Fourier Analysis Lecture Notes

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Contents

Chapter 1

Fourier Series

Fourier series provide a canonical way to represent periodic functions as superpositions of the basic characters *of the circle group, namely the complex exponentials e^{inx}, $n \in \mathbb{Z}$. Beyond their striking applications to boundary–value problems in physics, Fourier series form a central tool of analysis: they convert questions about a function into questions about its frequency spectrum $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$.*

Learning objectives.

- See how separation of variables in classical PDE produces trigonometric eigenfunctions.
- Define Fourier coefficients on $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ and interpret partial sums as convolutions with the Dirichlet kernel.
- Distinguish pointwise, uniform, and L^2 convergence; understand why summability kernels (Abel, Fejér) are useful.

1.1 Motivation: eigenfunction expansions in PDE

A guiding principle of Fourier analysis is that *translation-invariant* linear problems diagonalize in a basis of exponential functions. On $\mathbb{R}/2\pi\mathbb{Z}$ these exponentials are precisely e^{inx}, $n \in \mathbb{Z}$. One classical route to this conclusion is separation of variables in boundary–value problems.

The vibrating string on an interval

Consider the one-dimensional wave equation on the interval $(0, \pi)$ with Dirichlet boundary conditions,

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad u(0,t) = u(\pi,t) = 0. \tag{1.1.1}$$

Seeking separable solutions $u(x,t) = X(x)T(t)$ and dividing by XT yields

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

for some constant $\lambda \in \mathbb{R}$, hence

$$X'' + \lambda X = 0, \qquad T'' + \lambda T = 0.$$

The boundary conditions force $X(0) = X(\pi) = 0$, so nontrivial solutions occur exactly for $\lambda = n^2$ with $n \in \mathbb{N}$, with eigenfunctions

$$X_n(x) = \sin(nx), \qquad n \in \mathbb{N}.$$

By linearity, one is led to expansions of the form

$$f(x) \sim \sum_{n=1}^{\infty} A_n \sin(nx), \tag{1.1.2}$$

where $f(x) = u(x,0)$ is the initial displacement and the coefficients are determined using orthogonality:

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx.$$

A second initial condition $u_t(x,0) = g(x)$ determines the coefficients in front of $\sin(nx)\sin(nt)$ (or $\sin(nx)\cos(nt)$), and it already hints at a central theme: *regularity of the data controls decay of the coefficients.*

The Dirichlet problem on the disc

A second motivation comes from the Laplace equation on the unit disc

$$\Delta u = 0 \quad \text{on } D = \{(r,\theta) : 0 \le r < 1,\ 0 \le \theta < 2\pi\}, \qquad u(1,\theta) = f(\theta),$$

the classical Dirichlet problem. Writing $u(r,\theta) = F(r)G(\theta)$ leads to

$$G'' + \lambda G = 0, \qquad r^2 F'' + rF' - \lambda F = 0.$$

Periodicity in θ forces $\lambda = n^2$ with $n \in \mathbb{Z}$, so $G(\theta) = e^{in\theta}$, and boundedness at $r = 0$ selects the radial solutions $F(r) = r^{|n|}$. Thus a bounded harmonic function admits an expansion

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} a_n\, r^{|n|} e^{in\theta}, \qquad \text{so that} \qquad f(\theta) = u(1,\theta) \sim \sum_{n\in\mathbb{Z}} a_n e^{in\theta}.$$

Question 1.1.1 (Central question)**.** Given a function f on the circle (for instance $f \in L^1(S^1)$)

or $f \in C(S^1)$), in what sense does the Fourier series

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta}, \qquad \hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}\,d\theta,$$

recover f?

We now formalize the identification of functions on the unit circle with 2π-periodic functions on $\mathbb{R}$, and then develop the basic tools needed to answer the question above.

1.2 Functions on the circle

Throughout these notes we identify the unit circle

$$S^1 := \{e^{it} : t \in \mathbb{R}\}$$

with the quotient group $\mathbb{R}/2\pi\mathbb{Z}$ via the map $t \mapsto e^{it}$. Under this identification, a function $f : S^1 \to \mathbb{C}$ may be viewed as a 2π-periodic function (again denoted by f) on $\mathbb{R}$.

The Lebesgue measure on S^1 corresponds to the usual Lebesgue measure dt on $[0, 2\pi)$ and is the (unique) translation-invariant probability measure up to scaling (Haar measure) on the circle. We use the normalization

$$\int_{S^1} f(t)\,dt := \int_0^{2\pi} f(t)\,dt, \qquad \text{so that} \qquad \int_{S^1} 1\,dt = 2\pi.$$

In particular, for every $t_0 \in S^1$ and every integrable f we have the translation invariance

$$\int_{S^1} f(t - t_0)\,dt = \int_{S^1} f(t)\,dt,$$

which follows immediately from the substitution $s = t - t_0$ and 2π-periodicity.

A *trigonometric polynomialnomial* of degree at most N is an expression

$$P_N(t) = \sum_{k=-N}^{N} a_k e^{ikt},$$

and a *trigonometric series* is a formal sum $\sum_{k\in\mathbb{Z}} a_k e^{ikt}$.

Definition 1.2.1. For $n \in \mathbb{Z}$, and $f \in L^1(S^1)$, the nth Fourier coefficient of f is defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)\,dt$$

Definition 1.2.2. The Fourier Series of $f \in L^1(S^1)$ is the expression of

$$S(f) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}$$

Hence, the n'th partial sum of the Fourier Series (FS) is

$$S_n(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikt}$$

is a trigonometric polynomialnomial of degree n.

Lemma 1.2.3. *Let $f, g \in L^1(S^1)$, then*

(i) $\widehat{f+g}(n) = \hat{f}(n) + \hat{g}(n)$,

(ii) $\widehat{\alpha f}(n) = \alpha \hat{f}(n), \quad \alpha \in \mathbb{C}$,

(iii) $\hat{\bar{f}}(n) = \overline{\hat{f}(-n)}$,

(iv) If $\tau_{t_0} f(t) = f(t - t_0),\ t_0 \in S^1$, *then* $(\tau_{t_0} f)^\wedge (n) = e^{-int_0} \hat{f}(n)$

(v) $|\hat{f}(n)| \leq \frac{1}{2\pi} \int |f(t)| dt = \|f\|_1$

Corollary 1.2.4. *If $f_n \in L^1(S^1)$ and $\|f_j - f\|_1 \to 0$, then $\hat{f}_j(n) \to \hat{f}(n)$ absolutely (or even uniformly).*

Theorem 1.2.5. *Let $f : [0, 2\pi] \to \mathbb{C} \subset \mathbb{R}$. Then f is absolutely continuous if and only if f' exists a.e. and*

$$f(x) = f(0) + \int_0^x f'(t)\, dt.$$

(For a proof, see Carothers p.374.)

Theorem 1.2.6. *Let $f \in L^1(S^1)$ and $\hat{f}(0) = 0$. Define*

$$F(t) = \int_0^t f(s)\, ds.$$

Then F is continuous 2π-periodic function and

$$\hat{F}(n) = \frac{\hat{f}(n)}{in}, \quad n \neq 0.$$

Proof. For $t_k \to t_0$

$$F(t_k) - F(t_0) = \int_0^{2\pi} \chi_{[t_0, t_k)}(s) f(s)\, ds.$$

Since $\chi_{[t_0,t_k)}(s) f(s) \to 0$ point wise a.e. and $f \in L^1(S^1)$, by DCT, it follows that

$$F(t_k) - F(t_0) \to 0 \text{ as } k \to \infty.$$

Hence, F is continuous on S^1.

Notice that

$$\sum_{k=1}^{l} |F(t_k) - F(t_{k-1})| \leq \sum_{k=1}^{l} \int_0^{2\pi} \chi_{[t_{k-1},t_k)}(s)|f(s)|ds.$$

Hence, RHS tends to "0" when $l \to \infty$. This implies that F is absolutely continuous. Thus, F is differentiable a.e. Also

$$F(t+2\pi) - F(t) = \int_t^{t+2\pi} f(s)ds = \hat{f}(0) = 0.$$

Now, integrating by parts, we get

$$\hat{F}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}F(t)dt = -\frac{1}{2\pi}\int_0^{2\pi} F'(t)\left(\frac{e^{-int}}{-in}\right)dt = \frac{1}{in}\hat{f}(n).$$

□

Example 1.2.7. Let $f(\theta) = \theta$, $-\pi \leq \theta < \pi$. Then

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-in\theta}d\theta = \frac{(-1)^{n+1}}{in}, \quad n \neq 0.$$

$\hat{f}(0) = 0$. Thus,

$$f(\theta) \sim \sum \frac{(-1)^{n+1}}{in}e^{in\theta} = 2\sum \frac{(-1)^{n+1}\sin\theta}{n}$$

It's easy to see that Series on RHS is pointwise convergent, but showing it converges to $f(\theta)$ is not easy, as we see later!

Example 1.2.8. $f(\theta) = \frac{(\pi-\theta)^2}{4}$, $0 \leq \theta \leq 2\pi$

$$f(\theta) \sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2}$$

The Fourier Series is uniformly convergent, but it converges to $f(\theta)$ is not easy.

Theorem 1.2.9. *For $f, g \in L^1(S^1)$. Define convolution of f and g by*

$$h(t) = f * g(t) = \frac{1}{2\pi}\int_0^{2\pi} f(t-s)g(s)ds.$$

Then $h \in L^1(S^1)$ and $\|h\|_1 \leq \|f\|_1\|g\|_1$,
moreover, $\hat{h}(n) = \hat{f}(n)\hat{g}(n)$.

Proof.

$$\begin{aligned}\int |h(t)|dt &\leq \frac{1}{2\pi}\int\left(\int |f(t-s)||g(s)|ds\right)dt \\ &= \frac{1}{2\pi}\int(\int |f(t-s)|dt)|g(s)|ds \quad \text{(by Fubini's theorem)} \\ &= \frac{1}{2\pi}\int \|f\|_1|g(s)|ds = \|f\|_1\|g\|_1\end{aligned}$$

Further,

$$\begin{aligned}\hat{h}(n) &= \frac{1}{2\pi}\int h(t)e^{-int}dt \\ &= \frac{1}{4\pi^2}\int\left(\int f(t-s)e^{-in(t-s)}dt\right)\ g(s)e^{-ins}ds \\ &= \frac{1}{2\pi}\int \widehat{f}(n)g(s)e^{-ins}ds \\ &= \hat{f}(n)\hat{g}(n).\end{aligned}$$

□

Question 1.2.10. Does there exists $f, g \in L^1(S^1)$ such that $f * g(s) = 1$?

Let $f \in L^1(S^1)$ and $\varphi(t) = e^{int}$, then

$$\varphi * f(t) = \frac{1}{2\pi}\int f(s)e^{in(t-s)}ds = e^{int}\hat{f}(n).$$

Hence, if

$$P_N(t) = \sum_{n=-N}^{N} c_n e^{int},$$

then

$$P_N * f(t) = \sum_{n=-N}^{N} c_n\hat{f}(n)e^{int}.$$

that is convolution of a trigonometric polynomialnomial with any function is a trigonometric polynomialnomialnomial. Now, consider the Fourier series of $f \in L^1(S^1)$ as

$$f(t) \sim \sum_{n=-\infty}^{\infty}\hat{f}(n)e^{int}.$$

Let

$$D_N(t) = \sum_{n=-N}^{N} e^{int} \quad \text{and} \quad S_N(f)(t) = \sum_{n=-N}^{N}\hat{f}(n)e^{int}.$$

Then

$$S_N(f)(t) = D_N * f(t).$$

The function D_N is known as **Dirichlet kernel.** Further,

$$D_N(t) = \frac{\sin\left((N+\frac{1}{2})t\right)}{\sin(t/2)}, \quad t \neq 0$$

and $D_N(0) = 2N+1$. (Hint: put $\omega = e^{it}$, then $D_N(t)$ is the sum of two geometric series, etc.) Hence, the earlier question of convergence of Fourier series can be rephrased as:

Question 1.2.11. Whether the partial sum of the sequence $S_N(f)$ converges to f point wise. That is,

$$\lim_{N\to\infty} D_N * f(t) = f(t) \ ? \tag{4}$$

Recall back the heat-equation (steady-state):

$$\Delta U = 0, \quad U(r,\theta) = \sum a_m r^{|m|} e^{im\theta}$$

Let

$$P_r(\theta) = \sum_{m=-\infty}^{\infty} r^{|m|} e^{im\theta}, \quad 0 \leq r < 1,\, \theta \in [-\pi, \pi]$$

Then the series on RHS converges absolutely and uniformly. Hence,

$$\hat{P}_r(m) = r^{|m|} \quad \text{and we have}$$

$$P_r * f(\theta) = \sum_{m=-\infty}^{\infty} \hat{f}(m) r^{|m|} e^{im\theta}$$

The function $P_r(\theta)$ is known as **Poisson kernel** and can be represented as

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

(Hint: Series for $P_r(\theta)$ in terms of two geometric series, etc.)

Thus, we can ask when

$$\lim_{r\to 1} P_r * f(\theta) = f(\theta) \ ?$$

The function $P_r * f$ is called the **Abel mean** of Fourier series $S(f)$.

Now, the question is, does there exist a family of "good kernels" (i.e., weight functions or averaging functions) for the Fourier series that leads the series to the given function?

That is, if $f \in L^1(S^1)$, can we find a sequence $K_n \in L^1(S^1)$ such that $f * K_n \to f$?

Definition 1.2.12. A sequence of functions $\{K_n\}_{n=1}^{\infty}$ is "good kernels" if

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1, \quad$ for all $n \geq 1$.

(ii) There exists $M > 0$ such that $\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(t)|dt \leq M$, for all $n \geq 1$.

(iii) For each $\delta > 0$, $\int_{\delta<|t|\leq\pi} |K_n(t)|dt \to 0$ as $n \to \infty$.

Theorem 1.2.13. *Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of good kernels on $[-\pi, \pi]$ and $f \in R([-\pi, \pi])$ (Riemann integrable).*

Then $(f * K_n)(x) \to f(x)$ if x is a point of continuity of f, and the above limit is uniform if f is continuous on $[-\pi, \pi]$.

Proof. Since f is continuous at x, for $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x-y) - f(x)| < \epsilon$, for all $|y| < \delta$. Now

$$f * K_n(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)]dy \quad \text{(by property (i) of } K_n)$$

$$\begin{aligned}\Rightarrow |f * K_n(x) - f(x)| \leq \frac{1}{2\pi}\int_{|y|<\delta} |K_n(y)||f(x-y) - f(x)|dy \\ + \frac{1}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)||f(x-y) - f(x)|dy \\ \leq \frac{\epsilon}{2\pi}\int_{|y|<\delta} |K_n(y)|dy + \frac{2B}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)|dy,\end{aligned}$$

where $|f(x)| \leq B$, for all $x \in [-\pi, \pi]$. This implies

$$|f * K_n(x) - f(x)| < C\epsilon \quad \text{for large } n.$$

If f is continuous on $[-\pi, \pi]$, then we can find one $\delta > 0$ that serves for each x. Hence $f * K_n \to f$ uniformly in this case. □

Corollary 1.2.14. *If $\{K_n\}_{n=1}^{\infty}$ is a sequence of good kernels in $L^1(S^1)$ and $f \in L^1(S^1)$, then*

$$f * K_n \to f \quad \textit{in} \quad L^1(S^1).$$

Proof. Since $\overline{C([-\pi, \pi])} = L^1([-\pi, \pi])$, for $f \in L^1$ and $\epsilon > 0$, there exists g continuous such that $|f(x) - g(x)| < \epsilon$ for all $x \in [-\pi, \pi]$. That is,

$$\|f - g\|_1 < 2\pi\epsilon.$$

From the above result $g * K_n(x) \to g$ uniformly, that is

$$|g * K_n(x) - g(x)| < \epsilon \text{ for large } n, \text{ and for all } x$$

$$\Rightarrow \|g * K_n - g\|_1 < 2\pi\epsilon \tag{2}$$

This implies,

$$\begin{aligned}\|f * K_n - f\|_1 &\leq \|(f-g) * K_n\|_1 + \|g * K_n - g\|_1 + \|f - g\|_1 \\ &\leq \|f-g\|_1 \|K_n\|_1 + 4\pi\epsilon \\ &\leq \epsilon.1 + 4\pi\epsilon\end{aligned}$$

for large n. □

Remark 1.2.15. Dirichlet Kernel is **not a good kernel** for Fourier series.

$$D_n(t) = \frac{\sin\left((n+\frac{1}{2})t\right)}{\sin\left(\frac{t}{2}\right)}, \qquad t \neq 0,$$

Since $|\sin x| < |x|$, it follows that

$$\begin{aligned}\int_{-\pi}^{\pi} |D_n(t)| dt &\geq \frac{2}{\pi} \int_0^{\pi} \left|\sin\left((n+\frac{1}{2})t\right)\right| \frac{dt}{t} \\ &= \frac{2}{\pi} \int_0^{(n+\frac{1}{2})\pi} |\sin t| \frac{dt}{t} \\ &\geq \frac{2}{\pi} \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{t} dt \\ &\geq \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin t| dt \\ &= \frac{4}{\pi^2} \sum_{k=1}^{n} \frac{1}{k} \to \infty\end{aligned}$$

as $n \to \infty$. That is, Dirichlet Kernel D_n fails to satisfy property of a good kernel.

In fact, it is also clear from the above calculation that

$$\int_{\delta \leq |t| \leq \pi} |D_n(t)|\, dt \not\to 0 \quad \text{as } n \to \infty.$$

However,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} D_n(t) dt = 1.$$

Thus, if we write

$$F_n(t) = \frac{D_0(t) + D_1(t) + \ldots + D_{n-1}(t)}{n},$$

where

$$D_k(t) = \sum_{l=-k}^{k} e^{ilt},$$

then we can show that $\{F_n\}_{n=1}^{\infty}$ is a family of good Kernel. This is known as **Fejer Kernels**, and $F_n * f$ is known as **Cesàro sum** of the Fourier series for f.

