MA 101 (Mathematics I)

Practice Problem Set - 2

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1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) If (x_n) is a sequence in $\mathbb{R}$ which converges to 0, then the sequence (x_n^n) must converge to 0.
   (b) There exists a non-convergent sequence (x_n) in $\mathbb{R}$ such that the sequence $(x_n + \frac{1}{n}x_n)$ is convergent.
   (c) There exists a non-convergent sequence (x_n) in $\mathbb{R}$ such that the sequence $(x_n^2 + \frac{1}{n}x_n)$ is convergent.
   (d) If (x_n) is a sequence of positive real numbers such that the sequence $((-1)^n x_n)$ converges to $\ell \in \mathbb{R}$, then ℓ must be equal to 0.
   (e) If an increasing sequence (x_n) in $\mathbb{R}$ has a convergent subsequence, then (x_n) must be convergent.
   (f) If (x_n) is a sequence of positive real numbers such that $\lim\limits_{n\to\infty}(n^{\frac{3}{2}}x_n) = \frac{3}{2}$, then the series $\sum\limits_{n=1}^{\infty} x_n$ must be convergent.
   (g) If (x_n) is a sequence of positive real numbers such that the series $\sum\limits_{n=1}^{\infty} n^2x_n^2$ converges, then the series $\sum\limits_{n=1}^{\infty} x_n$ must converge.
   (h) If (x_n) is a sequence in $\mathbb{R}$ such that the series $\sum\limits_{n=1}^{\infty} x_n^3$ is convergent, then the series $\sum\limits_{n=1}^{\infty} x_n^4$ must be convergent.
   (i) If (x_n) is a sequence of positive real numbers such that the series $\sum\limits_{n=1}^{\infty} x_n^3$ is convergent, then the series $\sum\limits_{n=1}^{\infty} x_n^4$ must be convergent.
   (j) If (x_n) is a sequence of positive real numbers such that the series $\sum\limits_{n=1}^{\infty} x_n^4$ is convergent, then the series $\sum\limits_{n=1}^{\infty} x_n^3$ must be convergent.
   (k) If $f : \mathbb{R} \to \mathbb{R}$ is continuous at both 2 and 4, then f must be continuous at some $c \in (2, 4)$.
   (l) There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$ and $f(x) \in \mathbb{R} \setminus \mathbb{Q}$ for all $x \in \mathbb{Q}$.
   (m) If $f : [1, 2] \to \mathbb{R}$ is a differentiable function, then the derivative f' must be bounded on $[1, 2]$.
   (n) If $f : [0, \infty) \to \mathbb{R}$ is differentiable such that $f(0) = 0 = \lim\limits_{x\to\infty} f(x)$, then there must exist $c \in (0, \infty)$ such that $f'(c) = 0$.
   (o) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then for each $c \in \mathbb{R}$, there must exist $a, b \in \mathbb{R}$ with $a < c < b$ such that $f(b) - f(a) = (b - a)f'(c)$.
   (p) The function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x + \sin x$ for all $x \in \mathbb{R}$, is strictly increasing on $\mathbb{R}$.
   (q) There exists an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{(n)}(0) = n^3 - 5n + 2$ for all $n \geq 0$.
   (r) If $f : [0, 1] \to \mathbb{R}$ is a bounded function such that $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} f(\frac{k}{n})$ exists (in $\mathbb{R}$), then f must be Riemann integrable on $[0, 1]$.

2. Examine whether the sequences (x_n) defined as below are convergent. Also, find their limits if they are convergent.
   (a) $x_n = \frac{1}{n^2}(a_1 + \cdots + a_n)$, where $a_n = n + \frac{1}{n}$ for all $n \in \mathbb{N}$.

