A Project Report Submitted

in Partial Fulfilment of the Requirements

for the Degree of

MASTER OF SCIENCE

in

Mathematics and Computing

by

MONABBAR HOSSAIN

(Roll No. 11212323)

to the

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

GUWAHATI - 781039, INDIA

April 2013

CERTIFICATE

This is to certify that the work contained in this report entitled **"RIESZ-THORIN INTERPOLATION"** submitted by **MONABBAR HOSSAIN** (**Roll No:11212323**) to Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him under my supervision.

Guwahati - 781 039 — Dr. RAJESH KUMAR SRIVASTAVA

April 2013 — Project Supervisor

ABSTRACT

The main aim of the project to understand the the Riesz-Thorin interpolation theorem. This is a result about interpolation of bounded linear operators. This theorem can be used, for example, to prove the Hausdorff-Young inequality, which establishes that the Fourier transform can be extended in a unique way as a continuous linear map from L^p to L^q, where p and q are conjugate integers and $1 \leq p \leq 2$.

Contents

Chapter 1

INTRODUCTION

Before discuss Riesz-Thorin interpolation theorem, we discuss the important results and ideas which help us to understand it. We shall start with measure theory,then discuss some basic of dual of L^p space and after that we shall discuss Riesz-Thorin interpolation theorem.

Chapter 2

MEASURE THEORY AND LIMIT THEOREMS

2.1 Introduction

In this chapter, our main aim is to establish the limits theorems for integral. However, we take a quick review of some important basic notion, definitions and results measure theory.

In the very initial section, we shall define Measure and Outer measure and distinguish them by giving examples. Then we go to measurable function. After that we precisely establish the notion of integration and finally we end the chapter by discussing two limit theorems: monotone convergence theorem and Fatou's lemma.

2.2 σ-Algebra

Suppose X be an arbitary set and $\mathcal{A}$ be a collection of subsets of X. Then $\mathcal{A}$ is said to be σ-algebra if the following conditions holds:

1. $X \in \mathcal{A}$
2. $A \in \mathcal{A} \rightarrow A^c \in \mathcal{A}$
3. If $A_1, A_2, \ldots$ be a sequence in $\mathcal{A}$. Then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

2.2.1 Examples.

Let X be an uncountable set. Then $\mathcal{A} = \{A \subseteq X\colon\ A \text{ or } A^c \text{ is countable}\}$ Then $\mathcal{A}$ is a σ-algebra.

2.3 Measures

Let $\mathcal{A}$ be a σ- algebra on X. A measure $\mu : \mathcal{A} \rightarrow [0, +\infty]$ is a function such that

- $\mu(\phi) = 0$ and
- $\mu(\bigcup_{i=1}^{n} A_i) = \sum_1^n \mu(A_i)$, for any disjoint sequence $\{A_1, A_2, ...A_n\}$ in $\mathcal{A}$.

Hence measure is countably additive.

2.3.1 Measure space

Let $\mathcal{A}$ be a σ-algebra on X and μ be a measure on X. The triplet $(X, \mathcal{A}, \mu)$ is called measure space. If from the context there is no ambiguity about the σ-algebra, we simply say μ is a measure on X.

Examples.

Suppose $(X, \mathcal{A}, \mu)$ is a measure space.

1. $$\mu(A) = \begin{cases} 0 & \text{if } A = \phi \\ \infty & \text{if } A \neq \phi \ , \end{cases} \tag{2.1}$$

   where μ is a measure on X.

2. $$\mu(A) = \begin{cases} 0 & \text{if } A = \phi \\ 1 & \text{if } A \neq \phi \ , \end{cases} \tag{2.2}$$

   where μ is not a measure on X, since countable additivity fails to hold.

2.3.2 Outer Measures

Let X be a nonempty set. An outer measure $\mu^* : 2^X \longrightarrow [0, +\infty]$ is function such that

1. $\mu(\phi) = 0$
2. if $A \subseteq B \subseteq X$,then $\mu^*(A) \leq \mu^*(B)$

3. $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_1^{\infty} \mu(A_i)$, for each sequence $\{A_1, A_2, \ldots\}$ in 2^X.

Thus an outer measure is monotone and countably subadditive. Notice that an outer measure is countably subadditive and may not be countable additive, whereas measure must be countably additive. So an outer measure may not be a measure. But since countably additivity implies monotonocity and countably subadditivity, a measure is an outer measure if and only if domain of the measure is 2^X.

Examples.

suppose $(X, \mathcal{A}, \mu)$ is a measure space and $\mu : \mathcal{A} \longrightarrow [0, +\infty]$ is a function. Let us consider $\mathcal{A} = 2^X$.

1.

$$\mu(A) = \begin{cases} 0 & \text{if } A = \phi \\ \infty & \text{if } A \neq \phi \ , \end{cases} \tag{2.3}$$

where μ is a measure,as well as an outer measure X.

2.

$$\mu(A) = \begin{cases} 0 & \text{if } A = \phi \\ 1 & \text{if } A \neq \phi \end{cases} \tag{2.4}$$

Here μ is an outer measure X. But not a measure since countable additive fails.

3.

$$\mu(A) = \begin{cases} 0 & \text{if A is finite} \\ 1 & \text{if A is not finite} \end{cases} \tag{2.5}$$

Here μ is neither an outer measure nor a measure since countable sub-additivity fails.

4. Consider $\mathcal{A} \neq 2^X$

$$\mu(A) = \begin{cases} 0 & \text{if } A = \phi \\ \infty & \text{if } A \neq \phi \end{cases} \tag{2.6}$$

Here μ is a measure, but not an outer measure since domain of μ is a proper subset of 2^X.

2.3.3 Lebesgue outer measures on $\mathbb{R}^d$

Suppose $A \subseteq \mathbb{R}^d$ and $\mathcal{C}_A$ be the collection of all sequence (R_i) of bounded and open $d-\mathit{dimensional}$ intervals such that $A \subseteq \bigcup_{i=1}^{\infty} R_i$. Then Lebesgue outer measure of A is denoted by $\lambda^*(A)$ and defined by the infimum of the set $\{\sum_{i=1}^{\infty} vol(R_i) : (R_i) \in \mathcal{C}_A\}$.

For $d = 1$, we take $A \subseteq \mathbb{R}$,subintervals and $(R_i) = (a_i, b_i)$, bounded and open subintervals of $\mathbb{R}$. So

$$\lambda^*(A) = \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) : (R_i) \in \mathcal{C}_A\right\}.$$

We can verify that Lebesgue outer measures satisfies all the property of outer measure as well as measure.

2.3.4 Finite, σ-finite, semi-finite measures

Suppose $(X, \mathcal{A}, \mu)$ is a measure space.

1. μ is said to be finite if $\mu(X) < +\infty$
2. μ is said to be σ- finite if $A_1, A_2, \ldots$ be a sequence in $\mathcal{A}$ and $\bigcup_{i=1}^{\infty} A_i = X$ such that for $\mu(A_i) < +\infty$, for each A_i.