In general, for a sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers, let $S_n = a_1 + \ldots + a_n$. Then the series $\sum a_n$ is said to be **Cesàro summable** if

$$\sigma_n = \frac{S_1 + \cdots + S_n}{n}$$

is convergent.

Example 1.2.16.

$$1 - 1 + 1 - 1 + \cdots = \sum_{n=0}^{\infty} (-1)^n$$

then $S_n = 0$ (if n even), $S_n = 1$ (if n odd), and hence $\sigma_n = \frac{[n/2] \pm 1}{n} \to \frac{1}{2}$.

Let

$$\sigma_n(f)(x) = \frac{S_0(f)(x) + \cdots + S_{n-1}(f)(x)}{n}.$$

Since $S_n(f) = f * D_n$, it follows that $\sigma_n(f) = f * F_n$, where

$$F_n = \frac{D_0 + D_1 + \ldots + D_{n-1}}{n}.$$

Exercise 1.2.17. (i) $F_n(x) = \frac{1}{n} \frac{\sin^2\left(\frac{nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}$, if $n \neq 0$.

(ii) $F_n(0) = 1$ (since F_n continuous at $x = 0$).

(iii) $\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(t)\, dt = 1$.

Notice that for $\delta > 0$, there exists $c_\delta > 0$ such that

$$\sin^2\left(\frac{x}{2}\right) > c_\delta, \quad \delta \leq |x| \leq \pi.$$

Hence, $F_n(x) \leq \frac{1}{nc_\delta}$, $\forall x \geq \delta$. Therefore,

$$\int_{\delta \leq |x| \leq \pi} F_n(x)\, dx \leq \frac{(\pi - \delta)}{c_\delta} \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Hence $\{F_n\}_{n=1}^{\infty}$ is a family of **good kernels**.

Thus, if $f \in R[-\pi, \pi]$, then the Fourier series is Cesáro summable to f at the point of continuity of f, and uniformly Cesáro summable if f is continuous.

Remark 1.2.18. If $f \in R[-\pi, \pi]$ and $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$ at all points of continuity of f. Since

$$S_n(f)(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikt} = 0,$$

$$f * F_n(t) \equiv 0 \implies f(t) = 0,$$

if f is continuous at t.

1.3 Uniqueness Theorem

Theorem 1.3.1. *If $f \in L^1(S^1)$ is such that $\widehat{f}(m) = 0$ for all $m \in \mathbb{Z}$, then $f = 0$ on S^1 a.e.*

Proof. For $f \in L(S^1)$ and $\varepsilon > 0$, there exists $g \in C(S^1)$ such that $\|f - g\|_1 < \epsilon$. Now

$$\begin{aligned} \|f\|_1 &\leq \|f * F_n - f\|_1 \\ &\leq \|f * F_n - g * F_n\|_1 + \|g * F_n - g\|_1 + \|g - f\|_1 \\ &\leq \|f - g\|_1 \cdot 1 + \|g * F_n - g\|_1 + \|g - f\|_1. \end{aligned}$$

Since g is continuous, for $\varepsilon > 0$, $\|g * F_n - g\|_1 < \epsilon$ for $n \geq N_0$. Hence,

$$\|f\|_1 < 3\epsilon \text{ for all } \epsilon > 0.$$

Thus, $\|f\|_1 = 0 \iff f = 0$ a.e. □

Remark 1.3.2. A continuous function on S^1 can be uniformly approximated by trigonometric polynomialnomials. That is, if $f \in C[-\pi, \pi]$ and $f(-\pi) = f(\pi)$, then $\sigma_n(f) = f * F_n$ is a trigonometric polynomialnomialnomial and we know that $f * F_n \to f$ uniformly. That is, $\{f * F_n : n \in \mathbb{N}\}$ is dense in $\{f \in C[-\pi, \pi] : f(\pi) = f(-\pi)\}$.

We also mention that if $f \in L^1(S^1)$, then for $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\|f * F_n - f\|_1 < \epsilon, \quad n \geq N_0.$$

Hence, trigonometric polynomialnomials are dense in $L^1(S^1)$.

1.4 Riemann-Lebesgue Lemma

Lemma 1.4.1. *If $f \in L^1(S^1)$, then $\lim_{|n| \to \infty} \hat{f}(n) = 0$.*

Proof. For $\epsilon > 0$, there exists a trigonometric polynomialnomial P such that $\|f - P\|_1 < \epsilon$ (where $P = f * F_n$ etc.). Let $|n| > \deg P$. Then

$$|\hat{f}(n)| = |\hat{f}(n) - \hat{P}(n)| \leq \|f - P\|_1 < \epsilon, \quad \text{if } |n| > \deg P.$$

That is, $|\hat{f}(n)| < \epsilon$ for large n. Hence, $\lim_{|n| \to \infty} \hat{f}(n) = 0$. □

1.5 Abel Means Summability

A series $\sum_{n=0}^{\infty} a_n$ is said to be **Abel summable** to s if the series

$$A(r) = \sum_{n=0}^{\infty} a_n r^n$$

is convergent for each $0 \le r < 1$, and $\lim_{r\to 1} A(r) = s$.

Example 1.5.1. Every convergent series is Abel summable. Consider

$$1 - 2 + 3 - 4 + 5 - \cdots = \sum_{n=0}^{\infty} (-1)^n (n+1).$$

Then

$$A(r) = \sum_{n=0}^{\infty} (-1)^n (n+1) r^n = \frac{1}{(1+r)^2} \to \frac{1}{4}$$

Show that the above series is **not Cesaro summable**.

Now, consider the Fourier series of $f \in R[-\pi, \pi]$ as

$$f(t) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}$$

Let

$$A_r f(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{in\theta}$$

then

$$A_r f(\theta) = (f * P_r)(\theta)$$

where

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} \qquad (*)$$

Lemma 1.5.2. *$P_r(\theta)$ is a good kernel in the following sense:*

(i) $\frac{1}{2\pi} \int P_r(\theta)\, d\theta = 1$

(ii) $\lim_{r\to 1} \int_{\delta \le |\theta| \le \pi} P_r(\theta)\, d\theta = 0$, *for all* $\delta > 0$.

Proof. (i) easily follows from $(*)$, since the series converges uniformly for each $0 \le r < 1$.

To prove (ii), let $\frac{1}{2} \le r < 1$. Then

$$1 - 2r\cos\theta + r^2 = (1-r)^2 + 2r(1 - \cos\theta)$$

For $0 < \delta < |\theta| \leq \pi$, $1 - 2r\cos\theta + r^2 > c_\delta$. Hence,

$$P_r(\theta) < \frac{1-r^2}{c_\delta} \quad \text{for all } \delta > 0.$$

$$\Rightarrow \frac{1}{2\pi}\int_{\delta<|\theta|\leq\pi} P_r(\theta)\,d\theta \leq \frac{1-r^2}{c_\delta} \to 0 \text{ as } r \to 1.$$

□

Theorem 1.5.3. *Let $f \in R[-\pi, \pi]$. Then*

(i) $A_r f(\theta) = (P_r * f)(\theta) \to f(\theta)$, if θ is a point of continuity of f.

(ii) $A_r f \to f$ uniformly if f is continuous.

Proof. Proof of this result is same as for the Fejer kernel when we consider continuous parameter $r \in (0, 1)$. □

Corollary 1.5.4. *Since $\overline{C(S^1)} = L^1(S^1)$, it follows that*

$$\|P_r * f - f\|_1 \to 0 \ \text{as } r \to 1 \quad \text{for } f \in L^1(S^1)$$

Theorem 1.5.5. *Let $U(r, \theta) = f * P_r(\theta)$. Then*

(ii) U is twice differentiable on the unit disc $D = \{re^{i\theta} : 0 \leq r < 1, -\pi \leq \theta < \pi\}$

(iii) If θ is a point of continuity of f, then $U(r, \theta) \to f(\theta)$ as $r \to 1$, and the limit is uniform if f is continuous on $E = [-\pi, \pi]$.

(iii) If f is continuous on $E = [-\pi, \pi]$, then $U(r, \theta)$ is the unique solution of $\Delta U = 0$ with $\lim_{r\to1} U(r, \theta) = f(\theta)$.

Proof. (i)

$$U(r, \theta) = \sum_{n=-\infty}^{\infty} r^{|n|}\hat{f}(n)e^{in\theta}$$

Since the series and its derivative (with respect to r and θ), both are uniformly convergent, term-by-term differentiation is allowed. In fact, $U(r, \theta) \in C^\infty$-function on D. Since

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}$$

it is easy to verify $\Delta U = 0$, if $U = P_r * f$. A proof for (i) is followed by the previous result.

(iii) Let $v(r, \theta)$ be another solution of $\Delta U = 0$ with $\lim_{r\to1} v(r, \theta) = f(\theta)$. Then

$$v(r, \theta) = \sum_{n=-\infty}^{\infty} a_n(r)e^{in\theta} \qquad (\because \Delta v = 0)$$

where

$$a_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} v(r,\theta)\, d\theta$$

Since v is two times differentiable,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial^2}{\partial v^2} v(r,\theta) e^{-in\theta} d\theta = -n^2 a_n(r).$$

Hence, from

$$\Delta v = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0,$$

it follows that

$$a_n''(r) + \frac{1}{r} a_n'(r) - \frac{n^2}{r^2} a_n(r) = 0.$$

This gives

$$a_n(r) = A_n r^n + B_n r^{-n}, \text{ if } n \neq 0.$$

Since v is bounded on D, letting $r \to 0$ implies $B_n = 0$. That is,

$$v(r,\theta) = \sum A_n r^n e^{in\theta} \xrightarrow{\text{uniform}} f(\theta)$$

$$\implies A_n = \frac{1}{2\pi}\int f(\theta) e^{-in\theta}\, d\theta.$$

For $n = 0$, $A_0(r) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$. Thus for each $0 \leq r < 1$, Fourier series of v is same as for u. By uniqueness it follows that $u = v$.

□

Exercise 1.5.6. If $\{J_n\}_{n=1}^{\infty}$ and $\{K_n\}_{n=1}^{\infty}$ are two families of good kernels for $L^1(S^1)$, then $\{J_n * K_n\}_{n=1}^{\infty}$ is a good kernel for $L^1(S^1)$.

(i)

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} J_n * k_n(t)dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi}\int_{-\pi}^{\pi} J_n(t-s)k_n(s)dsdt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} J_n(t-s)dt\right) k_n(s)ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 \cdot k_n(s)ds \quad (\text{ since } L^1(S^1) \text{ is translation invariant}) \\
&= 1
\end{aligned}$$

(ii)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |J_n * k_n(t)|dt \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} M|k_n(s)|ds \leq MN < \infty$$

(iii) Let $\delta > 0$, then

$$\int_{\delta<|t|\leq\pi} |K_n * J_n(t)|dt \leq \int_{s=-\pi}^{\pi} \left(\int_{\delta<|t|\leq\pi} |K_n(t-s)|dt \right) |J_n(s)|ds$$

Let $|s| < \delta/2$, then $r = t - s \in (-\delta/2, \delta/2)$. Now

$$(**) \int_{|s|<\delta/2} \left(\int_{\delta/2<|r|<\pi} |K_n(r)|dr \right) |J_n(s)|ds \to 0 \text{ as } n \to \infty,$$

since $\int_{\delta/2<|s-t|<\pi} |K_n(t-s)|dt \to 0$ as $n \to \infty$. (Exercise)

Since $|s| < \delta/2$, (use the fact that $\tau_x f \to f$ is continuous on $L^1(S^1)$). That is, if

$$\int_{\delta<|t|\leq\pi} |K_n(t)|dt \to 0 \quad \text{for all } \delta > 0,$$

then

$$\left| \int_{\delta<|t|\leq\pi} (\tau_s K_n(t) - K_n(t))dt \right| < \int_{\delta<|t|\leq\pi} |(\tau_s K_n(t) - K_n(t))| \, dt \leq \epsilon$$

For $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $\int_{|t|>\delta} |K_n(t)|dt < \epsilon$ for all $n \geq n_0$ and for small $|s| < \delta^1$. However,

$$\int_{|s|>\delta/2} \int_{|t|>\delta} |K_n(t-s)||J_n(s)|ds \; dt \leq \int_{|s|>\delta/2} M|J_n(s)|ds \to 0 \text{ as } n \to \infty.$$

Lemma 1.5.7. *Let $f : [-\pi, \pi] \to \mathbb{C}$ be such that*

$$|f(x) - f(y)| \leq M|x - y| \quad \textit{for all } x, y \in [-\pi, \pi]$$

for some $M > 0$. Then $S_n(f) \to f$ uniformly. Note that $|x - y| = \min\{|x - y|, |x - y \pm 2\pi|\}$, that is, the distance between x and y modulo 2π.

Proof. Calculate

$$S_n(f)(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x))D_n(t)dt.$$

Since

$$D_n(t) = \frac{\sin((n + \frac{1}{2})t)}{\sin(t/2)}, \quad t \neq 0,$$

$$|S_n(f)(x) - f(x)| \leq \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x-t) - f(x)) \frac{\cos t/2}{\sin t/2} \sin nt \, dt \right| + \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x-t) - f(x)) \cos nt \, dt \right|.$$

Let

$$g(t) = \frac{f(x+t) - f(x)}{t/2} \cos\frac{t}{2}, \quad t \neq 0.$$

Then $|g(t)| \leq 2M|\frac{t/2}{\sin t/2}|$, if $t \neq 0$.

Since $\lim_{t\to 0} \frac{t/2}{\sin(t/2)} = 1$, it follows that g is a bounded function on $[-\pi, \pi]$ and continuous on $[-\pi, \pi] \setminus \{0\}$. Hence, $g \in R[-\pi, \pi]$.

Let $h(t) = f(x-t) - f(x)$. Then

$$\begin{aligned} |S_n(f)(x) - f(x)| &\leq \frac{1}{2\pi}|\int_{-\pi}^{\pi} g(t)\sin(nt)dt| + \frac{1}{2\pi}|\int_{-\pi}^{\pi} h(t)\cos(nt)dt| \\ &= \frac{1}{2}|\hat{g}(n) - \hat{g}(-n)| + \frac{1}{2}|\hat{h}(n) + \hat{h}(-n)| \to 0 \text{ (by R-L Lemma)} \end{aligned}$$

whenever $x \in [-\pi, \pi]$. □

Corollary 1.5.8. *If $f \in R[-\pi, \pi]$ and f is differentiable at x_0, then $S_n(f)(x_0) \to f(x_0)$.*

Define $g(t) = \begin{cases} \frac{f(x_0-t)-f(x_0)}{t}, t \neq 0; \\ -f'(x_0), \text{ otherwise} \end{cases}$

Corollary 1.5.9. *If $f \in C'[-\pi, \pi]$, then $S_n(f) \to f$ uniformly. (Hint: Use MVT.)*

Notice that if f is piecewise C^1-function, then $S_n(f) \to f$ uniformly too.

Question 1.5.10. Does every continuous function f on S^1 have a Fourier series which converges to f at each point of S^1?

To discuss this, we need the following lemma.

Lemma 1.5.11. *Let $f \in R[-\pi, \pi]$ and f is bounded on $[-\pi, \pi]$ by M. Then there exists a sequence f_n of continuous functions on $[-\pi, \pi]$ such that*

(i) $|f_n(x)| \leq M$ *for all* $n \in \mathbb{N}$, $x \in [-\pi, \pi]$.

(ii) $\int_{-\pi}^{\pi} |f_n(x) - f(x)|dx \to 0$ *as* $n \to \infty$.

Proof. First consider f as a real-valued function. For $\epsilon > 0$, there exists a partition P of $[-\pi, \pi]$ such that

$$U(P, f) - L(P, f) < \epsilon, \tag{1}$$

where

$$P = \{-\pi = x_0 < x_1 < \cdots < x_i < x_{i+1} < \cdots < x_N = \pi\}$$

For $x \in [x_{i-1}, x_i]$, define $g(x) = \sup\{f(y) : x_{i-1} \leq y \leq x_i\}$. Then g is bounded by M.

$$\int_{-\pi}^{\pi} |g(x) - f(x)|dx = \int_{-\pi}^{\pi} (g(x) - f(x))dx < \epsilon \quad \text{(by (1))}$$

Let $\delta > 0$ and $x \in (x_i - \delta, x_i + \delta)$, define $\tilde{g}(x)$ be the linear function joining $g(x-\delta)$ and $g(x+\delta)$, and $\tilde{g} = 0$ near $-\pi$ and π. Then $\tilde{g}$ is a continuous periodic function which differs with g on N many intervals, each of length less than 2δ surrounding the partitioning points. Hence,

$$\int_{-\pi}^{\pi} |g(x) - \tilde{g}(x)|\, dx \leq (2M)N(2\delta).$$

For δ sufficiently small,

$$\int_{-\pi}^{\pi} |g(x) - \tilde{g}(x)|\, dx < \epsilon.$$
$$\implies \int_{-\pi}^{\pi} |f(x) - \tilde{g}(x)|\, dx < 2\epsilon.$$

For $2\epsilon = \frac{1}{n}$, take $\tilde{g} = f_n$. Thus

$$\int_{-\pi}^{\pi} |f(x) - f_n(x)| dx \to 0 \quad \text{as} \quad n \to \infty.$$

□

Remark 1.5.12. If $f \in R[-\pi, \pi]$ has only finitely many points of discontinuity, then $\tilde{g}_n(x) \to f(x)$ point-wise.

Now, let $X = C(S^1)$ and define $\Lambda_n : X \to X$ by

$$\Lambda_n(f) = S_n(f)(0).$$

Then $\{\Lambda_n\}$ is a sequence of linear functionals on X and

$$|\Lambda_n(f)| \leq \|D_n\|_1 \|f\|_\infty \implies \|\Lambda_n\| \leq \|D_n\|_1.$$

We claim that $\|\Lambda_n\| = \|D_n\|_1$ that is $\|\Lambda_n\| = \int_{-\pi}^{\pi} |D_n(t)|\, dt$.

