx on $SU(2)$ is given by

$$ds \;=\; \frac{1}{2\pi^2}\sin^2\theta\sin\phi d\theta d\phi d\psi.$$

For each $n \geq 0$, let

$$H_n = \{f(z_1, z_2) = \sum_{k=0}^{n} \alpha_k z_1^k z_2^{n-k} : \alpha_k \in \mathbb{C}\}.$$

Then H_n is vector space and the set of monomial $\phi_k(z_1, z_2) = z_1^k z_2^{n-k}$, $0 \leq k \leq n$, form a basis for H_n and hence dimension of H_n is $n+1$. Define an inner product $\langle , \rangle$ on H_n by $\langle \phi_k, \phi_j \rangle = 0 \; if \; j \neq k$ and
$\langle \phi_k, \phi_k \rangle = k!(n-k)!$, otherwise.
Then, for any $g, h \in H_n$, $g(z) = \sum_{k=0}^{n} \alpha_k z_1^k z_2^{n-k}$ and $h(z) = \sum_{k=0}^{n} \beta_k z_1^k z_2^{n-k}$,
$\langle g, h \rangle = \sum_{k=0}^{n} k!(n-k)! \alpha_k \overline{\beta_k}$.

Definition 3.4.1. For every $n \geq 0$, suppose H_n is defined as above. If

$g \in SU(2)$, define a linear map $\pi_n(g)$ on H_n by,

$$\pi_n(g)f(z) = f(zg),\ f \in H_n$$

Then the map $g \longmapsto \pi_n(g)$ from $SU(2)$ to $\mathcal{B}(H_n)$ is a representation of $SU(2)$.

Proposition 3.4.2. *For each $n \geq 0$, the representation (π_n, H_n) of $SU(2)$ is an irreducible unitary representation.*

Proof. We will only prove the unitary part. To show (π_n, H_n) is unitary, consider the subset

$$U = \{\phi_a :\ \phi_a(z) = (za)^n,\ a \in \mathbb{C}^2\} \subseteq H_n.$$

Then, $\pi_n(g)\phi_a(z) = \phi(zg) = (zga)^n = \phi_{ga}(z)$, So if $a, b \in \mathbb{C}^2$, then,

$$\langle \pi_n(g)\phi_a, \pi_n(g)\phi_b \rangle = \langle \phi_{ga}, \phi_{gb} \rangle, \tag{3.2}$$

and $\langle \phi_a, \phi_b \rangle = n!\langle a, b\rangle^n$, which in turn implies that

$$\langle \pi_n(g)\phi_a, \pi_n(g)\phi_b \rangle = n!\langle ga, gb\rangle^n = n!\langle a, b\rangle^n = \langle \phi_a, \phi_b \rangle.$$

So it remains to prove U contains a basis for H_n. Suppose ω be the primitive n^{th} root of unity. Then the following $n+1$ polynomials $(z_1 + \omega^k z_2)^n$, where $0 \leq k \leq n-1$, and z_2^n are linearly independent and contained in U, hence π_n is a unitary representation. $\square$

Remark 3.4.3. Suppose (π, H) be an irreducible unitary representation of $SU(2)$. Then π is equivalent to (π_n, H_n) for some n.

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