Note that if μ σ-finite, we can choose a disjoint sequence $B_1, B_2, \ldots$ in $\mathcal{A}$ such that $\bigcup_{i=1}^{\infty} B_i = X$ and $\mu(B_i) < +\infty$. If $A_1, A_2, \ldots$ is a sequence in $\mathcal{A}$ such that $\bigcup_{i=1}^{\infty} A_i = X$ and $\mu(A_i) < +\infty$, for each A_i, then consider the sequence (B_n) given by $B_1 = A_1$ and $B_n = A_n - \bigcup_{i=1}^{n-1} A_i$.

A measure μ is said to be semi-finite if for each such A, $\mu(A) = \infty$, there exists a set $B \subseteq A$ such that $\mu(B) < \infty$.

Examples.

1. $X = \{1,2,3\}$, $\mathcal{A}$=$\{\phi, X, \{1\}, \{2,3\}\}$

$$\mu(A) = \begin{cases} n & \text{if A has n elements} \\ \infty & \text{if A is infinte,} \end{cases} \tag{2.7}$$

where μ is finite since $\mu(X) = 3 < \infty$.

2. $X = \mathbb{R}$, $\mathcal{A} = 2^X$ and $\mu = \lambda^*$, the Lebesgue outer measures. Write $A_n = (-n, -n+1] \bigcup [n-1, n)$. Here μ is both σ-finite and semi-finite measures but not finite.

3. $X = \mathbb{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$, the borel set on $\mathbb{R}$ and μ=counting measures defined by

$$\mu(A) = \begin{cases} n & \text{if A has n elements} \\ \infty & \text{if A is infinte,} \end{cases} \tag{2.8}$$

Here μ neither of finite, σ-finite or semi-finite measures.

2.3.5 μ Almost Everywhere

Let $(X, \mathcal{A}, \mu)$ be measure space.Let P be a property of points of X. Let F be the set of all points of X at each of P fails to hold. Then P said to be $\mu - \mathit{Almost}$ Everywhere in X if $\mu(F) = 0$.

If the measure is clearly understood, we simply say P is almost everywhere (a.e.) in x.

Examples.

For $X = \mathbb{R}$, $\mathcal{A} = 2^{\mathbb{R}}$ and $\mu = \lambda^*$, the Lebesgue outer measures

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{R} - \mathbb{N} \\ x & \text{if } x \in \mathbb{N} \end{cases}$$

Here $f(x) = x^2$ $a.e.$ on $\mathbb{R}$.

2.4 Measurable set

Suppose $\mu^* : 2^X \longrightarrow [0, +\infty]$ is an outer measure on X. A subset A of X is said to be μ^*- measurable if for each $B \subseteq X$,

$$\mu^*(B) = \mu^*(A \cap B) + \mu^*(A \cap B^c).$$

$i.e.$, A is measurable if the sum of measures of the portion of B bisected by A returned back the measure of B, for each $B \subseteq X$.

Example. $X = \mathbb{R}$, $\mathcal{A} = 2^X$ and $\mu = \lambda^*$. Then $A = (a, b)$, open subinterval of $\mathbb{R}$ is measurable.

2.5 Simple functions

Let X be a set and s be real-valued function defined on X.If the range of s is finite, we say s is a simple function. Let $E \subset X$. Define

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E, \end{cases} \tag{2.9}$$

χ_E is called the characteristic function of E. Let the range of s consists of the n distinct numbers $a_1, a_2, \ldots, a_n$. Let $E_i = \{x : s(x) = a_i\}, i = 1, 2, \ldots, n$. Then $s = \sum_{i=1}^{n} a_i \chi_{E_i}$, i.e., every simple function is a finite linear combination of characteristic functions.

Theorem 2.5.1. *Let $(X, \mathcal{A}, \mu)$ be measure space and $A \subseteq X$ such that $A \in \mathcal{A}$. Suppose $f : A \longrightarrow [0, +\infty]$ be a $\mathcal{A}$- measurable function.Then exist a sequence of $[0, \infty)$ $\mathcal{A}$- measurable simple functions (s_n) such that*

1. $s_1(x) \leq s_2(x) \leq \cdots$,
2. $f(x) = \lim_{n\to\infty} s_n(x)$, *for all* $x \in A$.

Proof. Define

$$E_{n,i} = \left\{ x : \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \right\},$$

and

$$F_n = \{x : f(x) \geq n\}$$

for all $n = 1, 2, \ldots$ and $i = 1, 2, \ldots, n2^n$. Now we define a sequence (s_n) of functions from A by

$$s_n(x) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}}(x) + n\chi_{F_n}(x).$$

For each n, the function s_n, defined above are simple and satisfy conditions 1 and 2. This completes the proof. $\square$

Corollary 2.5.2. *The result also holds for if the function f is of $[-\infty, +\infty]$ valued instead of $[0, +\infty]$ valued function.In this case decompose f as $f = f^+ - f^-, where f^+ = \max\{f, 0\}, f^- = -\min\{f, 0\}$ and apply the the preceding construction to f^+ and f^- separately since then f^+ and f^- both are $[0, +\infty]$ valued function.*

2.6 Integral

Let $(X, \mathcal{A}, \mu)$ be a measure space and Σ_X^+ be the collection of all non-negative simple function in X. Let $f \in \Sigma_X^+$, then we can write

$$f = \sum_{i=1}^{n} a_i \chi_{A_i},$$

where $a_1, a_2, ..., a_n$ are non-negative real numbers and $A_1, A_2, \ldots, A_n$ are disjoint subsets of X that belong to $\mathcal{A}$. We denote the integral of f on X by $\int_X f d\mu$ and defined by

$$\int_X f d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

Lemma 2.6.1. *Let $(X, \mathcal{A}, \mu)$ be measure space and $f, g \in \Sigma_X^+$ suich that*

$f(x) \leq g(x)$ holds at each x in X. Then

$$\int_X f d\mu \leq \int_X g d\mu.$$

Proof. Clearly $g - f \in \Sigma_X^+$ and $g = f + (g - f)$. It follows that

$$\int_X g d\mu = \int_X f d\mu + \int_X (g - f) d\mu \geq \int_X f d\mu$$

This completes the proof. $\square$

Theorem 2.6.2. *Let $(X, \mathcal{A}, \mu)$ be measure space. and $f \in \Sigma_X^+$ be a function. Let (f_n) be a sequence of $[0, \infty]$-valued $\mathcal{A}$-measurable simple functions on X such that*