For this, let $g(t) = \operatorname{sign} D_n(t)$. Then for each fixed n, g has only finitely many points of discontinuity. Hence, there exists $g_n \in C[-\pi, \pi]$ such that $|g_n(t)| \leq 1$ and $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in [-\pi, \pi]$ (by previous lemma). Therefore

$$\begin{aligned}\lim_{m\to\infty} \Lambda_n(g_m) &= \lim_{m\to\infty} \int_{-\pi}^{\pi} g_m(-t) D_n(t)\, dt \\ &= \int_{-\pi}^{\pi} g(-t) D_n(t)\, dt \quad \text{(by DCT)} \\ &= \int_{-\pi}^{\pi} |D_n(t)|\, dt = \|D_n\|_1\end{aligned}$$

Thus,

$$\|\Lambda_n\| = \|D_n\|_1 \to \infty \quad \text{as} \quad n \to \infty.$$

That is, $\{\Lambda_n\}_{n=1}^{\infty}$ is not a uniformly bounded sequence in $\mathcal{B}(X, D)$, hence by Uniform Boundedness Principle (UBP), there exists $f \in C([-\pi, \pi])$ such that $\Lambda_n(f) = S_n(f)(0)$ is not bounded. Therefore, the F.S. (Fourier Series) of f at 0 does not converge to $f(0)$.

Notice that by translation we can show that for each $x \in [-\pi, \pi]$, there exists a function $f \in C[-\pi, \pi]$ whosee Fourier series does not converge to $f(x)$ at x. In fact, for each $x \in [-\pi, \pi]$, we can create a dense class of continuous functions say E_x such that $S_n(f)(x) \to \infty$ (see Rudin, Real & Complex).

1.6 Convergence of Fourier Series in $L^2(S^1)$

We have seen that the Fourier series of $f \in C(S^1)$ need not converge to f uniformly. Similarly, we can also see that the Fourier series of $f \in L^1(S^1)$ need not converge to f in L^1-norm. (For this, define $\Lambda_n(f) = S_n(f)$, $f \in L^1(S^1)$ and use $\|F_n\|_1 = 1$). However, because of the self-duality of the space $L^2(S^1)$, for $f \in L^2(S^1)$, we shall see that $S_n(f) \to f$ in L^2-norm.

For $f, g \in L^2(S^1)$, define an inner product by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}\, d\theta$$

and

$$\|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2\, d\theta$$

Let $e_n(\theta) = e^{in\theta}$. Then $\{e_n : n \in \mathbb{Z}\}$ forms an orthonormal system (ONS) in $L^2(S^1)$, because

$$\langle e_n, e_m\rangle = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

Let

$$\langle f, e_n\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt = a_n.$$

Then

$$S_N(f) = \sum_{|n|\leq N} a_n e_n.$$

Note that

$$f - \sum_{|n|\leq N} a_n e_n \perp e_n \quad \text{for all } |n| \leq N$$

Hence,

$$\left(f - \sum_{|n|\leq N} a_n e_n\right) \perp \sum_{|n|\leq N} b_n e_n$$

whenever $b_n \in \mathbb{C}$.

By the Pythagorean theorem,

$$f = f - \sum_{|n|\leq N} a_n e_n + \sum_{|n|\leq N} a_n e_n,$$

it follows that

$$\|f\|_2^2 = \|f - \sum_{|n|\leq N} a_n e_n\|_2^2 + \sum_{|n|\leq N} |a_n|^2$$

or

$$\|f\|_2^2 = \|f - S_N(f)\|_2^2 + \sum_{|n|\leq N} |a_n|^2 \tag{1}$$

Since $f \in L^2(S^1)$, we get $\sum_{|n|\leq N} |a_n|^2 \leq \|f\|_2^2$ for each $N \in \mathbb{N}$ (**Bessel's inequality**).

1.7 Best Approximation Lemma

Lemma 1.7.1. *Let $f \in L^2[0, 2\pi]$ and $a_n = \hat{f}(n)$. Then*

$$\|f - S_N(f)\|_2 \leq \|f - \sum_{|n|\leq N} c_n e_n\|_2$$

for any sequence $(c_n) \subset \mathbb{C}$. Moreover, equality holds if $c_n = a_n$ for all $|n| \leq N$.

Proof.

$$f - \sum_{|n|\leq N} c_n e_n = f - S_N(f) + \sum_{|n|\leq N} (a_n - c_n) e_n$$

Let $a_n - c_n = b_n$. Then by orthogonality,

$$\left\| f - \sum_{|n|\leq N} c_n e_n \right\|_2^2 = \|f - S_N(f)\|_2^2 + \left\| \sum_{|n|\leq N} b_n e_n \right\|_2^2 \tag{1}$$

So,

$$\|f - S_N(f)\|_2 \leq \left\| f - \sum c_n e_n \right\|_2.$$

But equality holds if and only if $\|\sum b_n e_n\|_2^2 = 0$, if and only if $b_n = 0$. That is, Fourier approximation is best among any other approximation of the form $\sum_{|n|\leq N} c_n e_n$. □

1.8 Mean Square Convergence

Theorem 1.8.1. *If $f \in R[-\pi, \pi]$, then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - S_N(f)(x)|^2 dx \to 0 \text{ as } N \to \infty$$

(i.e. $\|f - S_N(f)\|_2 \to 0$*).*

Proof. First, we suppose f is continuous. Then for $\epsilon > 0$, there exists a trigonometric polynomialnomial P such that

$$|f(x) - P(x)| < \epsilon \quad \text{for all } x \in [-\pi, \pi].$$

Let $\deg P = k$. Then $\langle P, e_n \rangle \neq 0$ for $|n| = k$, and by the best approximation lemma,

$$\|f - S_N(f)\|_2^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - P(x)|^2 dx \leq \epsilon \quad \text{for all } N > k$$

Now, if $f \in R[-\pi, \pi]$, then for $\epsilon > 0$, there exists $g \in C[-\pi, \pi]$ such that

$$\sup |g(x)| \leq \sup |f(x)| \leq M$$

and

$$\int |f(x) - g(x)| dx < \epsilon^2$$

Hence,

$$\|f - g\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - g(x)||f(x) - g(x)| dx < \frac{2M}{2\pi} \epsilon^2 \tag{2}$$

Since

$$\|g - S_N(g)\|_2 < \epsilon \text{ for all } N > k, \tag{3}$$

from (2) and (3), we get

$$\begin{aligned}
\|f - S_N(f)\|_2 &\leq \|f - g\|_2 + \|g - S_N(g)\|_2 + \|S_N(g - f)\|_2 \\
&\leq \sqrt{\frac{2M}{2\pi}} \epsilon + \epsilon + \sum_{|n| \leq M} |(f - g)\hat{}(n)|^2 \\
&\leq \sqrt{\frac{M}{\epsilon}} \epsilon + \epsilon + \|f - g\|_2^2 \\
&\leq \sqrt{\frac{M}{\pi}} \epsilon + 2\epsilon \text{ for all } N > k.
\end{aligned}$$

□

Corollary 1.8.2. *If* $f \in L^2(S^1)$*, then* $\|f - S_N(f)\|_2 \to 0$*.*

Since

$$\overline{R[-\pi, \pi]} = L^2[-\pi, \pi]$$

Further,

$$\|f\|_2^2 = \|f - S_N(f)\|_2^2 + \sum_{|n| \leq N} |a_n|^2$$

implies

$$\|f\|_2^2 = \lim_{N\to\infty} \sum_{|n|\le N} |a_n|^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \quad \textbf{(Parseval's Identity)}.$$

The set $\{e_n : n \in \mathbb{Z}\}$ is a complete orthonormal system (ONS). For this, let $f \in L^2(S^1)$ and $\langle f, e_n\rangle = 0,$ for all $n \in \mathbb{N}$. Then, $f = 0$ by uniqueness of Fourier series, since $L^2(S^1) \subset L^2(S^1)$.

Now, for $f, g \in L^2(S^1)$

$$\langle f, g\rangle = \left\langle \lim_{N\to\infty} \sum_{|n|\le N} \langle f, e_n\rangle e_n, g \right\rangle = \lim_{N\to\infty} \sum \langle f, e_n\rangle\langle e_n, g\rangle = \sum \langle f, e_n\rangle \overline{\langle g, e_n\rangle}$$

that is

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

Exercise 1.8.3. Let $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$. Then there exists a unique $f \in L^2(S^1)$ such that $\hat{f}(n) = a_n$.

Proof. Consider

$$\sum a_n e_n(t) = \sum a_n e^{int}$$

then

$$\sum \left|a_n e^{int}\right|^2 = \sum |a_n|^2 \cdot 1 < \infty.$$

That is, $\sum a_n e^{int}$ is absolutely summable in $L^2(S^1)$. Set $f = \sum a_n e^{int}$. Then $f \in L^2(S^1)$ and $\langle f, e_n\rangle = a_n = \hat{f}(n)$. Since the Fourier series of any L^2 function is unique, it follows that f must be unique. □

Now we end the topic of Fourier series by the following optimal result about the convergence of the Fourier series.

Theorem 1.8.4. *Let $f \in R[-\pi, \pi]$ and $\hat{f}(n) = O(1/n)$. Then $S_n(f)(t) \to f(t)$ if t is a point of continuity of f; and the limit is uniform if f is continuous on $[-\pi, \pi]$.*

Proof. We know that

$$\sigma_n(f;t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \hat{f}(j)e^{ijt} = S_n(f)(t) - \sum_{|j|\le n} \frac{|j|}{n+1} \hat{f}(j)e^{ijt}$$

Since $\sigma_n(f;t) \to f(t)$ at the point of continuity of f, we need to show that the residual in the RHS is negligible. For $0 \le n < m$, define

$$\sigma_{m,n}(f;t) = \frac{S_{m+1}(f)(t) + \cdots + S_n(f)(t)}{n-m} = \frac{(n+1)\sigma_{n+1}(f;t) - (m+1)\sigma_{m+1}(f;t)}{n-m} \qquad (1)$$

Thus,

$$\sigma_{m,n} = S_m + \sum_{m<|j|\le n} \frac{n+1-|j|}{n-m} \hat{f}(j) e_j,$$

where $e_j(t) = e^{ijt}$. For each fixed $k \in \mathbb{N}$, from (1),

$$\begin{aligned}\sigma_{kn,(k+1)n}(f;t) &= \frac{\{(k+1)n+1\}\sigma_{(k+1)n+1}(f;t) - (kn+1)\sigma_{kn+1}(f;t)}{n} \\ &\to (k+1)f(t) - kf(t) = f(t) \text{ as } n \to \infty.\end{aligned}$$

Further, if $nk \le m < (k+1)n$, then

$$|\sigma_{kn,(k+1)n}(f;t) - S_m(f;t)| \le \sum_{kn<|j|\le(k+1)n} |\hat{f}(j)| \le 2 \sum_{j=nk+1}^{(k+1)n} \frac{A}{j} \le \frac{2nA}{kn} = \frac{2A}{k}.$$

Now, for fixed k_0, choose $n_0 \ge k_0$ such that for all $n \ge n_0$

$$|\sigma_{k_0 n,(k_0+1)n}(f;t) - f(t)| < \epsilon/2 \qquad (3)$$

For $\epsilon > 0$, select k_0 so large that $2A/k_0 < \epsilon/2$. Then for $m > k_0 n_0$, and for some $n \ge n_0$, $k_0 n_0 \le k_0 n \le m < (k_0+1)n$,

$$|\sigma_{k_0 n,(k_0+1)n}(f;t) - S_m(f)(t)| < \frac{2A}{k_0} < \frac{\epsilon}{2} \tag{4}$$

From (3) and (4), for $m \ge k_0 n_0 = N_0$ (say), we get $|S_m(f)(t) - f(t)| < \epsilon$.

□

1.9 Isoperimetric problem

Theorem 1.9.1. *Let γ be a simple closed curve in $\mathbb{R}^2$ of length l, and it encloses the area A. Then $A \le \frac{l^2}{4\pi}$. Equality holds if and only if γ is a circle.*

Proof. By using dilation, we can assume that $l = 2\pi$. Then $A \le \pi$. Let $\gamma : [0, 2\pi] \xrightarrow{C^1} \mathbb{R}^2$ be given by $\gamma(t) = (x(t), y(t))$, such that

$$(x'(t))^2 + (y'(t))^2 = 1.$$

(i.e. γ was traced by a particle with constant speed). Then

$$\frac{1}{2\pi} \int_0^{2\pi} ((x'(t))^2 + (y'(t))^2) dt = 1 \tag{1}$$

Since γ is closed, $x(t)$ and $y(t)$ are 2π-periodic. Hence,

$$x(t) \sim \sum a_n e^{int}, \qquad y(t) \sim \sum b_n e^{int}.$$

As γ is given smooth, γ can be considered to be a continuously differentiable curve, i.e. $\gamma \in C^1([0, 2\pi])$, and

$$x'(t) \sim \sum (in) a_n e^{int}, \qquad y'(t) \sim \sum (in) b_n e^{int}$$

By the Parseval identity, (1) gives

$$\sum_{n=-\infty}^{\infty} n^2(|a_n|^2 + |b_n|^2) = 1 \tag{2}$$

Since $x(t)$ and $y(t)$ are real-valued, we have $a_n = \overline{a_{-n}}$ and $b_n = \overline{b_{-n}}$. Now, by bilinear form of the Parseval identity,

$$A = \frac{1}{2}\left|\int_0^{2\pi} (x(t)y'(t) - x'(t)y(t))dt\right| = \pi \left|\sum_{n=-\infty}^{\infty} n(a_n\overline{b_n} - b_n\overline{a_n})\right| \tag{3}$$

Here,

$$|a_n\overline{b_n} - b_n\overline{a_n}| \leq 2|a_n||b_n| \leq |a_n|^2 + |b_n|^2$$

Since $|n| \leq n^2$, from (3) we get:

$$A \leq \pi \sum |n|^2(|a_n|^2 + |b_n|^2) = \pi \quad \text{(by (2))}$$

When $A = \pi$, it follows that

$$x(t) = a_{-1}e^{-it} + a_0 + a_1 e^{it} \text{ and } y(t) = b_{-1}e^{-it} + b_0 + b_1 e^{it} \qquad \text{(from (3))}$$

From (2),

$$2(|a_1|^2 + |b_1|^2) = 1, \qquad (\text{ since } a_{-1} = \overline{a_1},\ b_{-1} = \overline{b_1})$$

that is

$$a_1 = \frac{1}{2}e^{i\alpha}, \qquad b_1 = \frac{1}{2}e^{i\beta}$$

The fact that $1 = 2|a_1\overline{b_1} - b_1\overline{a_1}|$, we get

$$|\sin(\alpha - \beta)| = 1 \implies \alpha = \beta = k\pi/2$$

$$\Rightarrow \qquad x(t) = a_0 \pm \cos(\alpha + t), \qquad y(t) = b_0 \pm \sin(\alpha + t).$$

$\square$

1.10 Exercise

1. Determine whether each of the following statements is **TRUE** or **FALSE**, providing rigorous justification in each case.

   (a) Let D_n denote the Dirichlet kernel on S^1. Does the identity $D_n * D_n = D_n$ necessarily hold?

   (b) Does there exist a function $f \in L^1(S^1)$ such that $\sum_{n=-\infty}^{\infty} |n\hat{f}(n)|^2 = \infty$?

2. Suppose f is continuously differentiable on S^1. Show that

$$\widehat{f'}(n) = in\hat{f}(n) \quad \text{for all } n \in \mathbb{Z}.$$

   Deduce that there exists a constant $C > 0$ such that

$$|\hat{f}(n)| \leq \frac{C}{|n|}.$$

   Does this conclusion remain valid if f is absolutely continuous?

3. Let f be of bounded variation on $[-\pi, \pi]$. Prove that

$$|\hat{f}(n)| \leq \frac{\mathrm{Var}(f)}{2\pi|n|}$$

   for all $n \in \mathbb{Z}$.

4. For $f \in L^1(S^1)$, establish that

$$\hat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi}[f(x) - f\left(x + \frac{\pi}{n}\right)]e^{-inx}\, dx.$$

   Use this identity to prove the Riemann–Lebesgue lemma.

5. Let $f \in L^1(S^1)$ satisfy the Hölder condition

$$|f(x+h) - f(x)| \leq M|h|^{\alpha}$$

   for all $x, h \in S^1$, where $0 < \alpha < 1$ and $M > 0$. Show that

$$\hat{f}(n) = O\left(\frac{1}{|n|^{\alpha}}\right).$$

6. Demonstrate that Fejér's kernel F_n can be expressed as

$$F_n(t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n}\right) e^{ijt}.$$

7. Given $f \in L^1(S^1)$ and $m \in \mathbb{N}$, define $f_m(t) = f(mt)$. Prove that

$$\hat{f}_m(n) = \begin{cases} \hat{f}\left(\frac{n}{m}\right), & \text{if } m \mid n, \\ 0, & \text{otherwise.} \end{cases}$$

8. For $f : S^1 \to \mathbb{C}$, and for $x, y \in S^1$, define the translation operator $\tau_x f(y) = f(x - y)$. Prove that the map $x \mapsto \tau_x f$ is continuous in $L^p(S^1)$ for $1 \leq p < \infty$. That is,

$$\|\tau_x f - f\|_p \to 0 \quad \text{as } |x| \to 0.$$

Does this continuity hold for $p = \infty$?

9. Let $f \in L^1(S^1)$ and $g \in L^\infty(S^1)$. Show that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)g(nt)\, dt = \hat{f}(0)\hat{g}(0).$$

10. Given $f \in L^1(S^1)$, define the convolution operator $T_f : L^1(S^1) \to L^1(S^1)$ by $T_f(g) = f * g$. Prove that T_f is a bounded operator and that its operator norm satisfies

$$\|T_f\| = \|f\|_1.$$

11. Let P be a trigonometric polynomialnomial of degree n on S^1. Show that

$$\|P'\|_\infty \leq 2n\|P\|_\infty.$$

12. For $1 \leq p \leq \infty$ with $p^{-1} + q^{-1} = 1$, and $f \in L^p(S^1)$, $g \in L^q(S^1)$, prove that the convolution $f * g$ is continuous on S^1.