(b) $x_n = (n^2+1)^{\frac{1}{8}} - (n+1)^{\frac{1}{4}}$ for all $n \in \mathbb{N}$.
(c) $x_n = (n^2+n)^{\frac{1}{n}}$ for all $n \in \mathbb{N}$.
(d) $x_n = 5^n \left(\frac{1}{n^3} - \frac{1}{n!}\right)$ for all $n \in \mathbb{N}$.
(e) $x_n = \frac{1}{1.n} + \frac{1}{2.(n-1)} + \frac{1}{3.(n-2)} + \cdots + \frac{1}{n.1}$ for all $n \in \mathbb{N}$.
(f) $x_n = \frac{n}{3} - [\frac{n}{3}]$ for all $n \in \mathbb{N}$.
(g) $x_1 = 1$ and $x_{n+1} = (\frac{n}{n+1})x_n^2$ for all $n \in \mathbb{N}$.
(h) $x_1 = a$, $x_2 = b$ and $x_{n+2} = \frac{1}{2}(x_n + x_{n+1})$ for all $n \in \mathbb{N}$, where $a, b \in \mathbb{R}$.
(i) $0 < x_n < 1$ and $x_n(1 - x_{n+1}) > \frac{1}{4}$ for all $n \in \mathbb{N}$.

3. Let (x_n) be any non-constant sequence in $\mathbb{R}$ such that $x_{n+1} = \frac{1}{2}(x_n + x_{n+2})$ for all $n \in \mathbb{N}$. Show that (x_n) cannot converge.

4. Let (x_n) be a sequence in $\mathbb{R}$ and let $y_n = \frac{1}{n}(x_1 + \cdots + x_n)$ for all $n \in \mathbb{N}$. If (x_n) is convergent, then show that (y_n) is also convergent.
If (y_n) is convergent, is it necessary that (x_n) is (i) convergent? (ii) bounded?

5. If (x_n) is a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty}(x_{n+1} - x_n) = 5$, then determine $\lim_{n\to\infty} \frac{x_n}{n}$.

6. If $x_1 = \frac{3}{4}$ and $x_{n+1} = x_n - x_n^{n+1}$ for all $n \in \mathbb{N}$, then examine whether the sequence (x_n) is convergent.

7. Let $a > 0$ and let $x_1 = 0$, $x_{n+1} = x_n^2 + a$ for all $n \in \mathbb{N}$. Show that the sequence (x_n) is convergent iff $a \le \frac{1}{4}$.

8. For $a \in \mathbb{R}$, let $x_1 = a$ and $x_{n+1} = \frac{1}{4}(x_n^2 + 3)$ for all $n \in \mathbb{N}$. Examine the convergence of the sequence (x_n) for different values of a. Also, find $\lim_{n\to\infty} x_n$ whenever it exists.

9. If $x_n = (1 + \frac{1}{n})^n$ and $y_n = (1 + \frac{1}{n})^{n+1}$ for all $n \in \mathbb{N}$, then show that the sequence (x_n) is increasing, the sequence (y_n) is decreasing and both (x_n) and (y_n) are bounded.

10. Let (x_n) be a sequence in $\mathbb{R}$. If for every $\varepsilon > 0$, there exists a convergent sequence (y_n) in $\mathbb{R}$ such that $|x_n - y_n| < \varepsilon$ for all $n \in \mathbb{N}$, then show that (x_n) is convergent.

11. Let (x_n) be a sequence in $\mathbb{R}$. Which of the following conditions ensure(s) that (x_n) is a Cauchy sequence (and hence convergent)?
(a) $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$.
(b) $|x_{n+1} - x_n| \le \frac{1}{n}$ for all $n \in \mathbb{N}$.
(c) $|x_{n+1} - x_n| \le \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

12. Let (x_n) be a sequence in $\mathbb{R}$ such that each of the subsequences (x_{2n}), (x_{2n-1}) and (x_{3n}) converges. Show that (x_n) is convergent.

13. Examine whether the following series are convergent.
(a) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$
(b) $\sum_{n=1}^{\infty} \frac{2^n - n}{n^2}$
(c) $\sum_{n=1}^{\infty} \frac{\frac{1}{2} + (-1)^n}{n}$
(d) $\frac{1}{\sqrt{1}} - \frac{1}{2} + \frac{1}{\sqrt{3}} - \frac{1}{4} + \frac{1}{\sqrt{5}} - \frac{1}{6} + \cdots$

(e) $1+2x+x^2+2x^3+x^4+2x^5+x^6+2x^7+\cdots$, where $x \in \mathbb{R}$

14. If (x_n) is a sequence in $\mathbb{R}$ such that $\lim\limits_{n\to\infty} x_n = 0$, then show that the series $\sum\limits_{n=1}^{\infty} \frac{x_n}{x_n^2+n^2}$ is absolutely convergent.