1. $f_1(x) \leq f_2(x) \leq \cdots ,$

2. $f(x) = \lim f_n(x)$, *at each x in X. Then*

$$\int_X f d\mu = lim \int_X f_n d\mu.$$

Proof. By Lemma 2.6.1, it follows that

$$\int_X f_1 d\mu \leq \int_X f_2 d\mu \leq \cdots \leq \int_X f d\mu.$$

Hence $\lim \int_X f_n d\mu$ exists and satisfies

$$\lim \int_X f_n d\mu \leq \int_X f d\mu.$$

Since $f \in \Sigma_X^+$, we can write $f = \sum_{i=1}^k a_i \chi_{A_i}$, where $a_1, a_2, ..., a_k$ are non-negative real numbers and $A_1, A_2, ..., A_k$ are disjoint subsets of X that belong $\mathcal{A}$. Let$\epsilon > 0$ and $A_{n,i} = \{x \in A_i : f_n(x) \geq (1-\epsilon)a_i\}$, for each n. For each n, define $g_n = \sum_{i=1}^{k}(1-\epsilon)a_i\chi_{A_{n,i}}$. Then $g_n \leq f_n$ for each n and

$$\int_X g_n d\mu = (1-\epsilon)\int_X f d\mu.$$

Since this holds for each $\epsilon > 0$, therefore, $\int_X g_n d\mu = \int_X f d\mu$. Hence the result follows. ☐

Definition 2.6.3. Let $(X, \mathcal{A}, \mu)$ be measure space. We define the integral of an arbitrary $[0, \infty]$-valued $\mathcal{A}$- measurable f on X as

$$\int_X f d\mu = \sup\left\{\int_X g d\mu : g \in \Sigma_X^+ \text{ and } g \leq f\right\}. \tag{2.10}$$

We can define the integral of an arbitrary $[-\infty, +\infty]$- valued $\mathcal{A}$- measurable f on X by decomposing f as $f = f^+ - f^-, where f^+ = \max\{f, 0\}, f^- = -\min\{f, 0\}$ and apply (2.10) to f^+ and f^- separately since then f^+ and f^- both are $[0, +\infty]$ valued function.

Lemma 2.6.4. *Let $(X, \mathcal{A}, \mu)$ be measure space and f, g are $[0, \infty]$-valued $\mathcal{A}$-measurable on X such that $f(x) \leq g(x)$ holds at each x in X. Then*

$$\int_X f d\mu \leq \int_X g d\mu.$$

Proof. Any class of functions h in Σ_X^+ that satisfy $h \leq f$ also satisfy $h \leq g$. Hence the above definition readily gives the result. ☐

Lemma 2.6.5. *Let $(X, \mathcal{A}, \mu)$ be measure space and f, g are $[-\infty, +\infty]$- valued $\mathcal{A}$- measurable function on X such that $f(x) \leq g(x)$ holds at each x in*

X. Then

$$\int_X f d\mu \leq \int_X g d\mu.$$

Proof. Clearly $g - f \geq 0$ and $g = f + (g - f)$. This implies that

$$\int_X g d\mu = \int_X f d\mu + \int_X (g - f) d\mu \geq \int_X f d\mu.$$

This completes the proof. $\square$

Theorem 2.6.6. *Let $(X, \mathcal{A}, \mu)$ be measure space and f be $[0, +\infty]$- valued $\mathcal{A}$- measurable function on X. Let (f_n) be a sequence of non-negative $\mathcal{A}$-measurable simple functions such that*

1. $f_1(x) \leq f_2(x) \leq \cdots,$

2. $f(x) = \lim f_n(x)$, *for each x in X. Then*

$$\int_X f d\mu = \lim \int_X f_n d\mu.$$

Proof. By Lemma 2.6.4, it follows that

$$\int_X f_1 d\mu \leq \int_X f_2 d\mu \leq \cdots \leq \int_X f d\mu.$$

Hence $\lim \int_X f_n d\mu$ exists and satisfies

$$\lim \int_X f_n d\mu \leq \int_X f d\mu.$$

Now we have to show the reverse inequality

$$\lim \int_X f_n d\mu \geq \int_X f d\mu.$$

Recalling the above definition of integral, it is enough to show that any $g \in \Sigma_X^+$ that satisfies $g \leq f$ also satisfies

$$\int_X g d\mu \leq \lim \int_X f_n d\mu.$$

Let any $g \in \Sigma_X^+$. Then $(\min\{g, f_n\})$ is a non-decreasing sequence of $[0, \infty)$-valued simple function such that $g = \lim_{n\to\infty} \min\{g, f_n\}$. By Theorem 2.6.2, we have $\int_X g d\mu = \lim_{n\to\infty} \int_X \min\{g, f_n\} d\mu$. Since $\int_X \min\{g, f_n\} \leq \int_X f_n d\mu$ for each n, it follows that

$$\int_X g d\mu \leq \lim_{n\to\infty} \int_X f_n d\mu.$$

This completes the proof. □

2.7 Limit theorems

2.7.1 Monotone convergence theorem

Theorem 2.7.1. *Let $(X, \mathcal{A}, \mu)$ be measure space.Let f and $f_1, f_2, \ldots$ are $[0, \infty]$-valued $\mathcal{A}$- measurable function on X s.t.*

1. $f_1(x) \leq f_2(x) \leq \cdots$,

2. $f(x) = \lim f_n(x)$, *hold at a.e. in* X.

Then

$$\int_X f d\mu = \lim \int_X f_n d\mu. \tag{2.11}$$

Proof. First we suppose that the conditions 1 and 2 hold at each x in X. By Lemma 2.6.4, it follows from the condition 1 that

$$\int_X f_1 d\mu \leq \int_X f_2 d\mu \leq \cdots \leq \int_X f d\mu$$

Hence $\lim \int_X f_n d\mu$ exists and satisfies

$$\lim \int_X f_n d\mu \leq \int_X f d\mu.$$

Now we need to show the reverse inequality

$$\lim \int_X f_n d\mu \geq \int_X f d\mu.$$

For each n, choose a non-decreasing sequence $(g_{n,k})_{k=1}^{\infty}$ in Σ_X^+ such that $f_n = \lim\limits_{k\to\infty} g_{n,k}$. For each n, write $h_n = \max\{g_{1,n}, g_{2,n}, \ldots, g_{n,n}\}$. Then (h_n) is a non-decreasing sequence of $[0, \infty]$- valued $\mathcal{A}$- measurable function on X such that $h_n \leq f_n$ and $f = \lim\limits_{k\to\infty} h_n$. Now by Lemma 2.6.4 and Theorem 2.6.6, it follows that

$$\int_X f d\mu = \lim \int_X h_n d\mu \leq \lim \int_X f_n d\mu.$$

This established the result when condition 1 and 2 at each x in X.

Now suppose that the conditions 1 and 2 hold a.e. x in X. Also suppose that the conditions 1 and 2 fail to hold at $N \in \mathcal{A}$. Then $\mu(N) = 0$. Since the conditions 1 and 2 hold at N^c, by the above argument, it follows that

$$\int_X f\chi_{N^c} d\mu = \lim \int_X f_n \chi_{N^c} d\mu.$$

Hence the result follows.