13. Suppose $f \in L^\infty(S^1)$ satisfies

$$|\hat{f}(n)| \leq \frac{k}{|n|}$$

for some constant $k > 0$ and all $n \in \mathbb{Z} \setminus \{0\}$. Prove that

$$|S_n(f)(t)| \leq \|f\|_\infty + 2k,$$

where $S_n(f) = D_n * f$.

14. If f is a bounded monotone function on S^1, show that

$$\hat{f}(n) = O\left(\frac{1}{|n|}\right).$$

15. Let f be Riemann integrable on $[-\pi, \pi]$. Prove that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 < \infty,$$

from which it follows that $\hat{f}(n) = o(1)$.

16. Prove that if the series $\sum_{n=0}^{\infty} a_n$ of complex numbers converges to s, then it is both Cesàro and Abel summable to s.

17. Prove that if the series $\sum_{n=0}^{\infty} a_n$ is Cesàro summable to σ, then it is Abel summable to σ. Show by counterexample that the converse need not hold.

18. Suppose the series $\sum_{n=0}^{\infty} a_n$ is Cesàro summable to l. Show that

$$\lim_{n\to\infty} \frac{a_n}{n} = 0,$$

where $s_n = a_1 + \cdots + a_n$.

19. Define $u(r,\theta) = \frac{\partial P_r}{\partial \theta}(\theta)$, where $P_r(\theta)$ is the Poisson kernel on the open unit disk $\mathbb{D} = \{re^{i\theta} : 0 \le r < 1, \theta \in [-\pi, \pi)\}$. Prove that

$$\Delta u = 0 \quad \text{on } \mathbb{D}$$

and

$$\lim_{r\to 1} u(r,\theta) = 0$$

for every $\theta \in [-\pi, \pi)$.

20. Let f be Riemann integrable on $[-\pi, \pi]$ and define the Abel mean

$$A_r(f)(\theta) = f * P_r(\theta), \quad 0 \le r < 1.$$

If f has a jump discontinuity at θ, prove that

$$\lim_{r\to 1} A_r(f)(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}.$$

Provide justification for why

$$\lim_{r\to 1} A_r(f)(\theta) \neq \frac{f(\theta)}{2}$$

when f is continuous at θ.

21. Let f be Riemann integrable on $[-\pi, \pi]$ and $\sigma_n(f)(\theta) = f * F_n(\theta)$, where F_n is Fejér's kernel. If f has a jump discontinuity at θ, prove that

$$\lim_{n\to\infty} \sigma_n(f)(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}.$$

22. Suppose f is Riemann integrable on $[-\pi, \pi]$ such that

$$\hat{f}(n) = O\left(\frac{1}{|n|}\right) \quad \text{for all } n \in \mathbb{Z}.$$

Prove the following assertions:

(a) If f is continuous at θ, then

$$S_N(f)(\theta) = D_N * f(\theta) \to f(\theta) \quad \text{as } N \to \infty.$$

(b) If f has a jump discontinuity at θ, then

$$S_N(f)(\theta) \to \frac{f(\theta^+) + f(\theta^-)}{2} \quad \text{as } N \to \infty.$$

(c) If f is continuous on $[-\pi, \pi]$, then the convergence

$$S_N(f) \to f$$

is uniform.

23. Assume f is a Lebesgue measurable function on S^1 satisfying

$$\int_0^{2\pi} \frac{|f(t)|}{t}\, dt < \infty.$$

Show that

$$\lim_{n\to\infty} S_n(f; 0) = 0.$$

24. For $f \in L^2(S^1)$, prove that

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) \to \hat{f}(0)$$

in the L^2-metric as $n \to \infty$.

25. Does there exist a function $f \in L^1(S^1)$ such that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \infty?$$

26. Suppose $f \in L^1(S^1)$ vanishes on a neighborhood of $x = 0$. Prove that

$$S_N(f) \to 0$$

uniformly near $x = 0$.

27. Let f be a function on $[-\pi, \pi]$ satisfying the Lipschitz condition

$$|f(\theta) - f(\varphi)| \leq M|\theta - \varphi|,$$

for some $M > 0$ and all $\theta, \varphi \in [-\pi, \pi]$.

(a) For

$$u(r, \theta) = P_r * f(\theta),$$

show that $\frac{\partial u}{\partial \theta}$ exists for all $0 \leq r < 1$ and that

$$\left|\frac{\partial u}{\partial \theta}\right| \leq M.$$

(b) Demonstrate that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \leq |\hat{f}(0)| + 2M\sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}}.$$

28. If f is continuously differentiable on S^1, show that

$$\sum_{n=-\infty}^{\infty} (1 + |n|^2)|\hat{f}(n)|^2 < \infty.$$

29. Let $\{G_n\}_{n=1}^{\infty}$ be a family of good kernels on S^1. Prove that

$$\lim_{n\to\infty} \hat{G}_n(k) = 1.$$

30. Let f and g be Riemann integrable on $[-\pi, \pi]$. Define $\tilde{g}(x) = \overline{g(-x)}$.

(a) Show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(t)|^2 \, dt = (g * \tilde{g})(0).$$

(b) Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|(f*g)(x)|^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}|(f*\tilde{g})(x)|^2\,dx.$$

31. Let $f \in L^1(S^1)$ satisfy $\hat{f}(|n|) = -\hat{f}(-|n|) \geq 0$ for all $n \in \mathbb{Z}$. Show that

$$\sum_{n>0}\frac{\hat{f}(n)}{n} < \infty.$$

32. If $\{K_n\}_{n=1}^{\infty}$ and $\{J_n\}_{n=1}^{\infty}$ are families of good kernels on S^1, show that $\{K_n * J_n\}_{n=1}^{\infty}$ is also a family of good kernels.

33. Suppose f is absolutely continuous on S^1 with $f' \in L^2(S^1)$. Prove that

$$\sum_{n=-\infty}^{\infty}|\hat{f}(n)| \leq \|f\|_1 + 2\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}\,\|f'\|_2.$$

34. Show that there exists a function $f \in L^1(S^1)$ for which the partial sums $S_n(f)$ of its Fourier series fail to converge to f in the L^1-norm.

35. Let $f \in L^1(S^1)$ and $S_n(f)$ denote the n-th partial sum of the Fourier series of f. Show that

$$\left\|\frac{S_n(f)}{n}\right\|_1 \to 0 \quad \text{as } n \to \infty.$$

36. If f is Riemann integrable on $[-\pi,\pi]$ and differentiable at $t_0 \in [-\pi,\pi]$, prove that

$$S_n(f;t_0) \to f(t_0) \quad \text{as } n \to \infty.$$

37. Suppose $f \in C^1(S^1)$ satisfies

$$(f * (1+f))(t) = f'(t)$$

for all $t \in S^1$. Prove that f is a trigonometric polynomialnomialnomial.

Chapter 2

The Fourier Transform

On $\mathbb{R}^n$, translations form a non-compact abelian group, so the Fourier expansion of a non-periodic function is no longer discrete. The Fourier transform replaces the Fourier coefficients $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ by a continuous *frequency variable $\xi \in \mathbb{R}^n$. It linearizes convolution, converts differentiation into multiplication, and provides the natural L^2 isometry (Plancherel).*

Learning objectives.

- Understand characters of $\mathbb{R}^n$ and how they motivate the definition of the Fourier transform.
- Prove the basic identities: translation/modulation, scaling, convolution, and differentiation rules.
- Establish inversion and the Plancherel theorem, and see how L^p estimates (Hausdorff–Young, Young) fit into the picture.

Fourier analysis may be viewed as the systematic study of functions through the exploitation of their underlying symmetries. In the case of Fourier series, we observed that when a function is periodic on $\mathbb{R}$, it suffices to restrict attention to a single fundamental period. Each period contributes precisely one Fourier coefficient, so that the entire function is encoded by a countable collection of complex numbers. By contrast, when f is not periodic, a different framework is required, though the central idea remains the same: to understand how a function on $\mathbb{R}^n$ (or on $\mathbb{T}^n$) transforms under the action of translations.

Suppose the function f transforms under the translation by a multiplication of absolute value 1. That is,

$$f(x+y) = \varphi(x)f(y), \quad \text{where } |\varphi(x)| = 1.$$

Then

$$f(x) = \varphi(x)f(0).$$

That is, f is completely determined by φ. Moreover,

$$\varphi(x)\varphi(y)f(0) = \varphi(x)f(y) = f(x+y) = \varphi(x+y)f(0)$$

$$\implies \varphi(x+y) = \varphi(x)\varphi(y), \quad f \not\equiv 0.$$

Hence, to determine all such f that transform as above, it is enough to find out those φ such that

$$\varphi(x+y) = \varphi(x)\varphi(y).$$

Theorem 2.0.1 (Characters of $\mathbb{R}^n$)**.** *Let $\varphi : \mathbb{R}^n \to \mathbb{C}$ be measurable and satisfy*

$$\varphi(x+y) = \varphi(x)\varphi(y), \qquad |\varphi(x)| = 1, \qquad x, y \in \mathbb{R}^n.$$

Then there exists $\xi \in \mathbb{R}^n$ such that

$$\varphi(x) = e^{2\pi i\, x\cdot\xi}, \qquad x \in \mathbb{R}^n.$$

Proof. We first treat the case $n = 1$. Since $|\varphi| = 1$, we have $\varphi \in L^1_{\text{loc}}(\mathbb{R})$. Choose $a \in \mathbb{R}$ such that $\displaystyle\int_0^a \varphi(t)\,dt \neq 0$ and set $A^{-1} := \int_0^a \varphi(t)\,dt$. Using $\varphi(x+t) = \varphi(x)\varphi(t)$ we obtain

$$\varphi(x) = A\int_0^a \varphi(x+t)\,dt = A\int_x^{x+a} \varphi(t)\,dt.$$

In particular, φ is continuous (as a translate of an absolutely continuous primitive of φ), hence differentiable. Differentiating the identity above and using again the functional equation gives

$$\varphi'(x) = A\big(\varphi(x+a) - \varphi(x)\big) = A\big(\varphi(a) - 1\big)\varphi(x) =: B\,\varphi(x).$$

Solving the ODE yields $\varphi(x) = e^{Bx}$, and the condition $|\varphi(x)| = 1$ forces $B = 2\pi i\,\xi$ for some $\xi \in \mathbb{R}$.

For general n, let $e_1, \dots, e_n$ be the standard basis and define $\varphi_j(t) := \varphi(te_j)$. Each φ_j satisfies the one-dimensional hypotheses, hence $\varphi_j(t) = e^{2\pi i\,\xi_j t}$ for some $\xi_j \in \mathbb{R}$. Using the functional equation repeatedly,

$$\varphi(x) = \varphi\left(\sum_{j=1}^n x_j e_j\right) = \prod_{j=1}^n \varphi(x_j e_j) = \prod_{j=1}^n e^{2\pi i\,\xi_j x_j} = e^{2\pi i\, x\cdot\xi},$$

where $\xi = (\xi_1, \dots, \xi_n)$. □

Corollary 2.0.2. *If $\varphi : \mathbb{T} \to \mathbb{C}$ measurable and $\varphi(x+y) = \varphi(x)\varphi(y)$ with $|\varphi(x)| = 1$, then $\varphi(x) = e^{2\pi inx}$ for some $n \in \mathbb{Z}$.*

Proof. Notice that φ is periodic with period 1 if and only if $\varphi(0) = \varphi(1)$, if and only if $e^{2\pi i\xi} = 1$ if and only if $\xi \in \mathbb{Z}$. That is, $\varphi(x) = e^{2\pi inx}$, $n \in \mathbb{Z}$.

□

Exercise 2.0.3. If $\varphi : \mathbb{T}^n \to \mathbb{C}$ measurable and $|\varphi(x)| = 1$,

$$\varphi(s+t) = \varphi(s)\varphi(t),$$

then show that

$$\varphi(t) = e^{2\pi it.\alpha}, \quad \alpha \in \mathbb{Z}^n$$

Thus, we conclude that those functions which transform as above, satisfying

$$f(x+y) = e^{2\pi ix\cdot\xi} f(y), \text{ for some } \xi \in \mathbb{R}^n \text{ (or } \mathbb{Z}^n).$$

For the time being, we keep in mind the model eigenfunctions $x \mapsto e^{2\pi ix\cdot\xi}$, which satisfy $f(x+y) = e^{2\pi ix\cdot\xi} f(y)$.

2.1 Definition of the Fourier transform

Definition 2.1.1. Let $f \in L^1(\mathbb{R})$ (or $L^1(\mathbb{R}^n)$), then we define its Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

Lemma 2.1.2. *Let $f \in L^1(\mathbb{R}^n)$. Then*

(i) $(\tau_y f)^\wedge(\xi) = e^{-i\xi\cdot y}\hat{f}(\xi)$, *where* $\tau_y f(x) = f(x-y)$.

(ii) *If* $g(x) = e^{i\alpha\cdot x} f(x)$, *then* $\hat{g}(\xi) = \hat{f}(\xi - \alpha) = (\tau_\alpha \widehat{f})(\xi)$.

(iii) *If* $g(x) = \overline{f(-x)}$, *then* $\hat{g}(\xi) = \overline{\hat{f}(\xi)}$.

(iv) *If* $g(x) = f(\frac{x}{\lambda}), \lambda > 0$ *then* $\widehat{g}(\xi) = \lambda \widehat{f}(\lambda\xi)$

(v) $|\widehat{f}(\xi)| \le \|f\|_1$ *(uniformly bounded).*

(vi) *If* $f, g \in L^1(\mathbb{R}^n)$, *then* $(f * g)^\wedge(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.
(Hint: use Fubini's theorem and change of variable.)

Lemma 2.1.3. *Let $f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ is uniformly continuous on $\mathbb{R}^n$.*

Proof. Let $x_n, y_n \in \mathbb{R}^n$, be such that $|x_n - y_n| \to 0$. Then

$$|\hat{f}(x_n) - \hat{f}(y_n)| = \left|\int f(\xi)(e^{-ix_n\cdot\xi} - e^{-iy_n\cdot\xi})d\xi\right| \le \int |f(\xi)||e^{-i(x_n-y_n)\cdot\xi} - 1|d\xi$$

For each fixed ξ, $e^{-ix\cdot\xi}$ is uniformly continuous. It follows by Dominated Convergence Theorem (DCT) that

$$|\hat{f}(x_n) - \hat{f}(y_n)| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\hat{f}$ is uniformly continuous on $\mathbb{R}^n$.

$\square$

Lemma 2.1.4. *Let $f \in L^1(\mathbb{R})$ and f is uniformly continuous. Then*

$$\lim_{|x|\to\infty} f(x) = 0.$$

Proof. Suppose $\lim_{|x|\to\infty} f(x) \neq 0$, then for some $\epsilon_0 > 0$, there exists $x_0 \in \mathbb{R}$ such that $|f(x_0)| > \epsilon_0$, $|x_0| > \delta$ for all $\delta > 0$. By continuity at x_0, there exists $\delta_0 > 0$ such that if $|x - x_0| < \delta_0$ implies $|f(x) - f(x_0)| < \frac{\epsilon_0}{2}$ implies $|f(x)| > \epsilon_0/2$. By uniform continuity, $|f(x)| > \epsilon_0/2$ on each interval of length $2\delta_0$. Since $y \in (x_0 - 2\delta_0, x_0 - \delta_0)$, $|f(y)| > \epsilon_0/2 \implies |x_0 - y| < \delta_0 \implies |f(y)| > \frac{\epsilon_0}{2}$. Hence

$$\int_{|y|>\delta} |f(y)|dy = \sum \int_{x_0 - n\delta_0}^{x_0+(n+1)\delta_0} |f(y)|dy \geq \sum_{n\in\mathbb{Z}} \delta \cdot \epsilon_0/2 = \infty$$

$\square$

We use this fact to prove the following result.

Theorem 2.1.5. *Let $f \in L^1(\mathbb{R})$ and $xf(x) \in L^1(\mathbb{R})$, then $\hat{f}$ is differentiable and*

$$\frac{\partial}{\partial\xi}\hat{f}(\xi) = -\widehat{(ixf)}(\xi)$$

Proof.

$$\frac{\hat{f}(\xi + h) - \hat{f}(\xi)}{h} = \int f(x)e^{-ix\xi}\frac{(e^{-ixh} - 1)}{h}dx$$

Notice that

$$\left|\frac{e^{-ixh} - 1}{h}\right| \leq |x|, \quad \frac{e^{-ixh} - 1}{h} \to -ix \text{ as } h \to 0.$$

Hence, the integrand on the RHS is bounded by $|xf(x)| \in L^1(\mathbb{R})$. By DCT, it follows that

$$\frac{\partial}{\partial\xi}\hat{f}(\xi) = \int f(x)e^{-ix\xi}(-ix)dx = \widehat{(-ixf)}(\xi).$$

$\square$

Theorem 2.1.6. *Let $f \in L^1(\mathbb{R})$, and $F(x) = \int_{-\infty}^{x} f(y)dy$. If $F \in L^1(\mathbb{R})$ then $\hat{F}(\xi) = \frac{1}{i\xi}\hat{f}(\xi)$, $\xi \neq 0$.*
Equivalently, if $f, f' \in L^1(\mathbb{R})$ then $\widehat{f'}(\xi) = i\xi\hat{f}(\xi)$ f' is the derivative of f.