15. Let the series $\sum\limits_{n=1}^{\infty} x_n$ be convergent, where $x_n > 0$ for all $n \in \mathbb{N}$. Examine whether the following series are convergent.
(a) $\sum\limits_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$
(b) $\sum\limits_{n=1}^{\infty} \frac{x_n+2^n}{x_n+3^n}$

16. If $\sum\limits_{n=1}^{\infty} x_n$ is a convergent series, where $x_n > 0$ for all $n \in \mathbb{N}$, then show that it is possible for the series $\sum\limits_{n=1}^{\infty} \sqrt{\frac{x_n}{n}}$ to converge as well as not to converge.

17. Let (x_n) be a sequence in $\mathbb{R}$ with $\lim\limits_{n\to\infty} x_n = 0$. Show that there exists a subsequence (x_{n_k}) of (x_n) such that the series $\sum\limits_{k=1}^{\infty} x_{n_k}$ is absolutely convergent.

18. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then show that there exist non-negative continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f = g - h$.

19. Give an example (with justification) of a function from $\mathbb{R}$ onto $\mathbb{R}$ which is not continuous at any point of $\mathbb{R}$.

20. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, then show that $f(x) = f(1)x$ for all $x \in \mathbb{R}$.

21. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(\frac{1}{2}(x+y)) = \frac{1}{2}(f(x)+f(y))$ for all $x, y \in \mathbb{R}$. Show that there exist $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ for all $x \in \mathbb{R}$.

22. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that for each $x \in \mathbb{Q}$, $f(x)$ is an integer. If $f(\frac{1}{2}) = 2$, then find $f(\frac{1}{3})$.

23. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Show that f is a constant function.

24. If $f : [0,1] \to \mathbb{R}$ is continuous, then show that
(a) there exist $a, b \in [0,1]$ such that $a - b = \frac{1}{2}$ and $f(a) - f(b) = \frac{1}{2}(f(1) - f(0))$.
(b) there exist $a, b \in [0,1]$ such that $a - b = \frac{1}{3}$ and $f(a) - f(b) = \frac{1}{3}(f(1) - f(0))$.

25. Let $f : [a,b] \to \mathbb{R}$ be continuous. For $n \in \mathbb{N}$, let $x_1, ..., x_n \in [a,b]$ and let $\alpha_1, ..., \alpha_n$ be nonzero real numbers having same sign. Show that there exists $c \in [a,b]$ such that
$f(c) \sum\limits_{i=1}^{n} \alpha_i = \sum\limits_{i=1}^{n} \alpha_i f(x_i)$.
(In particular, this shows that if $f : [a,b] \to \mathbb{R}$ is continuous and if for $n \in \mathbb{N}$, $x_1, ..., x_n \in [a,b]$, then there exists $\xi \in [a,b]$ such that $f(\xi) = \frac{1}{n}(f(x_1) + \cdots + f(x_n))$.)

26. Let $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ be continuous such that $\sup\{f(x) : x \in [0, 1]\} = \sup\{g(x) : x \in [0, 1]\}$. Show that there exists $c \in [0, 1]$ such that $f(c) = g(c)$.

27. Let $f : (0, \infty) \to \mathbb{R}$ be continuous such that $\lim\limits_{x\to 0+} f(x) = 0$ and $\lim\limits_{x\to\infty} f(x) = 1$. Show that there exists $c \in (0, \infty)$ such that $f(c) = \frac{\sqrt{3}}{2}$.

28. Let $f : (a, b) \to \mathbb{R}$ be continuous. If both $\lim\limits_{x\to a+} f(x)$ and $\lim\limits_{x\to b-} f(x)$ exist (in $\mathbb{R}$), then show that f is bounded.

29. Consider the continuous function $f : (0, 1] \to \mathbb{R}$, where $f(x) = 1-(1-x)\sin\frac{1}{x}$ for all $x \in (0, 1]$. Does there exist $x_0 \in (0, 1]$ such that $f(x_0) = \sup\{f(x) : x \in (0, 1]\}$? Justify.

30. Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f(a) = f(b)$. Show that for each $\varepsilon > 0$, there exist distinct $x, y \in [a, b]$ such that $|x - y| < \varepsilon$ and $f(x) = f(y)$.

31. Give an example (with justification) of a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable only at 2.

32. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) - f(y) \le (x - y)^2$ for all $x, y \in \mathbb{R}$. Show that f is a constant function.

33. If $m, k \in \mathbb{N}$, then evaluate $\lim\limits_{n\to\infty} \left(\frac{(n+1)^m+(n+2)^m+\cdots+(n+k)^m}{n^{m-1}} - kn\right)$.

34. Let $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$ such that $f(c) = g(c)$ and $f(x) \le g(x)$ for all $x \in (a, b)$. Show that $f'(c) = g'(c)$.

35. Let $f : [0, 1] \to \mathbb{R}$ be differentiable such that $f(0) = f(1) = 0$. Show that there exists $c \in (0, 1)$ such that $f'(c) = f(c)$.

36. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(0) = 0$ and $f'(x) > f(x)$ for all $x \in \mathbb{R}$. Show that $f(x) > 0$ for all $x > 0$.

37. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $f(x) \neq 0$ for all $x \in [a, b]$. Show that there exists $c \in (a, b)$ such that $\frac{f'(c)}{f(c)} = \frac{1}{a-c} + \frac{1}{b-c}$.

38. Let $f : [0, 1] \to \mathbb{R}$ be differentiable such that $f(0) = 0$ and $f(1) = 1$. Show that there exist $c_1, c_2 \in [0, 1]$ with $c_1 \neq c_2$ such that $f'(c_1) + f'(c_2) = 2$.

39. Show that for each $a \in (0, 1)$ and for each $b \in \mathbb{R}$, the equation $a \sin x + b = x$ has a unique root in $\mathbb{R}$.

40. Find the number of (distinct) real roots of the following equations.
(a) $3^x + 4^x = 5^x$
(b) $x^{13} + 7x^3 - 5 = 0$

41. Show that for each $n \in \mathbb{N}$, the equation $x^n + x - 1 = 0$ has a unique root in $[0, 1]$.
If for each $n \in \mathbb{N}$, x_n denotes this root, then show that the sequence (x_n) converges to 1.

42. Let $f : (0, 1) \to \mathbb{R}$ be differentiable and let $|f'(x)| \le 3$ for all $x \in (0, 1)$. Show that the sequence $(f(\frac{1}{n+1}))$ converges.

43. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $\lim_{x\to\infty} f'(x) = 1$. Show that f is unbounded.

44. Let $f : [a, b] \to \mathbb{R}$ be twice differentiable and let $f(a) = f(b) = 0$ and $f(c) > 0$, where $c \in (a, b)$. Show that there exists $\xi \in (a, b)$ such that $f''(\xi) < 0$.

45. If $f : [0, 4] \to \mathbb{R}$ is differentiable, then show that there exists $c \in [0, 4]$ such that $f'(c) = \frac{1}{6}(f'(1) + 2f'(2) + 3f'(3))$.

46. Let $f(x) = \begin{cases} x & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q}). \end{cases}$

    Examine whether f is Riemann integrable on $[0, 1]$. Also, find $\int_0^1 f$, if it exists (in $\mathbb{R}$).

47. If $f : [0, 1] \to \mathbb{R}$ is Riemann integrable, then find $\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx$.

48. If $f : [0, 2\pi] \to \mathbb{R}$ is continuous such that $\int_0^{\frac{\pi}{2}} f(x)\, dx = 0$, then show that there exists $c \in (0, \frac{\pi}{2})$ such that $f(c) = 2\cos 2c$.

49. Prove that for each $a \geq 0$, there exists a unique $b \geq 0$ such that $a = \int_0^b \frac{1}{(1+x^3)^{1/5}}\, dx$.

50. Show that there exists a positive real number α such that $\int_0^\pi x^\alpha \sin x\, dx = 3$.

51. Determine all real values of p for which the integral $\int_0^\infty \frac{e^{-x}-1}{x^p}\, dx$ is convergent.