□

2.7.2 Fatou's Lemma

Let $(X, \mathcal{A}, \mu)$ be measure space.Let $f_1, f_2, ...$ are $[0, \infty]$-valued $\mathcal{A}$- measurable function on X. Then

$$\int_X \liminf \ f_n d\mu \leq \liminf \ \int_X f_n d\mu. \tag{2.12}$$

Proof. For each $k \geq 1$, we have $inf_{n\geq k} \ f_n \leq f_j$, for each $j \geq k$. This implies that

$$\int_X \inf_{n\geq k} \ f_n d\mu \leq \int_X f_j \ d\mu,$$

for each $j \geq k$. Hence we can write

$$\int_X \inf_{n\geq k} \ f_n d\mu \leq \inf_{j\geq k} \ \int_X f_j \ d\mu. \tag{2.13}$$

Now taking limit $k \to \infty$ of both the sides and apply monotone convergence theorem (see Theorem 2.7.1), we have

$$\int_X \lim_{k\to\infty} \inf_{n\geq k} f_n \ d\mu = \lim_{k\to\infty} \int_X \inf_{n\geq k} \ f_n d\mu$$

That is,

$$\int_X \liminf f_n d\mu \leq \lim_{k\longrightarrow\infty} \inf_{n\geq k} \int_X f_n d\mu = \liminf \int_X f_n d\mu$$

Hence the proof is completed. $\square$

Corollary 2.7.2. *If* $f_n \to f$ *then* $\liminf\ f_n = \lim\ f_n = f$. *Hence Fatou's Lemma 2.7.2 gives,*

$$\int_X f d\mu \leq \liminf\ \int_X f_n d\mu. \tag{2.14}$$

Chapter 3

Basic of L^p theory

3.1 Introduction

In this chapter we define the notion of norm of a function in dual of L^p and study some of its properties. We start with defining L^p space, then we quickly move to the dual space of L^p space and discuss some results of L^p space and on its dual.

3.2 Definition of L^p space

3.2.1 For $1 \leq p < \infty$

Let $(X, \mathcal{A}, \mu)$ is a measure space and $f : X \to \mathbb{C}$ is a measurable function. For $1 \leq p < \infty$, the space $L^p(X, \mathcal{A}, \mu)$ is a set defined by

$$L^p(X,\mathcal{A},\mu)=\left\{f:X\to\mathbb{C}:\text{f is measurable function and }\int_X |f|^p d\mu<\infty\right\}.$$

If $f\in L^p(X,\mathcal{A},\mu)$, we define L^p- norm of f by $\|f\|_{L^p(X,\mathcal{A},\mu)}=\left(\int_X |f(x)|^p\,\mathrm{d}\mu\right)^{1/p}$.
If there is no ambiguity about the measure space, we simply denote the L^p-norm of f by $\|f\|_p$ instead of $\|f\|_{L^p(X,\mathcal{A},\mu)}$

3.2.2 For $p=\infty$

Let $(X,\mathcal{A},\mu)$ is a measure space and $f:X\longrightarrow\mathbb{C}$ is a measurable function. Then we define the space $L^\infty(X,\mathcal{A},\mu)$ as

$$L^\infty(X,\mathcal{A},\mu)=\{f:X\to\mathbb{C}:\text{f is measurable function and }\exists M>0\text{ such that }|f(x)|<M\text{ a.e. on X}$$

If $f\in L^\infty(X,\mathcal{A},\mu)$, we define L^∞-norm to be the infimum of all such M in the above definition, i.e.,

$$\|f\|_{L^\infty(X,\mathcal{A},\mu)}=\inf\left\{M\geq 0:\mu(x:|f(x)|>M)=0\right\}.$$

$\|f\|_{L^\infty(X,\mathcal{A},\mu)}$ is also called essential supremum of f.

If there is no ambiguity about the measure space, we simply denote the L^∞-norm of f by $\|f\|_\infty$ instead of $\|f\|_{L^\infty(X,\mathcal{A},\mu)}$.

3.3 Results-I:

Conjugate Exponent:

Let $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.Then p, q are called conjugate exponent to each other.

Holder's Inequality:

R1:

For $A, B \geq 0$ and $0 \leq \theta \leq 1$, we have

$$A^{\theta} B^{1-\theta} \leq \theta A + (1-\theta) B \tag{3.1}$$

The equality holds if and only if $A = B$ for $0 < \theta < 1$. For $\theta = 0$ or 1, the equality holds trivially.

Proof. Assume that $B \neq 0$, otherwise the inequality readily follows. Replacing A by AB in equation (3.1), we get

$$\begin{aligned} A^{\theta} B^{1-\theta} &\leq \theta A + (1-\theta) B \\ A^{\theta} B^{\theta} B^{1-\theta} &\leq \theta AB + (1-\theta) B \\ A^{\theta} &\leq \theta A + (1-\theta). \end{aligned}$$

This implies that

$$A^{\theta} - \theta A - (1-\theta) \leq 0. \tag{3.2}$$

Let $f(x) = x^{\theta} - \theta x - (1-\theta)$

$$f^{'}(x) = \theta(x^{\theta-1} - 1) = \begin{cases} \geq 0 & \text{if } x \geq 1 \\ \leq 0 & \text{if } 0 \leq x \leq 1 \end{cases} \tag{3.3}$$

Therefore, $f(x)$ is increasing for $x \geq 1$ and decreasing for $0 \leq x \leq 1$, Hence $f(1)$ is maximum. But as $f(1) = 0, f(x) \leq 0, \forall x$. We can assume $A = x$. This completes the proof. Clearly for $A = B$, the equality holds.

□

R2: (Holder's Inequality)

Suppose $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (i.e. p, q are conjugate exponent). If $f \in L^p$ and $g \in L^q$,then $fg \in L^1$ and

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q} \tag{3.4}$$

The equality holds if and only if $\alpha|f|^p = \beta|g|^q$, some constant $\alpha, \beta \in \mathbb{C}$ with $\alpha\beta \neq 0$.