Proof. By Fundamental theorem of calculus (FTC), it follows that $F' = f$ a.e. on $\mathbb{R}$. Since $F \in L^1(\mathbb{R})$, we have

$$\int_{-\infty}^{\infty} F(x)e^{-ix\cdot y}dx = \frac{F(x)e^{-ixy}}{-iy}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)e^{-ixy}\frac{dx}{-iy}$$

Since $F(x)e^{-ixy} \in L^1(\mathbb{R})$ and uniformly continuous, by the previous Lemma 2.1.4,

$$\hat{F}(y) = \frac{1}{iy}\hat{f}(y), \qquad y \neq 0$$

or

$$\widehat{f'}(y) = iyf'(y), \quad \text{if } f, f' \in L^1(\mathbb{R}).$$

□

Lemma 2.1.7. *Let $C_c^\infty(\mathbb{R})$ be the space of all infinitely differentiable functions on $\mathbb{R}$ having compact support. Then*

$$\overline{C_c^\infty(\mathbb{R})} = L^1(\mathbb{R}).$$

Proof. Let $f \in L^1(\mathbb{R})$. Since $\overline{C_c(\mathbb{R})} = L^1(\mathbb{R})$, for $\epsilon > 0$, there exists $g \in C_c(\mathbb{R})$ such that $\|g - f\|_1 < \epsilon$. Now, consider $\varphi \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \varphi = 1$. For $t > 0$, let $\varphi_t(x) = t^{-1}\varphi(x/t)$. Then $\int \varphi_t = 1$. Hence, $g * \varphi_t \in C_c^\infty(\mathbb{R})$ (exercise). Now

$$g * \varphi_t(x) - g(x) = \int (g(x-y) - g(x))\varphi_t(y)dy = \int (g(x-tz) - g(x))\varphi(z)dz \tag{2.1.1}$$

$$\Rightarrow \|g * \varphi_t - g\|_1 \leq \int \|\tau_{tz}g - g\||\varphi(z)|dz$$

For small t, $\|\tau_{tz}g - g\| < \epsilon$. By DCT it follows that $\|g * \varphi_t - g\|_1 < \epsilon$ for all $|t| < \delta$. So $\|g * \varphi_t - f\|_1 < 2\epsilon$ for all $|t| < \delta$.

□

Exercise 2.1.8. For $1 \leq p < \infty$, show that

$$\overline{C_c^\infty(\mathbb{R})} = L^p(\mathbb{R}), \qquad \overline{C_c^\infty(\mathbb{R})} = C_0(\mathbb{R}).$$

(Hint: use Minkowski integral inequality in (2.1.1).)

2.2 Riemann-Lebesgue Lemma

Theorem 2.2.1. *If $f \in L^1(\mathbb{R})$, then $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.*

Proof. Since $f \in L^1(\mathbb{R})$, for $\epsilon > 0$, there exists $g \in C_c^\infty(\mathbb{R})$ such that $\|g - f\|_1 < \epsilon$. Given g is differentiable, $\widehat{g'}(x) = (ix)\widehat{g}(x)$, by Theorem 2.1.6. So $|x\widehat{g}(x)| \leq \|g'\|_1 < \infty$. Hence $|\hat{g}(\xi)| \to 0$ as

$|\xi| \to \infty$.

Now

$$|\hat{f}(\xi) - \hat{g}(\xi)| \le \|f - g\|_1 < \epsilon.$$

Letting $|x| \to \infty$, then $|\widehat{f}(x)| \le \epsilon$, for all $\epsilon > 0$. Which implies

$$\lim_{|x|\to\infty} \widehat{f}(x) = 0.$$

□

Notice that $(L^1(\mathbb{R}))^\wedge \subsetneq C_0(\mathbb{R})$. In fact, the inclusion is injective but **not** surjective. That is, every continuous function vanishing at ∞ need not be the Fourier transform (FT) of an L^1 function. This is based on the fact that F.T. of an L^1 function can't too far from being an L^1 function.

Suppose $g \in C_0(\mathbb{R})$ is an odd function such that $g = \hat{f}$, for some $f \in L^1(\mathbb{R})$. Then $\left|\int_1^b \frac{\hat{f}(x)}{x}\, dx\right| \le A < \infty$, where A is independent of b. This follows by the fact that $\int_\alpha^\beta |\frac{\sin t}{t}|dt \le B < \infty$, where B is free of choice of $\alpha, \beta \in \mathbb{R}$. Since $\hat{f}$ is odd (as g is odd):

$$\hat{f}(x) = -i \int_{\mathbb{R}} f(t) \sin tx\, dt$$

Consider

$$\begin{aligned} \left|\int_{-n}^{n} f(t) \left(\int_1^b \frac{\sin tx}{x} dx\right) dt\right| &= \left|\int_{-n}^{n} f(t) \left(\int_1^b \frac{\sin tx}{x} dx\right) dt\right| \\ &\le \int_{-n}^{n} |f(t)| B \le \|f\|_1 B < \infty. \end{aligned}$$

Notice that, by Fubini's theorem we can interchange the integrals in above. Hence

$$\left|\int_1^b \frac{\hat{f}(x)}{x} dx\right| \le \|f\|_1 B < \infty$$

But for

$$g(x) = \begin{cases} \frac{1}{\log x} & x > 0 \\ \frac{1}{\log |x|} & x < 0 \\ 0 & x = 0 \end{cases}$$

Then $g \in C(\mathbb{R})$ and g is odd. However,

$$\left|\int_1^b \frac{1}{x \log x} dx\right| = \infty.$$

Example 2.2.2. Let $f(x) = e^{-\pi x^2}$, the Gaussian. Then

$$F(\xi) = \hat{f}(\xi) = \int e^{-2\pi i x\xi} f(x)dx = f(\xi)$$

We know that

$$\int e^{-\pi x^2} dx = 1 \quad \text{(Exercise)}$$

Now

$$\begin{aligned} F'(\xi) &= \int (-2\pi i\xi)\, f(x) e^{-2\pi i x\xi} d\xi \\ &= (-2\pi i x f)^{\wedge}(\xi) \quad (\text{ since } f, xf \in L^1(\mathbb{R})) \\ &= i(f')^{\wedge}(\xi)(\text{ since } f'(x) = -2\pi x e^{-\pi x^2}) \\ &= i(2\pi i\xi)\widehat{f}(\xi) \\ &= -2\pi\xi F(\xi) \end{aligned}$$

That is

$$\begin{aligned} F'(\xi) &= -2\pi\xi F(\xi) \\ \implies \frac{d}{d\xi}\left(F(\xi)e^{\pi\xi^2}\right) &= 0 \\ \implies F(\xi)e^{\pi\xi^2} &= \text{const.} \end{aligned}$$

Since $F(0) = 1$, hence $F(\xi) = e^{-\pi\xi^2}$.

Remark 2.2.3. For $\delta > 0$, let $f_\delta(x) = \delta^{1/2} e^{-\pi x^2/\delta}$. Then $\widehat{f_\delta}(x) = e^{-\pi\delta x^2} \to 0$ as $\delta \to 0$, however, $f_\delta(x) \to 1$ as $\delta \to 0$. Hence, we cannot see both f_δ & $\hat{f}_\delta$ exist together. That is, f_δ and $\hat{f}_\delta$ cannot be localized together. (This is known as the Heisenberg uncertainty principle; we elaborate later.)

Example 2.2.4. If $f(x) = e^{-\pi x^2}$ then show that $|f(x)| \leq \frac{M}{1+x^2}$

Lemma 2.2.5. *Let $f, h \in L^1(\mathbb{R})$ and*

$$f(x) = \int_{\mathbb{R}} H(\xi) e^{ix\xi} d\xi$$

for some $H \in L^1(\mathbb{R})$, then

$$(h * f)(x) = \int H(\xi)\widehat{f}(\xi) e^{ix\xi} d\xi$$

Proof.

$$\begin{aligned} h * f(x) &= \int h(x-y)f(y)dy \\ &= \int\int H(\xi)e^{-i(x-y)\xi}f(y)dyd\xi \\ &= \int H(\xi)\left(\int e^{-iy\xi}f(y)dy\right)e^{ix\xi}d\xi \\ &= \int H(\xi)\widehat{f}(\xi)e^{ix\xi}d\xi \end{aligned}$$

□

2.3 Good Kernels on $\mathbb{R}$

Next, we shall consider seq. of good kernel on $\mathbb{R}$. Some more of it is known as summability kernel (or approximation of identity).

Definition 2.3.1. A seq. of functions $\{K_\lambda\} \subset L^1(\mathbb{R})$ is said to be "good kernels" if

(i) $\int K_\lambda(x)dx = 1$

(ii) $\int |K_\lambda(x)|dx \leq M$ as $\lambda \to \infty$.

(iii) $\int_{|x|>\delta} |K_\lambda(x)|dx \to 0$ as $\lambda \to \infty$, for all $\delta > 0$.

We can easily construct a sequence of good kernels in the following way. Let $f \in L^1(\mathbb{R})$ be such that $\int_{\mathbb{R}} f(x)dx = 1$. Write $K_\lambda(x) = \lambda f(\lambda x), \lambda > 0$. Then

(i) $\int K_\lambda(x)dx = \int f(y)dy = 1$ (put $y = \lambda x$)

(ii) $\|K_\lambda\|_1 = \|f\|_1 < \infty$ for all $\lambda > 0$

(iii) $\int_{|x|>\delta} |K_\lambda(x)|dx = \int_{|y|>\lambda\delta} |f(y)|dy = \int_{\mathbb{R}}(f - \chi_{\{|y|\leq\delta\lambda\}}f)dy$,

Since $f(x) - \chi_{\{|y|\leq\delta\lambda\}}(x) \to 0$ as $\lambda \to \infty$ and $|f - \chi_{\{|y|\leq\delta\lambda\}}f| \leq 2|f| \in L^1$ by DCT $\int_{|\lambda|>\delta} |K_\lambda(x)| \to 0$ as $\lambda \to \infty$. Hence, $\{K_\lambda\}_{\lambda>0}$ is a family of good kernels.

Theorem 2.3.2. *Let $f \in L^1(\mathbb{R})$ (or $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$). Then*

$$\lim_{\lambda\to\infty} \|f - K_\lambda * f\|_p = 0.$$

If $f \in L^\infty(\mathbb{R})$ and f is continuous at x, then

$$\lim_{\lambda\to\infty} (f * K_\lambda)(x) = f(x).$$

Proof.

$$|K_\lambda * f(x) - f(x)| \leq \int_{\mathbb{R}} |K_\lambda(y)(f(x-y) - f(x))|dy \qquad (1)$$

By Minkowski's integral inequality (exercise), (if $p > 1$)

$$\|K_\lambda * f - f\|_p \leq \int_{\mathbb{R}} |K_\lambda(y)| \|\tau_y f - f\|_p \, dy$$

For small $|y| < \delta$,

$$\|\tau_y f - f\|_p < \epsilon$$

Hence,

$$\begin{aligned}
\|K_\lambda * f - f\|_p &\leq \int_{|y|<\delta} |K_\lambda(y)|\epsilon dy + \int_{|y|\geq\delta} |K_\lambda(y)| \|\tau_y f - f\|_p \, dy \\
&\leq \epsilon M + \int_{|y|>\delta} |K_\lambda(y)| 2\|f\|_p dy \\
&\leq \epsilon M + 2\|f\|_p \epsilon, \text{ for } \delta > 0
\end{aligned}$$

If $f \in L^\infty(\mathbb{R})$, continuous at x, then from (1)

$$|K_\lambda * f(x) - f(x)| \leq \int_{\mathbb{R}} |K_\lambda(y)||f(x-y) - f(x)|dy$$

For small $|y| < \delta$, $|f(x-y) - f(x)| < \epsilon$. Hence,

$$|K_\lambda * f(x) - f(x)| < \epsilon M + 2\|f\|_\infty \epsilon, \quad \text{for } \delta > 0.$$

Therefore,

$$K_\lambda * f(x) \to f(x) \text{ as } \lambda \to \infty.$$

□

2.4 The Fejer Kernel on $\mathbb{R}$

The Fejer Kernel on $\mathbb{R}$ is given by

$$K_\lambda(x) = \lambda K(\lambda x), \quad \text{where}$$

$$K(x) = \frac{1}{2\pi} \left(\frac{\sin(x/2)}{x/2} \right)^2 = \int_{-1}^{1} (1 - |\xi|) e^{ix\xi} d\xi.$$

(It can be seen by evaluating the integral)

$$K_\lambda(x) = \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{ix\xi} d\xi$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \left(1 - \frac{|\xi|}{\lambda}\right) \chi_{[-\lambda,\lambda]}(\xi) e^{ix\xi} d\xi$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} G_\lambda(\xi) e^{ix\xi} d\xi$$

where

$$G_\lambda(\xi) = \left(1 - \frac{|\xi|}{\lambda}\right) \chi_{[-\lambda,\lambda]}(\xi)$$

is compactly supported.

To show K_λ is a good kernel, we need to show that

$$\int_{\mathbb{R}} K(x) dx = 1$$

For this, we use the fact that the Fejer kernel for the circle is

$$F_n(x) = \frac{1}{n+1}\left(\frac{\sin((n+1)x/2)}{\sin(x/2)}\right)^2$$

and

$$\lim_{x\to\infty} \frac{1}{2\pi}\int_{-\delta}^{\delta} F_n(x) dx = 1$$

We know that

$$\lim_{x\to 0}\left(\frac{\sin(x/2)}{x/2}\right)^2 = 1$$

For $\varepsilon = 1 - \frac{\sin(\delta)}{\delta}$, for some small $\varepsilon > 0$, there exists $\delta > 0$,

$$\left|\left(\frac{\sin(x/2)}{x/2}\right)^2 - 1\right| < \left|1 - \left(\frac{\sin\delta}{\delta}\right)^2\right|$$

That is,

$$\left(\frac{\sin(\delta)}{\delta}\right)^2 < \left(\frac{\sin(x/2)}{x/2}\right)^2$$

for $|x| < \delta$ (small). Hence,

$$\begin{aligned} \frac{1}{2\pi(n+1)}\left(\frac{\sin\delta}{\delta}\right)^2\left(\frac{\sin((n+1)x/2)}{x/2}\right)^2 \quad &\le \frac{1}{2\pi(n+1)}\left(\frac{\sin(x/2)}{x/2}\frac{\sin(n+1)x/2}{\sin x/2}\right)^2 \\ &\le \frac{1}{2\pi(n+1)}\left(\frac{\sin(n+1)x/2}{x/2}\right)^2. \end{aligned}$$

Let $K_n(x) = \frac{1}{2\pi(n+1)}\left(\frac{\sin(n+1)x/2}{x/2}\right)^2$. Then

$$\frac{1}{2\pi}\left(\frac{\sin\delta}{\delta}\right)^2\int_{-\delta}^{\delta} F_n(x)\,dx \le \int_{-\delta}^{\delta} K_n(x)\,dx \le \frac{1}{2\pi}\int_{-\delta}^{\delta} F_n(x)\,dx.$$

Since,

$$\lim_{n\to\infty}\int_{-\delta}^{\delta} K_n(x)\,dx = \int_{-\infty}^{\infty} K(x)\,dx,$$

it follows that

$$\left(\frac{\sin\delta}{\delta}\right)^2 .1 \le \|K\|_1 \le 1, \quad \forall\,\delta > 0 \quad \text{(small)}$$

$$\implies \|K\|_1 = 1.$$

Hence, $\{K_\lambda\}_{\lambda>0}$ is a family of good kernels.

2.5 Fourier uniqueness theorem

Let $f \in L^1(\mathbb{R})$. Then, by the fact that

$$f * K_\lambda(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\left(1-\frac{|\xi|}{\lambda}\right)\chi_{[-\lambda,\lambda]}(\xi)\hat{f}(\xi)e^{ix\xi}d\xi$$

it follows that

$$f = \lim_{\lambda\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}}\left(1-\frac{|\xi|}{\lambda}\right)\chi_{[-\lambda,\lambda]}(\xi)\hat{f}(\xi)e^{ix\xi}d\xi \qquad (*)$$

in the L^1-norm. Thus, if $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}$, then by $(*)$

$$\|f\|_1 = 0 \implies f = 0 \quad \text{a.e.}$$

2.6 Fourier Inversion

Theorem 2.6.1. *Let $f, \hat{f} \in L^1(\mathbb{R})$. Then*

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(\xi)e^{ix\xi}d\xi$$

holds for almost all $x \in \mathbb{R}$.

Proof. We know that

$$f(x) = \lim_{\lambda\to\infty}\int_{\mathbb{R}}\left(1-\frac{|\xi|}{\lambda}\right)\hat{f}(\xi)e^{ix\xi}d\xi \tag{2.6.1}$$

holds in L^1-norm. Hence, it follows that there is a subsequence such that (2.6.1) holds. Therefore, w.l.o.g., we can assume (2.6.1) holds a.e. Since

$$|\chi_{[-\lambda,\lambda]}\left(1-\frac{|\xi|}{\lambda}\right)\widehat{f}(\xi)| \le 2|\widehat{f}(\xi)| \in L^1(\mathbb{R})$$

and $\chi_{[-\lambda,\lambda]}(\xi)(1-\frac{|\xi|}{\lambda})\hat{f}(\xi) \to \hat{f}(\xi)$ as $\lambda \to \infty$. By Dominated Convergence Theorem, we get

$$f(x) = \lim_{\lambda\to\infty} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi \quad \text{a.e.}$$

That is, if $f, \hat{f} \in L^1(\mathbb{R})$, then

$$f = (\hat{f})^{\vee} \quad \text{a.e.}$$

□

Notice that Fejer Kernel $K_\lambda \in L^1(\mathbb{R})$ (as $\int K_\lambda(x)dx = \int K(x)dx = 1$) and

$$K_\lambda(x) = \int_{\mathbb{R}} G_\lambda(\xi) e^{ix\xi} d\xi = G_\lambda^{\vee}(x) \tag{1}$$

where $G_\lambda(\xi) = \chi_{[-\lambda,\lambda]}(\xi)\left(1-\frac{|\xi|}{\lambda}\right) \in L^1(\mathbb{R})$. In fact, $K_\lambda \in L^1(\mathbb{R})$. Therefore, by inversion formula,

$$G_\lambda = (G_\lambda^{\vee})^{\wedge} = \hat{K}_\lambda(x) \qquad \text{(from (1))}$$

That is,

$$\hat{K}_\lambda(x) = \chi_{[-\lambda,\lambda]}(x)\left(1-\frac{|x|}{\lambda}\right).$$

2.7 Plancherel Theorem

We know that if $f \in L^1(\mathbb{R})$, then $\hat{f} = \mathcal{F}(f)$ is a uniformly continuous function on $\mathbb{R}$. However, for $f \in L^2(\mathbb{R})$, $\hat{f}$ exists uniquely as a function in $L^2(\mathbb{R})$ and satisfies the isometry

$$\|\hat{f}\|_2 = \|f\|_2$$

This can be seen using the fact that $\mathcal{F}$ is a continuous linear function on dense set $L^1 \cap L^2$ to L^2.