Proof. If we take $\theta = \frac{1}{p}$ then $1 - \theta = \frac{1}{q}$. Put $A = \frac{|f|^p}{\|f\|_{L^p}}$, $A = \frac{|f|^p}{\|f\|_{L^p}}$, $\theta = \frac{1}{p}$, $1 - \theta = \frac{1}{q}$ in equation (3.1), we have

$$\frac{|f|}{\|f\|_{L^p}}\frac{|g|}{\|g\|_{L^q}} \leq \frac{1}{p}\frac{|f|^p}{\|f\|^p_{L^p}} + \frac{1}{q}\frac{|g|^q}{\|g\|^q_{L^q}}. \tag{3.5}$$

Integrating both sides, we get

$$\frac{\|fg\|_{L^1}}{\|f\|_{L^p}\|g\|_{L^q}} \leq \frac{1}{p} + \frac{1}{q} = 1 \rightarrow \|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q} \tag{3.6}$$

Clearly the equality condition $A = B$ give rises to $\alpha|f|^p = \beta|g|^q$ and α, β are constants with $\alpha\beta \neq 0$. Hence the proof is completed. □

3.4 Linear functional

3.4.1 Definition

Let $(X, \|.\|)$ be a normed linear space over a field K. A linear functional on X is a linear mapping from X to K, i.e., $l : X \to K$ such that

$$l(\alpha x + y) = \alpha l(x) + l(y); \forall x, y \in x, \alpha \in K.$$

3.4.2 Norm of a linear functional

Let $(X\|.\ \|)$ be a normed linear space over a field K and $l : X \to K$ be linear functional on X. A linear functional l on X is called bounded if there exists a $M > 0$ such that $|l(x)| \leq M\|x\|, \forall x \in X$. The norm of a linear functional l on X is denoted by $\|l\|$ and defined as the infimum of all such $M > 0$ such that $|l(x)| \leq M\|x\|, \forall x \in X$. That is,

$$\|l\| = \inf \{M \geq 0 : |l(x)| \leq M\|x\|, \forall x \in X\}$$

It can be seen that

$$|l(x)| \leq \|l\|\|x\|, \forall x \in X.$$

This implies that

$$|l(x)| < \|l\|, \text{ for } \|x\| \leq 1.$$

$$\|l\| = \sup_{x \in X} \{|l(x)| : \|x\| \leq 1\} \tag{3.7}$$

3.5 Dual Space of a Normed Linear Space

Let $(X\|.\|)$ be a normed linear space over a field K. The collection of all bounded linear functionals on X is denoted by $BL(X,K)$. This space is a Banach space with operator norm. Norm of $f \in BL(X,K)$ can be expressed as

$$\|f\| = \sup_{x\in X} \{|f(x)| : \|x\| = 1\}$$

3.6 Dual of $L^p, 1 \leq p < \infty$

Let $(X, \mathcal{A}, \mu)$ be a measure space and p, q are conjugate exponent. For each $g \in L^q$, we define $\phi_g : L^p \to \mathbb{C}$ by

$$\phi_g(f) = \int_X fg \ \mathrm{d}\mu \tag{3.8}$$

Linearity of integration asserts that ϕ_g is linear. Hence ϕ_g is a linear functional and

$$\|\phi_g\| = \sup_{f\in L^p} \{|\phi_g(f)| : \|f\|_{L^p} = 1\} \tag{3.9}$$

By equation (3.4), we have $|\int_X fg \ \mathrm{d}\mu| = \|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}$. Therefore, ϕ_g is bounded.

See that for any $g \in L^q$, the map given by (3.9) is a bounded linear functional on L^p. It is of interest that can any bounded linear functional onL^p be expressed in the form (3.9). Riesz representation theorem gives a affirmative answer to it . Thus the linear functional of the form (3.8) are the all bounded linear functional on L^p. This collection of all such ϕ_g with norm given by (3.9) forms a normed linear space.This normed linear space is the

dual space of L^p and it is denoted by $(L^p)^*$.

$$(L^p)^* = \left\{ \phi_g : g \in L^q \text{ and } \phi_g(f) = \int_X fg \, \mathrm{d}\mu, \forall f \in L^p \right\}$$

Since each $g \in L^q$ gives rise a bounded linear functional in $(L^p)^*$ of the form (3.8) and to each bounded linear functional in $(L^p)^*$,there exists a $g \in L^q$ that satisfies (3.8),$(L^p)^*$ is isomorphic to L^q.

3.7 Results-II

R-3: Let $(X, \mathcal{A}, \mu)$ measure space.Suppose p, q be conjugate exponent with $1 \leq q < \infty$. If $g \in L^q$,then

$$\|g\|_q = \|\phi_g\| = \sup_{f \in L^p} \{|\phi_g(f)| : \|f\|_{L^p} = 1\} \tag{3.10}$$

the result also holds for $q = \infty$ if μ is semi-finite (where ϕ_g, bears the same meaning as in section (3.6).

Proof. Recall from equation 3.9,

$$\|\phi_g\| = \sup_{f \in L^p} \{|\phi_g(f)| : \|f\|_{L^p} = 1\} = \sup_{f \in L^p} \left\{ \left| \int_X fg \, \mathrm{d}\mu \right| : \|f\|_{L^p} = 1 \right\} \tag{3.11}$$

For $1 < q < \infty$, using the equation 3.4

$$\left| \int_X fg \, \mathrm{d}\mu \right| = \|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}, \text{ for all } f \in L_p.$$

This implies

$$|\phi_g(f)| \leq \|g\|_q, \text{ for all } f \in L_p \text{ such that } \|f\|_{L^p} = 1.$$

This implies

$$\sup_{\|f\|_{L^p}=1} |\phi_g(f)| \leq \|g\|_q, \text{ for all } f \in L_p.$$

This implies

$$\|\phi_g\| \leq \|g\|_q$$

For $q = 1,$ for all $f \in L_p$ such that $\|f\|_{L^\infty} = 1,$

$$|\phi_g(f)| = \left| \int_X fg \, \mathrm{d}\mu \right| \leq \int_X |fg| \, \mathrm{d}\mu \leq \|f\|_\infty \int_X |g| \, \mathrm{d}\mu = \|f\|_\infty \|g\|_1 = \|g\|_1$$

Similarly,

$$\|\phi_g\| \leq \|g\|_\infty.$$

So we have,

$$\|\phi_g\| \leq \|g\|_q, 1 \leq q \leq \infty. \tag{3.12}$$

Now for the reverse inequality of equation 3.12,we go through the following cases. Here the trick is to find out one $f \in L_p$ such that $\|f\|_{L^\infty} = 1$ and $|\phi_g(f)| = |\int_X fg \, \mathrm{d}\mu| \geq \|g\|_q$. If we can do this we are done.
For $1 < q < \infty$,take

$$f = \mathrm{sign}(g) \frac{|g|^{q-1}}{\|g\|_q^{q-1}}.$$

Then,

$$\|f\|_p^p = \int_X |f|^p \, \mathrm{d}\mu = \int_X \frac{|g|^{(q-1)p}}{\|g\|_q^{(q-1)p}} \, \mathrm{d}\mu = \frac{\|g\|_q^{(q-1)p}}{\|g\|_q^{(q-1)p}} = 1$$

and

$$|\phi_g(f)| = |\int_X fg \,\mathrm{d}\mu| = \int_X \mathrm{sign}(g)\frac{|g|^{q-1}}{\|g\|_q^{q-1}} g \,\mathrm{d}\mu = \int_X \frac{|g|^q}{\|g\|_q^{q-1}} \,\mathrm{d}\mu = \frac{\|g\|_q^q}{\|g\|_q^{q-1}} = \|g\|_q.$$

Hence

$$\|\phi_g\| \geq \|g\|_q, \quad 1 < q < \infty.$$

For q=1, take $f = \mathrm{sign}(g)$. Then by the above argument we can show that

$$\|f\|_\infty = 1 \text{ and } |\phi_g(f)| = |\int_X fg \,\mathrm{d}\mu| = \|g\|_1.$$

So $\|\phi_g\| \geq \|g\|_1$. For $q = \infty$, take $\epsilon > 0$ and $A = \{x : |g(x)| > \|g\|_\infty - \epsilon\}$. Then by definition of $\|g\|_\infty$, $\mu(A) > 0$. Since $\mu(A)$ is semi-finite, there exists $B \subseteq A$ such that $0 < \mu(B) < \infty$. Take,

$$f = \mathrm{sign}(g)\frac{\chi_B}{\mu(B)}.$$

Then,

$$\|f\|_1 = 1 \text{ and } |\phi_g(f)| = |\int_X fg \,\mathrm{d}\mu| = \frac{1}{\mu(B)}\int_B |g| \,\mathrm{d}\mu \geq \|g\|_\infty - \epsilon$$

Since $\epsilon > 0$ is arbitrary, we have $\|\phi_g\| \geq \|g\|_\infty$. Thus we have

$$\|\phi_g\| \geq \|g\|_q, \; 1 \leq q \leq \infty \tag{3.13}$$

Therefore, from equation (3.12) and (3.13), we have

$$\|\phi_g\| = \|g\|_q; 1 \leq q \leq \infty \tag{3.14}$$

Thus the proof is completed. $\square$

R-4: Let $(X, \mathcal{A}, \mu)$ measure space. Suppose p, q be conjugate exponent with $1 \leq q < \infty$. Also Suppose $g : X \to \mathbb{C}$ is measurable on x. If

$$\sup\left\{\left|\int_X fg \ \mathrm{d}\mu\right| : f \in \Sigma_X, \|f\|_{L^p} = 1\right\} = M < \infty$$

and $S_g = \{x : g(x) \neq 0\}$ is σ- finite or μ is semi-finite. Then $g \in L^q$. Also $\|g\|_q = M$.

Proof. If $\mu(S_g) = 0$, there is nothing to do, the result follows readily. Consider the case $\mu(S_g) \neq 0$. For $1 < q < \infty$. Choose a sequence of simple functions g_n such that $|g_n(x)| < g(x)$, $g_n \to g$ and $\|g_n\|_q \neq 0$. For $1 < q < \infty$, take

$$f = \operatorname{sign}(g)\frac{|g_n|^{q-1}}{\|g_n\|_q^{q-1}}.$$

Then,

$$\|f_n\|_p^p = \int_X |f_n|^p \ \mathrm{d}\mu| = \int_X \frac{|g_n|^{(q-1)p}}{\|g_n\|_q^{(q-1)p}} \ \mathrm{d}\mu = \frac{\|g_n\|_q^{(q-1)p}}{\|g_n\|_q^{(q-1)p}} = \frac{\|g_n\|_q^q}{\|g_n\|_q^q} = 1$$

and

$$|\int_X f_n g_n \ \mathrm{d}\mu| = \int_X \operatorname{sign}(g)\frac{|g_n|^{q-1}}{\|g_n\|_q^{q-1}} g_n \ \mathrm{d}\mu = \int_X \frac{|g_n|^q}{\|g_n\|_q^{q-1}} \ \mathrm{d}\mu = \frac{\|g_n\|_q^q}{\|g_n\|_q^{q-1}} = \|g_n\|_q.$$

Now, by Fatou's Lemma

$$\begin{aligned}
\|g\|_q \leq \liminf \|g_n\|_q &= \liminf \int_X |f_n g_n| \, \mathrm{d}\mu \\
&\leq \liminf \int_X |f_n g| \, \mathrm{d}\mu \\
&= \liminf \int_X f_n g \, \mathrm{d}\mu < M.
\end{aligned}$$

Therefore $g \in L^q$ and also $\|g\|_q < M$, for $1 < q < \infty$. For q= 1, take

$$f_n(x) = \mathrm{sign}(g(x))$$

Then by preceding argument we can show that $g \in L^1$ and also $\|g\|_1 < M$. For $q = \infty$, let $\epsilon > 0$. Suppose $A = \{x : |g(x)| > M + \epsilon\}$ has positive measure (otherwise the result follows trivially). SinceS_g is σ-finite (or $\mu(A)$ is semi-finite), there exists $B \subseteq A$ such that $0 < \mu(B) < \infty$.

Take,

$$f = \mathrm{sign}(g)\frac{\chi_B}{\mu(B)}.$$

Then f is simple and

$$\|f\|_1 = \frac{1}{\mu(B)} \int_B 1 \, \mathrm{d}\mu = 1.$$

$$\left| \int_X fg \, \mathrm{d}\mu \right| = \frac{1}{\mu(B)} \int_B |g| \, \mathrm{d}\mu > M + \epsilon.$$

If $g(x) > M$ on B then

$$\int_B g(x) \mathrm{d}\mu > \int_B M \, \mathrm{d}\mu.$$

It implies that
$$\frac{1}{\mu(B)}\int_B g(x)\ \mathrm{d}\mu > M.$$

This not possible. Hence $\mu(A)$ cannot be greater than 0 and consequently $|g(x)| \leq M + \epsilon$. Since $\epsilon > 0$ is arbitary, $|g(x)| \leq M$, a.e. in x. This implies $\|g\|_\infty \leq M$. Therefore, $g \in L^\infty$ and also $\|g\|_\infty < M$. We prove that $\|g\|_q = M$. From above discussion, we have $\|g\|_q < M$. For $1 \leq q \leq \infty$. The reverse inequality follows readily by Holder's inequality for $1 \leq q < \infty$ and for $q = \infty$ is trivial (see proof of 1st part of **R-3** for $q = \infty$. Hence $\|g\|_q = M$. Thus the proof is completed.

□

R-5: Let $1 \leq p \leq \infty$. The set of simple functions $f = \sum_1^n a_j \chi_{E_i}$, where $\mu(E_j) < \infty$, for all $i = 1, 2, \ldots, n$ is dense in L^p.