Further, using Riesz-Thorin interpolation theorem, for $f \in L^p(\mathbb{R})$, $1 \le p \le 2$, $\hat{f}$ exists as function in $L^q(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = 1$ (This we see later). Finally, for $p > 2$, we shall see that $\hat{f}$ exists as a distribution. That is, $\hat{f}$ defined by the relation

$$\langle \hat{f}, \varphi \rangle = \int f(x)\varphi(x)dx, \quad \varphi \in C_c^{\infty}(\mathbb{R}).$$

Theorem 2.7.1. *There exists a unique operator $\mathscr{F}$ from $L^2(\mathbb{R})$ onto $L^2(\widehat{\mathbb{R}})$ having the following properties:*

$$\mathscr{F} f = \widehat{f} \text{ for } f \in L^1 \cap L^2(\mathbb{R}),$$

$$\|\mathscr{F} f\|_2 = \|f\|_2$$

Proof. For $f \in L^1 \cap L^2(\mathbb{R})$, we define

$$\hat{f}(\xi) = \int e^{-2\pi i x\xi} f(x) dx$$

Then

$$f * K_\lambda(x) = \int_{\mathbb{R}} G_\lambda(\xi) \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

where $G_\lambda(\xi) = \left(1 - \frac{|\xi|}{\lambda}\right) \chi_{[-\lambda,\lambda]}(\xi)$.

Let $\tilde{f}(x) = \overline{f(-x)}$, and $g = f * \tilde{f}$. Then $g \in L^1(\mathbb{R})$ and

$$\tilde{g}(x) = \hat{f}(x)\overline{\hat{f}(x)} = |\hat{f}(x)|^2.$$

Further,

$$g(x) = \int f(x-y)\overline{f(-y)}dy = \int f(x+y)\overline{f(y)}dy = \langle f_{-x}, f \rangle$$

As $x \mapsto f_{-x}$ is continuous from $\mathbb{R} \to L^2(\mathbb{R})$ and $< . >$ is continuous, it follows that g continuous and $|g(x)| \le \|f_x\|_2 \|f\|_2$ that is $|g(x)| \le \|f\|_2^2$.

Notice that $g \in L^\infty$ and g is continuous.

$$g * K_\lambda(0) = \int G_\lambda(\xi)\hat{g}(\xi) d\xi \to g(0) \quad \text{as } \lambda \to \infty.$$

That is,

$$\lim_{\lambda\to\infty} \int G_\lambda(\xi)\widehat{g}(\xi)d\xi = \|\widehat{f}\|_2^2 = g(0)$$

Then,

$$\lim_{\lambda\to\infty} \int G_\lambda(\xi)|\hat{f}(\xi)|^2 d\xi = \|\hat{f}\|_2^2$$

Since $G_\lambda(\xi) \uparrow 1$, by monotone convergence theorem, it follows that

$$\int |\hat{f}(\xi)|^2 d\xi = \|f\|_2^2$$

that is $||\hat{f}||_2 = ||f||_2$ for $f \in L^1 \cap L^2$.

Let $Y = \{\hat{f} \mid f \in L^1 \cap L^2\}$, then

$$\mathscr{F} : L^1 \cap L^2(\mathbb{R}) \xrightarrow{\text{onto}} Y$$

isometry. We claim that $\overline{Y} = L^2(\mathbb{R})$. By Hahn-Banach theorem, it is enough to show that $Y^\perp = \{0\}$. If $y \in Y^\perp \subset L^2$, then the fact that $G_\lambda e_x$ where $e_x(\xi) = e^{2\pi i x\xi}$ belongs to $L^1 \cap L^2$,

$$(G_\lambda e_x)^\wedge = (G_\lambda e_{-x})^\vee = \tau_x G_\lambda^\vee = \tau_x K_\lambda \in Y$$

for each $x \in \mathbb{R}$. This holds, by applying Fourier inversion to $G_\lambda = \hat{K}_\lambda(x)$ as $G_\lambda \in L^1(\mathbb{R})$. Hence, we get

$$\langle \tau_x K_\lambda, h \rangle = 0 \implies K_\lambda * \overline{h}(x) = 0$$

But $||K_\lambda * \overline{h} - \overline{h}||_2 \to 0$ as $\lambda \to \infty$

$$\implies ||h||_2 = 0 \implies Y^\perp = \{0\}$$

Hence, $\mathscr{F}$ can be extended on L^2 onto L^2 with $||\mathscr{F}f||_2 = ||f||_2$. For this, $\mathscr{F} : L^1 \cap L^2 \subset L^2 \to Y \subset L^2$. Let $g \in L^2(\mathbb{R})$, then there exists $\mathscr{F}(g_n) \in Y$ with $g_n \in L^1 \cap L^2$ such that $\mathscr{F}g_n \xrightarrow{L^2} g$ and

$$||\mathscr{F}(g_n)\|_2 = ||g_n||_2$$

It implies that g_n is Cauchy sequence in $L^1 \cap L^2(\mathbb{R})$. Hence, there exists $f \in L^2$ such that $g_n \xrightarrow{L^2} f$ and it implies that $\mathscr{F}g_n \xrightarrow{L^2} \mathscr{F}f$. Then

$$\|\mathscr{F}(f)\|_2 = \|g\|_2.$$

□

Remark 2.7.2. Let $f \in L^2(\mathbb{R})$, then $\chi_{[-n,n]}f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. If we write

$$\hat{\varphi}_n(x) = \int_{-n}^{n} e^{-2\pi i x\xi} f(\xi) d\xi$$

then

$$||\hat{\varphi}_n - \hat{f}||_2 = ||(\chi_{[-n,n]}f)^\wedge - \hat{f}||_2 = ||\chi_{[-n,n]}f - f||_2 \to 0$$

Thus,

$$\hat{f}(\xi) = \lim_{n\to\infty} \int_{-n}^{n} e^{-2\pi i x\xi} f(x) dx$$

exists in the L^2-norm.

Example 2.7.3. Let $H(x) = e^{-|x|}$. Show that

$$\hat{H}(x) = \int_{\mathbb{R}} H(t)e^{itx} dt = \frac{2}{1+x^2}$$

Note that if $f \in L^2(\mathbb{R})$, then $||\hat{f}||_2 = ||f||_2$. By polarization identity

$$\int f\overline{g} = \int \hat{f}\hat{\overline{g}}$$

where $f, g \in L^2(\mathbb{R})$.

2.8 More on Convolution

Theorem 2.8.1. *Let $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ and $\frac{1}{p}+\frac{1}{q}=1$. Then $f*g$ is an uniformly continuous and bounded function on $\mathbb{R}$ with $\|f*g\|_\infty \le \|f\|_p\|g\|_q$. In particular, if $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, then $f*g \in C_0(\mathbb{R})$.*

Proof. By Hölder's inequality, we get

$$|f*g(x)| \le \int |f(x-y)||g(y)|\,dy \le \|\tau_x f\|_p\|g\|_q = \|f\|_p\|g\|_q.$$

Therefore, $f*g$ is bounded. Further,

$$|(\tau_x(f*g))(y) - (f*g)(y)| \le \int |\tau_x f(y-\xi) - f(y-\xi)||g(\xi)|d\xi \le \|\tau_x f - f\|_p\|g\|_q.$$

Hence,

$$\|\tau_x(f*g) - (f*g)\|_\infty \le \|\tau_x f - f\|_p\|g\|_q.$$

Since $x \mapsto \tau_x f$ is uniformly continuous on $\mathbb{R} \to L^1(\mathbb{R})$, it follows that $f*g$ is uniformly continuous on $\mathbb{R}$.

Let $1<p<\infty$, then $1<q<\infty$ since $\frac{1}{p}+\frac{1}{q}=1$.

For given $\epsilon>0$, there exists f_n, g_n in $C_c^\infty(\mathbb{R})$ such that

$$\|f_n - f\|_p < \epsilon, \|g_n - g\|_p < \epsilon.$$

(since $\overline{C_c(\mathbb{R})} = L^p(\mathbb{R})$ if $1 \le p < \infty$). Hence,

$$\|f_n * g_n - f*g\|_\infty \le \|f_n - f\|_p\|g\|_q + \|f\|_p\|g_n - g\|_q.$$

Since $g_n \to g$ in L^q, there exists $M_q > 0$ such that $\|g_n\|_q \le M_q$.

Therefore,

$$\|f_n * g_n - f*g\|_\infty \le \epsilon M_q + \|f\|_p\epsilon$$

Thus, $f_n * g_n \to f*g$ uniformly, but $C_0(\mathbb{R})$ is a complete space, hence $f*g \in C_0(\mathbb{R})$. ☐

2.9 Riesz-Thorin Interpolation Theorem

Theorem 2.9.1. *Let (X,S,μ) and (Y,T,ν) be two σ-finite measure spaces. Let $p_i, q_i \in [1,\infty]$, $i = 0,1$ and define*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

where $0 \le t \le 1$. If T is a linear map from

$$L^{p_0}(\mu) + L^{p_1}(\mu) \to L^{q_0}(\nu) + L^{q_1}(\nu)$$

such that

$$\|Tf\|_{q_i} \le M_i \|f\|_{p_i}, \quad i = 0, 1,$$

then

$$\|Tf\|_{q_t} \le M_0^{1-t} M_1^t \|f\|_{p_t}$$

(For a proof, see Real Analysis *by G.B. Folland.)*

Using R-T theorem we see that F.T. of a function $f \in L^p(\mathbb{R}),\ 1 \le p \le 2$, exists as a function in L^q, $\frac{1}{p} + \frac{1}{q} = 1$.

2.10 Hausdorff-Young Inequality

Theorem 2.10.1. *Let $1 \le p \le 2$. Then for $f \in L^p(\mathbb{R})$, $\hat{f} \in L^q(\mathbb{R})$, with $\|\hat{f}\|_q \le \|f\|_p$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Note that if $1 \le p < 2$, then $q \in [2, \infty]$.

Similarly, if $f \in L^p(S^1)$, $1 \le p \le 2$, then $\hat{f} \in l^q(\mathbb{Z})$, with $\frac{1}{p} + \frac{1}{q} = 1$ and $\|\hat{f}\|_q \le \|f\|_p$.

Proof. We know that $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ satisfies

$$\|\mathcal{F}(f)\|_\infty \le \|f\|_1$$

and $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with $\|\mathcal{F}(f)\|_2 = \|f\|_2$.

Let

$$\frac{1}{p_t} = \frac{1-t}{1} + \frac{t}{2}, \quad \frac{1}{q_t} = \frac{1-t}{\infty} + \frac{t}{2}$$

Note that

$$\frac{1}{p_t} + \frac{1}{q_t} = 1, \frac{1}{p} + \frac{1}{q} = 1.$$

so we can choose $t \in (0, 1)$ such that $\frac{1}{q} = \frac{t}{2}$ and $\frac{1}{p} = \frac{1-t}{1} + \frac{t}{2}$. Hence by R-T inequality, we get

$$\|\mathcal{F}(f)\|_q \le \|f\|_p$$

Thus, F.T. is a bounded linear function from L^p to L^q. □

2.11 Young's Inequality

Theorem 2.11.1. *Let $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and*

$$\|f * g\|_r \le \|f\|_p \|g\|_q$$

Proof. Case I: if $p = 1, q = r$, then

$$\|f * g\|_r = \|f * g\|_q \le \|f\|_1 \|g\|_q$$

(by Minkowski integral inequality).

Case II: if $p = \frac{q}{q-1}, r = \infty$, ($\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$) then

$$\|f * g\|_r = \|f * g\|_\infty \leq \|f\|_p \|g\|_q$$

(since $f * g \in C_0(\mathbb{R})$).

Case III: $1 \leq q \leq \infty$, fix $g \in L^q$ and consider $T_g(f) = f * g$. Then

(i) $T_g : L^1 \to L^q$ satisfies $\|T_g(f)\|_q \leq \|f\|_1 \|g\|_q$,

(ii) $T_g : L^q \to L^\infty$ satisfies $\|T_g(f)\|_\infty \leq \|f\|_{q'} \|g\|_q$, when $\frac{1}{q} + \frac{1}{q'} = 1$.

For Riesz-Thorin interpolation theorem, let $p_0 = 1$, $q_0 = q$; $p_1 = q'$, $q_1 = \infty$ and $M_0 = \|g\|_1; M_1 = \|g\|_q$. Then

$$\|T_g(f)\|_{q_t} \leq M_0^{1-t} M_1^t \|f\|_{p_t}$$

where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} = 1 - t + \frac{t}{q'}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1} = \frac{1-t}{q}$$

If we want $q_t = r$, then $\frac{1}{r} = \frac{1-t}{q}$. Hence $\frac{q}{r} = 1 - t$, $t = 1 - \frac{q}{r}$. Thus $\frac{1}{p_t} = \frac{1}{p}$. So,

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \text{ and } \frac{1}{q} + \frac{1}{q'} = 1.$$

Hence,

$$\|T_g f\|_r \leq \|f\|_p \|g\|_q.$$

□

Notice that, by the Hausdorff-Young inequality, if $1 \leq p \leq 2$, then for $f \in L^p(\mathbb{R})$, $\hat{f} \in L^q(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{q} = 1$. Hence by continuity we can define

$$\hat{f}(\xi) \overset{L^2}{:=} \lim_{n \to \infty} \int_{-n}^{n} e^{-ix\xi} f(x)\, dx.$$

However, if $1 < p < 2$, we do not know how the $\hat{f}$ looks like. For example, if $f \in L^1(\mathbb{R})$, then

$$\lim_{\lambda \to \infty} \|f * K_\lambda - f\|_1 = 0$$

and

$$f(x) = \lim_{\lambda \to \infty} \frac{1}{2\pi} \int_{\mathbb{R}} G_\lambda(\xi) \hat{f}(\xi) e^{ix\xi} d\xi \tag{*}$$

holds in $L^1(\mathbb{R})$.

For $1 < p < 2$, we can generalize (*). For this, we need to verify the following: If $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R}), 1 < p < 2$, then $f * g \in L^p$ and $(f * g)^\wedge = \hat{f}\hat{g}$. Since $C_0^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, for $\epsilon > 0$, there exists $g_n \in C_0^\infty(\mathbb{R})$ so that $\|g - g_n\|_{L^p} < \epsilon$.

Note that $\widehat{g_n} \in L^1(\mathbb{R})$ (since second derivative of g satisfies $\hat{g^2}_n(x) = (ix)^2\hat{g}_n(x)$) and

$$\mathcal{F}(g_n * f) = \mathcal{F}(g_n)\mathcal{F}(f). \tag{**}$$

As $\mathcal{F} : L^p \to L^q$, is a continuous linear map, from (**) it follows that

$$\mathcal{F}(g * f) = \mathcal{F}(g)\mathcal{F}(f).$$

Now, consider $f = K_\lambda$ (Fejer kernel on $\mathbb{R}$), then

$$(K_\lambda * g)^\wedge = \hat{K}_\lambda \hat{g} = G_\lambda \hat{g},$$

where

$$G_\lambda(\xi) = (1 - |\xi|/\lambda)\chi_{[-\lambda,\lambda]}(\xi)$$

Since $\hat{g} \in L^q(\mathbb{R})$, $q > 2$, it is easy to see that $G_\lambda \hat{g} \in L^2(\mathbb{R})$. By inversion formula,

$$K_\lambda * g(x) = \frac{1}{2\pi}\int G_\lambda(\xi)\hat{g}(\xi)e^{ix\xi}d\xi,$$

and $K_\lambda * g \in L^2(\mathbb{R})$. Since K_λ is a good kernel and $K_\lambda * g \to g$ in $L^p(\mathbb{R})$, we can write the following result:

Theorem 2.11.2. *Let $1 \le p \le 2$ and $g \in L^p(\mathbb{R})$. Then*

$$g(x) = \lim_{\lambda\to\infty} \frac{1}{2\pi}\int_{\mathbb{R}} G_\lambda(\xi)\hat{g}(\xi)e^{ix\xi}d\xi$$

in $L^p(\mathbb{R})$.

Corollary 2.11.3. *$\{f \in L^p, 1 \le p \le 2 \ \operatorname{supp}\hat{f}$ is compact $\}$, is dense in $L^p(\mathbb{R})$.*

Notice that, if $f, g \in L^1(\mathbb{R})$, then $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$ where $\mathcal{F}$ is the Fourier transform.

Question 2.11.4. Does F is unique that satisfies $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$?

Note that if we write

$$\mathcal{F}(f) = \int f(x)e^{-it_0x}dx = \hat{f}(t_0),$$

then $\mathcal{F}$ is a continuous linear functional on $L^1(\mathbb{R})$. We then shall see that such any continuous linear functional is only F.T.

2.12 Riesz Theorem

Theorem 2.12.1. *Let $1 \leq p < \infty$ and (X, S, μ) be a σ-finite measure space. Then for every continuous linear functional T on $L^p(\mu)$, there exists a unique $g \in L^q(X)$, where $1/p + 1/q = 1$, such that*

$$Tf = \int fg$$

Fourier Transform is unique. Now, suppose φ is a continuous linear functional on $L^1(\mathbb{R})$ with $\|\varphi\| \leq 1$ and $\varphi(f * g) = \varphi(f)\varphi(g),$ for all $f, g \in L^1(\mathbb{R})$. Then by the Riesz theorem, there exists $\beta \in L^\infty(\mathbb{R})$ such that

$$\varphi(f) = \int f(x)\beta(x)dx.$$

Then

$$\varphi(f * g) = \int (\int f(x-y)g(y)dy)\beta(x)dx = \int g(y)\varphi(f_y)dy$$

where $f_y(x) = f(x-y)$. On the other hand,

$$\varphi(f * g) = \varphi(f)\varphi(g) = \varphi(f)\left(\int g(y)\beta(y)dy\right)$$

Hence

$$\int (\varphi(f_y) - \varphi(f)\beta(y))g(y)dy = 0, \quad \text{for all } g \in L^1(\mathbb{R}). \tag{*}$$

By uniqueness in the Riesz theorem, it follows that

$$\varphi(f)\beta(y) = \varphi(f_y), \quad \text{a.e. } y$$

Since $y \to f_y$ is continuous on $\mathbb{R}$ to $L^1(\mathbb{R})$ and φ is continuous on $L^1(\mathbb{R}) \to \mathbb{C}$, it follows that RHS of (*) is continuous. Hence, we can assume $\beta(y)$ is continuous, except on a set of measure zero.

By replacing $y \to x + y$, we get

$$\varphi(f)\beta(x+y) = \varphi(f_{x+y}) = \varphi((f_x)_y) = \varphi(f_x)\beta(y) = \varphi(f)\beta(x)\beta(y).$$

Since φ is non-zero, we can find $f \in L^1(\mathbb{R})$ such that $\varphi(f) \neq 0$. Hence

$$\beta(x+y) = \beta(x)\beta(y)$$

By using Theorem 2.0.1, there exists $t_0 \in \mathbb{R}$ such that $\beta(x) = e^{-it_0x}$. Hence

$$\varphi(f) = \int f(x)e^{-it_0x}dx = \hat{f}(t_0).$$

□

Notice that for every φ (except $\varphi = 0$), there exists unique $t \in \mathbb{R}$ such that $\varphi(f) = \widehat{f}(t)$, because if $s \neq t$, then there exists $f \in L^1(\mathbb{R})$ such that $\widehat{f}(t) \neq \widehat{f}(s)$.

2.13 Poisson Summation Formula

For $f \in L^1(\mathbb{R})$, write

$$\varphi(t) = 2\pi \sum_{j=-\infty}^{\infty} f(t + 2\pi j).$$

Then φ is a 2π-periodic function on $\mathbb{R}$ and $\|\varphi\|_{L^1(S^1)} \leq \|f\|_{L^1(\mathbb{R})}$. This can be seen by the fact that

$$\int_0^{2\pi} |\varphi(t))|dt = 2\pi \sum_{j=-\infty}^{\infty} \int_0^{2\pi} |f(t + 2\pi j)|dt$$

$$= 2\pi \sum_{j=-\infty}^{\infty} \int_{2\pi j}^{2\pi(j+1)} |f(s)|ds = \int_{-\infty}^{\infty} |f(s)|ds.$$

Theorem 2.13.1. *Let $f \in L^1(\mathbb{R})$. Then*

$$\sum_{j=-\infty}^{\infty} f(t + 2\pi j) = \sum_{j=-\infty}^{\infty} \hat{f}(j)e^{ijt}, \quad \forall t \in \mathbb{R}, \tag{2.13.1}$$

where $\hat{f}(j)$ is the Fourier transform.

Proof. To prove this identity, it is enough to show the Fourier coefficients of LHS is $\hat{f}(j)$.

$$\frac{1}{2\pi} \int_0^{2\pi} \sum_{j=-\infty}^{\infty} f(t + 2\pi j)e^{-int}dt = \sum_{j=-\infty}^{\infty} \int_0^{2\pi} f(t + 2\pi j)e^{-int}dt$$

by Beppo-Levi theorem.

$$= \int_{\mathbb{R}} f(t)e^{-int}dt = \hat{f}(n)$$

Hence, by uniqueness of the Fourier series, we get the required identity. □

Example 2.13.2. Prove that

$$\sum \frac{1}{(n+x)^2} = \frac{\pi^2}{(\sin \pi x)^2}$$

(Hint: Take $g(x) = 1 - |x|$ for $|x| < 1$, $= 0$ otherwise in the Poisson summation formula (2.13.1)).

2.14 L^p-Derivative of a Function on $\mathbb{R}$

For $h \in \mathbb{R}$ and f a function on $\mathbb{R}$, define

$$D_h f(x) = \frac{f(x+h) - f(x)}{h}$$

Definition 2.14.1. A function $f \in L^p(\mathbb{R})$ is said to be differentiable in L^p sense if there exists $g \in L^p(\mathbb{R})$ such that

$$\lim_{h \to 0} \|D_h f - g\|_p = 0.$$

Lemma 2.14.2. *Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $f \in L^p$, has derivatives f' in L^p sense , then $(f * g)'$ exists in the ordinary sense when $g \in L^q$ and*

$$(f * g)' = f' * g.$$

Proof. We know that $f * g$ is continuous and $f' \in L^p$, therefore $f' * g$ is also continuous. Thus

$$|D_h(f * g)(x) - f' * g(x)| = |(D_h f - f') * g(x)| \leq \|D_h f - f'\|_p \|g\|_q \to 0 \text{ as } |h| \to 0$$

Hence

$$(f * g)' = f' * g$$

□

Theorem 2.14.3. *Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Then f has derivative in L^p sense if and only if f is absolutely continuous on each bounded interval $[a, b]$ (except on a set of measure zero) and its pointwise derivative $f' \in L^p(\mathbb{R})$.*

To prove this, we need a fact that $AC[a, b]$ is a complete space under the norm:

$$\|f\|_{AC} = |f(a)| + \int_a^b |f'(t)| dt.$$

We know that $f \in AC[a, b]$ if and only if f' exists a.e.,

$$f' \in L^1[a, b] \text{ and } f(x) = f(a) + \int_a^x f'(t) dt$$

Hence, $\|f\|_{AC} < \infty$ and $\|f'\|_{AC} = 0 \implies f(a) = 0, f'(t) = 0$ a.e. $\implies f(t) = f(a) = 0$.
($f' = 0$ a.e. $\implies f$ is constant, a non-trivial result (referred to Rayden book).) Hence, $(AC[a, b], \|\cdot\|_{AC})$ is a normed linear space.

If f_n is a Cauchy sequence, then $f_n(a)$ and f_n' are Cauchy sequences in $\mathbb{C}$ and $L^1([a, b])$, respectively. Let $f_n(a) \to f_a$, $f_n' \to g$ in L^1. Write

$$f(x) = f_a + \int_a^x g(t)\, dt$$

Then f is absolutely continuous and

$$\|f_n - f\|_{AC} \leq |f_n(a) - f_a| + \int_a^b |g(t) - f_n'(t)| dt$$

Hence, $f_n \to f \in AC[a,b]$.

Proof of Theorem 2.14.3. For simplicity, consider $p = 1$, $q = \infty$.

Suppose f has L^1-derivative (or derivative in L^1 sense). Then there exists $g \in L^1(\mathbb{R})$ such that $\lim_{h\to 0} \|D_h f - g\|_1 = 0$. By the previous lemma, $(f * K_\lambda)'$ exists ordinarily and satisfies

$$(f * K_\lambda)' = f' * K_\lambda$$

Note that for each fixed λ, the function $f * K_\lambda$ is smooth on $\mathbb{R}$. Hence by MVT, $f * K_\lambda \in AC[a,b], \quad \forall a,b \in \mathbb{R}$ That is,

$$f * K_\lambda(x) = f * K_\lambda(x_0) + \int_{x_0}^{x} (f * K_\lambda)'(t)\,dt \tag{1}$$

for some $x_0 \in [a,b]$. Since $f * K_\lambda \xrightarrow{L^1} f$, it follows that

$$f * K_\lambda(x) \to f(x) \text{ a.e.}$$

(as a subsequence of $f * K_\lambda$). Hence, we can choose $x_0 \in [a,b]$.

As $(f * K_\lambda)' = g * K_\lambda \to g$ (in L^1), we can take limit in (1) and hence

$$f(x) = f(x_0) + \int_{x_0}^{x} g(t)\,dt \quad \text{a.e.,} \quad x \in \mathbb{R}.$$

This implies $f' = g$ a.e. on $\mathbb{R}$, and $f' = g \in L^1(\mathbb{R})$.

Conversely, suppose $f \in AC[a,b]$, for all $a, b \in \mathbb{R}$ and pointwise derivative f' exists and belongs to $L^1(\mathbb{R})$. Then

$$\frac{f(x+h) - f(x)}{h} - f'(x) = \frac{1}{h}\int_0^h (f'(x+t) - f'(x))\,dt$$

(since $f \in AC[a,b]$, etc.)

Since $f' \in L^1(\mathbb{R})$, by Minkowski integral inequality, it follows that

$$\begin{aligned} \|D_h f - f'\|_1 &\le \frac{1}{|h|}\int_0^{|h|} \|\tau_t f' - f'\|_1\,dt \\ &< \|\tau_t f' - f'\|_1 < \epsilon \end{aligned}$$

whenever $|h| < \delta$, as $|t| < |h| < \delta$. Thus, f' is the L^1-derivative of f.

If $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$, then $L^p(\mathbb{R}) \subset L^1_{\text{loc}}(\mathbb{R})$. Hence, all the above calculations make sense, and same conclusion is followed by Minkowski integral inequality. □

2.15 C^∞ form of Urysohn lemma

Lemma 2.15.1. *Let K be a compact set that is contained in an open set $\mathcal{O} \subset \mathbb{R}$. Then there exists $f \in C_c^\infty(\mathbb{R})$ such that $0 \leq f \leq 1, f|_K = 1$ and $\text{supp} f \subset \mathcal{O}$.*

Proof. Let $\delta = d(K, \mathcal{O}^c)$. Then $\delta > 0$, and let

$$V = \{x : d(x, K) < \delta/3\}.$$

Suppose $\varphi \in C_c^\infty(\mathbb{R})$ such that $\int \varphi = 1, \varphi(x) = 0$ if $|x| > \delta/3$. Write $f = \chi_V * \varphi$. Then $f|_K = 1$, $0 \leq f \leq 1$, and $\text{supp}(f) \subset \{x : d(x, K) < 2\delta/3\} \subset \mathcal{O}$, and $f \in C_c^\infty(\mathbb{R})$. Note that φ can be constructed by choosing

$$\varphi(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

□

2.16 Exercise

1. (a) Let $f \in C_c^\infty(\mathbb{R})$ be nonzero and let P be a polynomial of degree $n \geq 1$. Determine whether the function $P\hat{f}$ is bounded on $\mathbb{R}$.

   (b) Is the subspace
$$\{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \text{ is compact}\}$$
dense in $L^2(\mathbb{R})$?

2. Suppose f is continuously differentiable on $[-R, R]$. Prove that there exists a constant $C > 0$ such that
$$|\hat{f}(\xi)| \leq \frac{C}{|\xi|}, \qquad \xi \neq 0.$$

3. Let $f, g \in L^2(\mathbb{R})$. Show that the convolution $f * g$ is a bounded continuous function on $\mathbb{R}$, and that
$$\lim_{|x|\to\infty} (f * g)(x) = 0.$$

4. Let $f \in L^1(\mathbb{R})$ satisfy $f(x) > 0$ for all $x \in \mathbb{R}$. Prove that there exists $\delta > 0$ such that
$$|\hat{f}(\xi)| < \hat{f}(0), \qquad |\xi| > \delta.$$

5. For $n \in \mathbb{N}$, define
$$F_n(x) = \chi_{[-1,1]} * \chi_{[-n,n]}(x).$$
Verify that $F_n \in C_c(\mathbb{R})$ with $\|F_n\|_\infty = 2$. Does the sequence $\{F_n(x)\}$ converge uniformly to 2 on $\mathbb{R}$?

6. For $1 \leq p < \infty$, let $f \in L^p(\mathbb{R})$ and set
$$F(x) = \int_x^{x+1} f(t)\, dt.$$
Show that $F \in C_0(\mathbb{R})$. Does this conclusion remain valid for $f \in L^\infty(\mathbb{R})$?

7. For $f \in L^1(\mathbb{R})$, prove the identity
$$2\hat{f}(\xi) = \int_{\mathbb{R}} \left[f(x) - f\left(x - \tfrac{\pi}{\xi}\right) \right] e^{-i\xi x}\, dx,$$
and deduce the Riemann–Lebesgue lemma.

8. Let $f, g \in L^1(\mathbb{R})$. Prove that
$$\int_{\mathbb{R}} f(y)\hat{g}(y)\, dy = \int_{\mathbb{R}} \hat{f}(\xi) g(\xi)\, d\xi.$$

If $\hat{f} \in L^1(\mathbb{R})$, deduce the Fourier inversion formula for f.

9. For $n \in \mathbb{N}$, define
$$f(x) = \frac{x^n}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$
Show that
$$\hat{f}(\xi) = P_n(\xi) e^{-\frac{\xi^2}{2}},$$
where P_n is a polynomial of degree n.

10. A continuous function $f : \mathbb{R} \to \mathbb{C}$ is of **moderate decrease** if there exists $A > 0$ such that
$$|f(x)| \leq \frac{A}{1+x^2}, \qquad x \in \mathbb{R}.$$
Suppose f is of moderate decrease and satisfies
$$\int_{\mathbb{R}} f(y) e^{-y^2} e^{2xy}\, dy = 0 \quad \forall x \in \mathbb{R}.$$
Prove that $f \equiv 0$.

11. Let f be of moderate decrease and define
$$f * K_\lambda(x) = \tfrac{1}{2\pi} \int_{-\lambda}^{\lambda} \left(1 - \tfrac{|\xi|}{\lambda}\right) \hat{f}(\xi) e^{i\xi x}\, d\xi.$$
Show that $f * K_\lambda \to f$ uniformly as $\lambda \to \infty$.

12. Let $\{k_\lambda\} \subset L^1(\mathbb{R})$ be a family of good kernels. If $f \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, prove that $f * k_\lambda \to f$ uniformly on every compact subset of $\mathbb{R}$.

13. For $1 \leq p \leq 2$, prove that
$$\{f \in L^p(\mathbb{R}) : \operatorname{supp} \hat{f} \text{ compact}\}$$
is dense in $L^p(\mathbb{R})$.

14. Show that
$$X = \{\hat{f} : f \in L^1(\mathbb{R})\}$$
is dense in $C_0(\mathbb{R})$.

15. Let $f \in C_c^2(\mathbb{R})$. Prove that there exists $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\hat{g} = f$.

16. For $f \in L^2(\mathbb{R})$, define the translation operator $\tau_x f(y) = f(y - x)$. Show that
$$X = \{\tau_x f : x \in \mathbb{R}\}$$

is dense in $L^2(\mathbb{R})$ if and only if $\hat{f}(\xi) \neq 0$ almost everywhere.

17. Let $f \in L^1(\mathbb{R})$ with compact support. Prove that $\hat{f}$ is real-analytic on $\mathbb{R}$. Does $\hat{f} \in L^1(\mathbb{R})$? What additional conclusion holds if $f \in C_c^2(\mathbb{R})$?

18. Let $f \in L^1(\mathbb{R})$ with $f \geq 0$. Show that

$$\|\hat{f}\|_\infty = \hat{f}(0) = \|f\|_1.$$

19. Suppose $f \in L^1(\mathbb{R})$ is continuous at 0 and $\hat{f}(\xi) \geq 0$ for all ξ. Prove that $\hat{f} \in L^1(\mathbb{R})$ and

$$f(0) = \int_{\mathbb{R}} \hat{f}(\xi)\, d\xi.$$

20. For $n \in \mathbb{N}$, let $g_n = \chi_{[-1,1]} * \chi_{[-n,n]}$. Show that g_n is the Fourier transform of

$$f_n(x) = \frac{\sin x \sin nx}{\pi^2 x^2} \in L^1(\mathbb{R}),$$

and that $\|f_n\|_1 \to \infty$. Conclude that the Fourier transform maps $L^1(\mathbb{R})$ into a proper subspace of $C_0(\mathbb{R})$.

21. For $f \in L^1(\mathbb{R})$, define $f_\lambda(x) = \lambda f(\lambda x)$ and

$$\varphi_\lambda(t) = 2\pi \sum_{j=-\infty}^{\infty} f_\lambda(t + 2\pi j).$$

Show that

$$\lim_{\lambda\to\infty} \|\varphi_\lambda\|_{L^1(S^1)} = \|f\|_{L^1(\mathbb{R})}.$$

22. For $f \in L^1(\mathbb{R})$, define

$$g(t) = 2\pi \sum_{n=-\infty}^{\infty} f(t + 2\pi n).$$

Show that g is periodic and

$$\|g\|_{L^1(S^1)} \leq \|f\|_{L^1(\mathbb{R})}.$$

23. For $1 \leq p < \infty$, suppose $f \in L^p(\mathbb{R})$ and $h \in \mathbb{R}$. Define

$$\Delta_h f(x) = \frac{f(x+h) - f(x)}{h}.$$

Show that there exists $g \in L^p(\mathbb{R})$ such that

$$\lim_{h\to 0} \|\Delta_h f - g\|_p = 0$$

iff f is absolutely continuous on bounded intervals (modulo null sets) and $f' \in L^p(\mathbb{R})$. Does this remain true for $f \in L^\infty(\mathbb{R})$?

24. Suppose $f \in L^\infty(\mathbb{R})$ satisfies

25. Give an example of $f \in L^\infty(0,\infty)$ such that f' exists pointwise on $(0,\infty)$ but $f' \notin L^\infty(0,\infty)$.