Proof. Let $f \in L^p$. Choose sequence $(f_n)_{n=1}^\infty$ of simple functions such that $|f_n| < |f|$ and $f_n \to f$. Therefore $|f_n - f| \to 0$ as $n \to \infty$. Now,

$$|f_n - f| \leq 2\max(|f_n|, |f|) = 2|f|.$$

This implies $|f_n - f|^p \leq 2^p|f|^p \in L^1$. Hence, by dominated convergence theorem,
$$\lim \int_X |f_n - f|^p = \int_X \lim |f_n - f|^p = \int_X 0 = 0.$$
That is $\|f_n - f\|_p \to 0$. For $p = \infty$, since $|f_n - f| \to 0$ as $n \to \infty$, it follows that there exists a $N \in \mathbb{N}$ such that $|f_n - f| < \epsilon$, whenever $n > N$. Hence $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. This completes the proof. □

R-6-:

If $0 < p_0 < p < p_1 \leq \infty$, then $L_0^p \cap L_1^p \subseteq L^p$ and

$$\|f\|_p \leq \|f\|_{p_0}^{1-t} \|f\|_{r_0}^{t}, \tag{3.15}$$

where t satisfies

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}.$$

Proof. For $t = 0$ or 1, the result follows readily. So we consider the case $t \in (0,1)$. Case-I. First we consider the case $p_1 = \infty$. Now $p_1 = \infty$ gives

$$t = 1 - \frac{p_0}{p}.$$

For $p_1 = \infty$,we see that

$$|f|^p = |f|^{p-p_0} |f|_0^p \leq \|f\|_\infty^{p-p_0} |f|_0^p$$

Integrating both sides, we get

$$\|f\|_p^p \leq \|f\|_\infty^{p-p_0} \|f\|_{p_0}^{p_0}.$$

This implies

$$\|f\|_p \leq \|f\|_\infty^t \|f\|_{p_0}^{1-t}$$

For $p_1 < \infty$, we take conjugate exponents $\frac{p_0}{(1-t)p}$ and $\frac{p_1}{tp}$. Now,

$$|f|^p = |f|^{(1-t)p} |f|^{tp}$$

Taking integration on both sides

$$\|f\|_p^p = \||f|^{(1-t)p}\|_{\frac{p_0}{(1-t)p}} \||f|^{tp}\|_{\frac{p_1}{tp}} = \|f\|_{p_0}^{1-t} \|f\|_{p1}^t.$$

This completes the proof. $\square$

Chapter 4

RIESZ-THORIN INTERPOLATION THEOREM

4.1 Introduction

Let $1 \leq p < q < r \leq \infty$. A linear operator T on $L^p + L^r$ is bounded on L^p and L^r. Now it is interesting to ask that is it bounded on L^q? Riesz-Thorin gives a positive response to this question. Here first we establish a familiar result of complex analysis, called Three-line lemma and then we prove Riesz-Thorin interpolation theorem

4.2 The Three-Line Lemma

Lemma 4.2.1. *Suppose* $S = \{z \in \mathbb{C} : 0 < Re z < 1\}$. *Let* ϕ *be continuous and bounded function on* $\bar{S}$ *and analytic on* S. *If*

$$\sup_{y\in\mathbb{R}} |\phi(iy)| = M_0 \text{ and } \sup_{y\in\mathbb{R}} |\phi(1+iy)| = M_1,$$

then

$$\sup_{y\in\mathbb{R}} |\phi(t+iy)| = M_0^{1-t} M_1^t,\ 0 \le t \le 1. \tag{4.1}$$

Proof. For $t = 0$ or $t = 1$, the result follows by the given conditions. We first proof the result in some special situation. Let ϕ satisfies the hypothesis of the lemma for $M_0 = M_1 = 1$ and also suppose $|\phi(x+iy)| \to 0$ as $|y| \to \infty$, where $z = x + iy$. Since ϕ is bounded on $\bar{S}$, let

$$M = \sup_{z\in\bar{S}} |\phi(z)|.$$

Clearly $0 \le M < \infty$. If $M = 0$, the result is obvious. Hence we consider the case $0 < M < \infty$. Since $|\phi(z)| \to 0$ as $|y| \to \infty$, for $\epsilon > 0, \exists$ a $k > o$ such that $||\phi(z)| - 0| < \epsilon,\ \forall\ |y| \ge k$. That is, $|\phi(z)| < \epsilon,\ \forall\ |y| \ge R$, for each $R > k$. So, by maximum modulus principle, ϕ attains M at Re $z = 0$ or 1. Hence $M = 1$ and so

$$\sup_{z\in\bar{S}} |\phi(z)| \le 1 = M_0^{1-t} M_1^t.$$

Let

$$\phi_{\epsilon,\lambda}(z) = \phi(z) \exp^{\epsilon z^2 + \lambda z}.$$

Then

$$\begin{aligned}
\phi_{\epsilon,\lambda}(z) &= \phi(x+iy)\exp^{\epsilon(x^2-y^2+2ixy)+\lambda(x+iy)} \\
&= \phi(x+iy)\exp^{\epsilon x^2+\lambda(x)}\exp^{-\epsilon y^2}\exp^{i(\epsilon 2xy+\lambda y)} \qquad (4.2)
\end{aligned}$$

Therefore,

$$|\phi_{\epsilon,\lambda}(z)| = |\phi(x+iy)|\exp^{\epsilon x^2+\lambda x}\exp^{-\epsilon y^2} \to 0, \text{ as } |y| \to \infty.$$

Also,

$$|\phi_{\epsilon,\lambda}(iy)| = |\phi(iy)|\exp^{-\epsilon y^2} \leq M_0$$

$$|\phi_{\epsilon,\lambda}(1+iy)| = |\phi(1+iy)|\exp^{\epsilon+\lambda}\exp^{-\epsilon y^2} \leq M_1 \text{ as } (\epsilon,\lambda) \to (0,0)$$

Finally, let

$$\tilde{\phi}_{\epsilon,\lambda}(z) = \frac{\phi(z)\exp^{\epsilon z^2+\lambda z}}{M_o^{1-z}M_1^z}.$$

We see that,

$$|M_o^{1-z}M_1^z| = M_o^{1-x}M_1^x \leq \max\{1, M_o\}\max\{1, M_1\}.$$

Hence $\tilde{\phi}_{\epsilon,\lambda}(z)$ is bounded. Also

$$|M_o^{1-iy}M_1^{iy}| = M_0|M_o^{-iy}||M_1^{iy}| = M_0$$

and

$$|M_o^{iy}M_1^{1+iy}| = M_1|M_o^{-iy}||M_1^{iy}| = M_1.$$

Hence $\tilde{\phi}_{\epsilon,\lambda}(z)$ will satisfy all the conditions that $\tilde{\phi}(z)$ satisfies in the first part of the proof, when $(\epsilon, \lambda) \to (0, 0)$. So, we have

$$\sup_{z \in \bar{s}} |\tilde{\phi}_{\epsilon,\lambda}(z)| \leq 1, \text{ as } (\epsilon, \lambda) \to (0, 0).$$

This implies,

$$\sup_{z \in \bar{s}} |\frac{\phi(z)}{M_o^{1-z} M_1^z}| \leq 1.$$

Thus,

$$\sup_{z \in \bar{s}} |\phi(z)| \leq |M_o^{1-z}||M_1^z| = |M_o^{1-x}||M_1^x| = M_o^{1-x} M_1^x.$$

$$\sup_{y \in \mathbb{R}} |\phi(t + iy)| \leq M_o^{1-t} M_1^t, \text{ for } t \in (0, 1).$$

Hence the proof is completed. □

4.3 Riesz-Thorin Interpolation

Theorem 4.3.1. *Let $(X, \mathcal{M}, \mu)$ and $(X, \mathcal{N}, \nu)$ be two measures spaces. Let $p_0, p_1, q_0, q_1 \in [1, \infty]$. Also let if $q_0 = q_1 = \infty$, then ν is semi-finite. Suppose a linear map*

$$T : L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$$

such that $T : L^{p_0} \to L^{q_0}$ and $T : L^{p_1} \to L^{q_1}$ are bounded, i.e.,

$$\|Tf\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}}$$

and

$$\|Tf\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}}.$$

Then

$$\|Tf\|_{L^{q_t}} \leq M_0^{1-t} M_1^t \|f\|_{L^{p_t}} \tag{4.3}$$

where,

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \text{ and } \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1};\ t \in [0,1].$$

Proof. For $t = 0$ or 1, the result follows readily. So we fix our attention now on $t \in (0,1)$.

Case-I:

First we consider the case $p_0 = p_1$. For $p_0 = p_1$, we have $p_0 = p_1 = p_t$. So, by equation (3.15), we have

$$\|Tf\|_{L^{q_t}} \leq \|Tf\|_{L^{q_0}}^{1-t}\|Tf\|_{L^{q_1}}^{t} \leq M_0^{1-t}\|f\|_{L^{p_0}}^{1-t}M_1^t\|f\|_{L^{p_1}}^t \leq M_0^{1-t}M_1^t\|f\|_{L^{p_t}}$$

Case-II: Now we consider the case when $p_0 \neq p_1$. As $p_0 \neq p_1$,p_t cannot be ∞.

Aim:

$$\|Tf\|_{L^{q_t}} \leq M_0^{1-t}M_1^t\|f\|_{L^{p_t}}.$$

To show the above inequality it is enough to show that

$$\|Tf\|_{L^{q_t}} \leq M_0^{1-t}M_1^t, \text{ for } \|f\|_{L^{p_t}} = 1.$$

We first prove the result for simple function, after that we shall extent it.

By **R-4**, we can write,

$$\|Tf\|_{L^{q_t}} = \sup\left\{\left|\int_Y (Tf)g\mathrm{d}\nu\right| : g \in \Sigma_Y \text{ and } \|g\|_{L{q_t}'} = 1\right\},$$

where ${q_t}'$ is the conjugate exponent of q_t.

Let $f = \sum_{j=1}^m c_j\chi_{E_j}$ and $g = \sum_{k=1}^n d_k\chi_{F_k}$ be two arbitrary simple function, where the $E_j's$ and $F_k's$ are disjoint in X and Y and $c_j's$ and $d_k's$ are non-zero such that

$$\|f\|_{p_t} = 1 \quad \text{and } \|g\|_{q_t'} = 1 \tag{4.4}$$

Let us write,

$$\alpha(z) = \frac{1-z}{p_0} + \frac{z}{p_1} \text{ and } \beta(z) = \frac{1-z}{p_0} + \frac{z}{p_1}.$$

We write c_j's and d_k's in polar for as;

$$c_j = |c_j| \exp(i\theta_j) \text{ and } d_k = |d_k| \exp(i\psi_k).$$

Since for $p_0 \neq p_1$, p_t cannot be ∞, and $\alpha(t) > 0$. For $\alpha(t) > 0$, let us define

$$f_z = \sum_{j=1}^{m} |c_j|^{\frac{\alpha(z)}{\alpha(t)}} \exp(i\theta_j) \chi_{E_j}.$$

and for $\beta(t) < 1$, we define

$$g_z = \sum_{k=1}^{n} |d_k|^{\frac{1-\beta(z)}{1-\beta(t)}} \exp(i\psi_k) \chi_{F_k}$$

For $\beta(t) = 1$. We define $g_z = g$ and shall proceed as we shall proceeds now for $\beta(t) < 1$. Now our concentration is only on $\beta(t) < 1$. Clearly $f_t = f$ and $g_t = g$. Let us define

$$\phi(z) = \int_Y (Tf_z) g_z \, d\mu. \tag{4.5}$$

Therefore,

$$\phi(z) = \sum_{j=1}^{m} \sum_{k=1}^{n} |c_j|^{\frac{\alpha(z)}{\alpha(t)}} |d_k|^{\frac{1-\beta(z)}{1-\beta(t)}} A_{j,k},$$

where

$$A_{j,k} = \exp(i\theta_j + i\psi_k) \int_Y (T\chi_{E_j}) \chi_{F_k} d\mu.$$

Notice that

$$\phi(t) = \int_Y (Tf)(t)g(t)\ d\mu.$$

We see that ϕ is analytic and bounded on $\bar(S)$. So by Three-Line Lemma, it is suffice to prove that $|\phi(z)| \leq M_0$ at $\mathrm{Re}(z) = 0$ and $|\phi(z)| \leq M_1$ at $Re(z) = 1$, we have

$$\alpha(iy) = \frac{1}{p_0} + iy\left(\frac{1}{p_1} - \frac{1}{p_0}\right)$$

and

$$1 - \beta(iy) = 1 - \frac{1}{q_0} - iy\left(\frac{1}{q_1} - \frac{1}{q_0}\right).$$

For $y \in \mathbb{R}$, we have

$$|f_{iy}| = |f|^{\mathrm{Re}\left(\frac{\alpha(iy)}{\alpha(t)}\right)} = |f|^{\left(\frac{p_t}{p_0}\right)}$$

and

$$|g_{iy}| = |g|^{\mathrm{Re}\left(\frac{1-\beta(iy)}{1-\beta(t)}\right)} = |g|^{\left(\frac{q_t'}{q_0'}\right)}.$$

So, by Holder's inequality, we have

$$|\phi(iy)| = \|Tf_{iy}\|_{q_0}\|g_{iy}\|_{q_0'} \leq M_0\|f_{iy}\|_{p_0}\|g_{iy}\|_{q_0'} = M_0\|f_{iy}\|_{p_t}\|g_{iy}\|_{q_t'} = M_0.$$

Calculating in an analogous way, we can show that $|\phi(1+iy)| \leq M_1$. Thus by Three-Line Lemma, we can say that

$$\|Tf\|_{L^{q_t}} \leq M_0^{1-t}M_1^t\|f\|_{p_t},\ \text{for}\ f \in \Sigma_X.$$

Let $f \in L^p$ is arbitary. Since Σ_X is dense in L^p, we can choose a sequence

(f_n) in Σ_X such that $|f_n| \leq |f|$ and $f_n \to f$ satisfying

$$\lim_n T(f_n) = T(f).$$

Hence, by Fatou's Lemma, we have

$$\|Tf\|_{q_t} \leq \liminf \|Tf_n\|_{q_t} \leq \liminf \ M_0^{1-t} M_1^t \|f_n\|_{p_t} = M_0^{1-t} M_1^t \|f\|_{p_t}.$$

This completes the proof. $\square$

Bibliography