26. For $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, $1 < p < 2$, prove that $f * g \in L^p(\mathbb{R}^n)$ and deduce that

$$\widehat{f * g} = \hat{f}\hat{g}.$$

Chapter 3

Distributions

Many operations in analysis — differentiation, convolution, Fourier transformation — extend well beyond smooth functions. The language of distributions (generalized functions) provides a precise framework for these extensions while remaining compatible with classical calculus whenever the latter makes sense. In this chapter we introduce test function spaces, distributions, and their basic operations, with an eye toward applications in Fourier analysis.

Learning objectives.

- Define the spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ and interpret distributions as continuous linear functionals.
- Understand distributional derivatives and multiplication by smooth functions.
- See how the Fourier transform extends naturally to the Schwartz space and to tempered distributions.

We know from the previous section that there are functions in L^p-spaces which are differentiable in L^p-sense. That is, there exists $g \in L^p$ such that $\|D_h f - g\|_p \to 0$ as $|h| \to 0$. However, there is a large class of functions which are neither differentiable nor their L^p-derivative exist. Though, there is a large sub-class of such functions whosee derivative can be realized with the help of certain class of differentiable functions, known as "test functions".

For example, suppose f is differentiable and g is a compactly supported differentiable function on $\mathbb{R}$. Then

$$\int_{-\infty}^{\infty} f'g = -fg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} fg' = -\int_{-\infty}^{\infty} fg',$$

because g is compactly supported. Therefore, this gives way to realize the derivative of $f \in L^1_{loc}(\mathbb{R})$. For $g \in C_c^\infty(\mathbb{R})$, write

$$\Lambda_f(g) = \int_{\mathbb{R}} fg,$$

then the derivative of Λ_f can be defined by

$$\Lambda'_f(g) = -\int_{\mathbb{R}} fg'.$$

In fact, functional Λ_f is all time differentiable and its k-th derivative is given by

$$D^k\Lambda_f(g) = (-1)^k \int_{\mathbb{R}} fD^k g,$$

where $D = \frac{d}{dx}$.

In order to discuss "distributions" in detail, we need to derive a complete topology on $C_c^\infty(\mathbb{R}^n)$. Since the space $C_c^\infty(\mathbb{R}^n)$ cannot be made complete under sup norm, a complete topology on $C_c^\infty(\mathbb{R}^n)$ will be derived from a family of semi-norms (defined on compact subsets of $\mathbb{R}^n$). Thus, the space $\mathcal{E}(\mathbb{R}^n)$ becomes a locally convex topological space.

3.1 Locally Convex Topology

Let $\{p_i : i \in I\}$ be a family of semi-norms on a topological vector space X. For a finite set $F \subset I$, let

$$U_{F,\epsilon} = \bigcap_{i\in F}\{x \in X : p_i(x) < \epsilon\} = \bigcap_{i\in F} V_{i,\epsilon}.$$

Then each $V_{F,\epsilon}$ is convex and balanced. Let

$$\mathcal{B} = \{U_{F,\epsilon} : \epsilon > 0, F \subset I, \#(F) < \infty\}.$$

Then the collection

$$\mathcal{T} = \{O \subset X : \text{ for all } x \in O, \text{there exists } U \in \mathcal{B} \text{ such that } x + U \subset O\}$$

is a topology on X.

Obviously, $\mathcal{T}$ contains $\emptyset$ and X, and is closed under arbitrary unions. Now, let

$$O = \bigcap_{i=1}^{k} O_i, \quad O_i \in \mathcal{T}$$

If $x \in O$, then $x \in O_i$ and there exists $U_{F_i,\epsilon_i} \in \mathcal{B}$ such that $x + U_{F_i,\epsilon_i} \subset O_i$. Write $\epsilon = \min_{1\le i\le k} \epsilon_i$ and $F = \bigcup_{i=1}^k F_i$.Then $\epsilon > 0$ and F is finite and hence

$$x + U_{F,\epsilon} \subset \bigcap_{i=1}^{k}(x + U_{F_i,\epsilon_i}) \subset O.$$

The space $(X, \mathcal{T})$ is known as **locally convex topological space**.

Example 3.1.1. Show that a locally convex topological vector space X is Hausdorff if and only if $\{p_i : i \in I\}$ separates points in X *i.e.,* given $x \in X, x \neq 0$, there exists $i \in I$ such that $p_i(x) \neq 0$.

Example 3.1.2. Let X be a locally convex Hausdorff space whosee topology is induced by $\{p_i : i \in I\}$. Define

$$d(x, y) = \sum 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}$$

Show that topology τ_d coincides with $\mathcal{T}$.

Note that, in general settings, $U_{F,\epsilon}$ plays the role of $B_\epsilon(0)$ in $\mathbb{R}^n$ as $B_\epsilon(0), \epsilon > 0$ forms a local base at 0. Therefore,

$$\mathcal{B} = \{U_{F,\epsilon} : \epsilon > 0, F \subset I, \#(F) < \infty\}$$

is a local base at $0 \in X$.

Definition 3.1.3. (i) A sequence $(x_i)_{i=1}^{\infty} \subset X$ is said to *converge* to $x \in X$ if for all $U \in \mathcal{B}$ there exists $N = N_0 \in \mathbb{N}$ such that $x - x_j \in U$, for all $j \geq N$.

(ii) $(x_i)_{i=1}^{\infty} \subset X$ is called a *Cauchy sequence* if for all $U \in \mathcal{B}$, there exists $N = N_0 \in \mathbb{N}$ such that $x_k - x_\ell \in U$ for all $k, \ell \geq N$.

(iii) X is called *sequentially complete* if every Cauchy sequence in X has a limit in X.

Lemma 3.1.4. *A sequence $(x_i)_{i=1}^{\infty} \subset X$ converges to $x \in X$ if and only if $\lim_{n\to\infty} p_n(x_i - x) = 0$ for all $n \in I$.*

Proof. Let $U_{j,\epsilon} = \{x \in X : p_j(x) < \epsilon\}$. Then there exists $N \in \mathbb{N}$ such that $p_j(x_j - x) < \epsilon$ for all $j \geq N$, etc. □

Theorem 3.1.5. *Let $\{p_i\}_{i\in I}$ be a separating family of semi-norms on a vector space X, and set*

$$V_{p,n} = \{x \in X : p(x) < 1/n\}.$$

Then $J = \{V_{p_i,n} : i \in I, n \in \mathbb{N}\}$ forms a convex balanced local base for a topology $\mathscr{T}$ on X, which makes X into a locally convex space such that

(i) each p_i is continuous, and

(ii) A set $E \subset X$ is bounded if and only if for all $i \in I$, $p_i(E)$ is bounded.

Proof. Let $x \in X$ and $x \neq 0$. Then there exists p_i such that $p_i(x) > 0$. Therefore, for some x, $np_i(x) > 1$, implies $x \notin V(p_i, n)$, a neighborhood of 0. Hence, $\{0\}$ is closed. Since $\mathscr{T}$ is translation invariant, each $\{x\} \subset X$ is closed in $(X, \mathscr{T})$.

Addition is continuous: Let U be a neighborhood of 0 in X. Then $\bigcap_{i\in I} V(p_i, n_i) \subset U$ (by the definition of topology $\mathscr{T}$). Let

$$V = \bigcap_{i\in I} V(p_i, 2n_i).$$

Then $V + V \subset U$.

Consider $(x_1, x_2) \mapsto x_1 + x_2$, and let U be an open set containing $x_1 + x_2$. Then $U - (x_1 + x_2)$ is a neighborhood of 0. Hence, there exists a neighbourhood V of 0 such that

$$V + V \subset U - (x_1 + x_2)$$

then

$$(V + x_1) + (V + x_2) \subset U.$$

Thus, addition is continuous.

Scalar multiplication is continuous: Let $x \in X$ and $\alpha \in \mathbb{C}$, U and V as above. Then $x \in sV$ for some $s > 0$. Write $t = \frac{s}{1+|\alpha|s}$, and $y = x + tV$, with $|\beta - \alpha| < 1/s$. Then

$$\beta y - \alpha x = \beta(y - x) + (\beta - \alpha)x \in |\beta|tV + |\beta - \alpha|sV \subset V + V \subset U$$

Since $|\beta|t < (|\alpha| + \frac{1}{s})t = 1$, and V is balanced, thus $\beta(x + tV) \subset \alpha x + U$, this implies scalar multiplication is continuous.

(ii) Suppose E is a bounded subset of X. Since each $V(p_i, 1)$ is a neighborhood of 0, there exists $k_i > 0$ such that

$$E \subset k_i V(p_i, 1) = V(p_i, 1/k_i)$$

$$\Rightarrow p_i(x) < k_i, \quad \forall i, \forall x \in E.$$

Conversely, suppose $p_i(x) < M_i$, for all $x \in E$, for all $i \in I$, then for each neighborhood V of 0,

$$U \supset \bigcap_{i=1}^{m} V(p_i, n_i)$$

which implies

$$E \subset \bigcap_{i=1}^{m} V(p_i, 1/M_i) = \bigcap_{i=1}^{m} M_i n_i V(p_i, n_i)$$

If $n > M_i n_i$ for all $i = 1, 2, \ldots, m$, then

$$E \subset n \bigcap_{i=1}^{m} V(p_i, n_i) \subset nU$$

Hence E is bounded in $(X, \mathscr{T})$. □

3.2 Topology of the spaces $C^\infty(\Omega)$ and $\mathcal{D}_K$

We define a topology on $C^\infty(\Omega)$ which makes $C^\infty(\Omega)$ a Fréchet space with the Heine-Borel property, such that the space

$$\mathscr{D}_K = \{\varphi \in C^\infty(\mathbb{R}^n) : \operatorname{supp}(\varphi) \subset K\}$$

where K is a compact set in Ω, is a closed subspace of $C^\infty(\Omega)$.

Define a sequence of compact sets in Ω such that $K_i \subset K_{i+1}$

$$K_i = \{x \in \Omega : d(x, \mathscr{D}(\Omega) \geq 1/i\} \cap \overline{B_i},$$

where $B_i = \{x \in \mathbb{R}^n : |x| < i\}$.

For $f \in C^\infty(\Omega)$, define

$$p_N(f) = \sup\{|D^\alpha f(x)| : x \in K, |\alpha| \leq N\}.$$

These $\{p_N\}_{N=1}^\infty$ form a separating family of seminorms that makes $C^\infty(\Omega)$ a metrizable locally convex topological space (exercise: use the previous theorem).

For $x \in \Omega$, define $\delta_x(f) = f(x)$. Then each δ_x is a continuous linear functional in the topology induced by $\{p_N\}_{N=1}^\infty$. That is, $p_N(f_i) \to 0 \implies |f_i(x)| \leq p_N(f_i) \to 0$. It is easy to see that

$$\mathscr{D}_K = \bigcap_{x \in \Omega \setminus K} \ker \delta_x$$

Hence $\mathscr{D}_K$ is a closed subspace of $C^\infty(\Omega)$. Notice the collection

$$V_N = \{f \in C^\infty(\Omega) : p_N(f) < 1/N\}, \quad N = 1, 2, \dots$$

forms a convex balanced local base at $0 \in C^\infty(\Omega)$.

If $\{f_j\}$ are a Cauchy sequence in $C^\infty(\Omega)$, then for each V_N, there exists $l_N \in \mathbb{N}$ such that

$$\begin{aligned} &f_i - f_j \in V_N \text{ for all } i, j > l_N \\ &\implies p_N(f_i, f_j) < 1/N, \\ &\implies |D^\alpha f_i(x) - D^\alpha f_j(x)| < 1/N, \quad x \in K_N \end{aligned}$$

That is, $D^\alpha f_i \to g_\alpha$ on each compact set K_N in Ω. In particular, $f_i(x) \to g_0(x)$. Thus $g_0 \in C^\infty(\Omega)$ and $g_\alpha = D^\alpha g_0$. This implies that $f_i \to g_0$ in the topology of $C^\infty(\Omega)$. Hence $C^\infty(\Omega)$ is a Fréchet space and the same is true for $\mathscr{D}_K$.

Suppose $E \subset C^\infty(\Omega)$ is closed and bounded. Then, by the previous theorem A, there exists $0 < M_N < \infty$ such that $p_N(f) < M_N$ for all $N = 1, 2, \dots$, $f \in E$.

Thus, $|D^\alpha f| < M_N$ on K_N, $|\alpha| \leq N$. Hence,

$$\left\{D^\beta f : f \in E\right\}$$

is an equicontinuous family on K_{N-1}, if $|\beta| \leq N-1$. By the Mean Value Theorem (MVT),

$$|f(x) - f(y)| < N\|D^1 f\|_\infty |x-y| \tag{1}$$

Replacing $f \to D^\beta f$ in (1), we get

$$|D^\beta f(x) - D^\beta f(y)| \leq \|D^{\beta+1} f\|_\infty \|x-y\| \leq \|f\|_N \|x-y\|$$

By Arzelà-Ascoli Theorem, every sequence (f_n) in E has a convergent subsequence. Hence, E is compact in $C^\infty(\Omega)$. Thus, $C^\infty(\Omega)$ has the Heine–Borel property. Since

$$d(f,0) \leq \sum 2^{-n} \frac{p_N(f)}{1+p_N(f)} < 2,$$

the topology on $C^\infty(\Omega)$ is not normable.

Now, for each fixed $K \subset \Omega$, $\mathscr{D}_K$ is a Fréchet space and

$$\mathscr{D}(\Omega) = C_c^\infty(\Omega) = \bigcup_{K \subset \Omega} \mathscr{D}_K$$

It is known as the space of test functions.

For $\varphi \in \mathscr{D}(\Omega)$, define

$$\|\varphi\|_N = \sup\{|D^\alpha \varphi(x)| : x \in \Omega,\ |\alpha| \leq N\}$$

for $N = 0, 1, 2, \ldots$.

Note: Restriction of these norms to $\mathscr{D}_K$ gives the same topology as do the semi-norms $\{p_N\}_{N=1}^\infty$. For this, let $K \subset \Omega$ compact. Then there exists $N_0 \in \mathbb{N}$ such that $K \subset K_N$, $N \geq N_0$, add for these $N \geq N_0$,

$$\|\varphi\|_N = p_N(\varphi),\ \forall \varphi \in \mathscr{D}_K$$

Since $\|\varphi\|_N \leq \|\varphi\|_{N+1} \leq \ldots$ and

$$p_N(\varphi) \leq p_{N+1}(\varphi) \leq \ldots$$

the topology given by either sequence $\{\|p_N\}_{N=N_0}^\infty$ or $\{\|\cdot\|_N\}_{N=N_0}^\infty$ will be the same. Thus, the topology on $\mathscr{D}_K$ coincides. Therefore,

$$V_N = \left\{\varphi \in \mathscr{D}_K : \|\varphi\|_N < \frac{1}{N}\right\}$$

form a local base for $\mathscr{D}_K$.

Notice that $\|\cdot\|_{N=0}^{\infty}$ can be used to define a locally convex metrizable topology on $\mathscr{D}(\Omega)$, but this topology is not complete.

For $\varphi \in \mathscr{D}(\Omega)$, $\operatorname{supp}\varphi \subset [0,1]$, $\varphi > 0$ on $(0,1)$,

$$\varphi_m(x) = \varphi(x-1) + \frac{1}{2}\varphi(x-2) + \frac{1}{m}\varphi(x-m)$$

is a Cauchy sequence in this topology, but (φ_m) is not completely supported. This happens because $\{p_N\}_{N=0}^{\infty}$ is not enough to prevent Cauchy sequences "leaking" toward the boundary of Ω, so that we can add more semi-norms to the family $\{p_N\}_{N=0}^{\infty}$ that allows more functions on $\mathscr{D}(\Omega)$ to be continuous.

Now, we define another topology τ on $\mathscr{D}(\Omega)$ (in which Cauchy sequences do converge), however τ is not metrizable.

(i) Let $\mathcal{B} = \{W \subset \mathscr{D}(\Omega) : W$ is convex, balanced; sets with $\mathscr{D}_K \cap W \in \tau_K$, $\forall K$ compact $\subset \Omega\}$.

(ii) $\Sigma =\{$ unions of the form $\varphi + W$, $\varphi \in \mathscr{D}(\Omega)$, $W \in \mathcal{B}\}$

NOte that The topology τ is different than the topology generated by the p_N's as the topologies τ includes more seminorms. For example, let $\varphi \in \mathscr{D}(\Omega))$, and $\{x_i\} \subset \Omega$: the sequence having no limit point, for any $C_i > 0$,

$$p(\varphi) = \sup_i C_i |\varphi(x_i)| < \infty \text{ (since there exist only finitely many } i \text{ for each } \varphi)$$

is a semi-norm on $\mathscr{D}(\Omega)$ and p restricted to each $\mathscr{D}_k$ is continuous. In fact,

$$W = \{\varphi \in \mathscr{D}(\Omega) : p(\varphi) < C\}$$

is convex balanced and belongs to $\mathcal{B}$ as a τ-neighborhood of $0 \in \mathscr{D}(\Omega)$. This forces every τ-bounded set (or Cauchy Sequence) in $\mathscr{D}(\Omega))$ to be concentrated on a common compact set $K \subset \Omega$. This will be formalized in the next theorem. That is, a sequence $(\varphi_i) \in \mathscr{D}(\Omega)$ converges to 0 if $\operatorname{supp}\varphi_i \subset K$, $\forall i = 1, 2 \ldots$.

Theorem 3.2.1. *(a) τ is a topology on $\mathscr{D}(\Omega)$, and $\mathcal{B}$ is a local base for τ.*

(b) Σ makes $\mathscr{D}(\Omega)$ into a locally convex topological vector space.

Proof. To prove (a), it is enough to show that for $V_1, V_2 \in \tau$ and $\varphi \in V_1 \cap V_2$, there exists $W \in \mathcal{B}$ such that $\varphi + W \subset V_1 \cap V_2$. By definition, there exists $\varphi_i + W_i \in \tau$ such that $\varphi \in \varphi_i + W_i \subset V_i$, $i = 1, 2$.

Choose $K \subset \Omega$ compact so that $\varphi_1, \psi_2, \varphi \in \mathscr{D}_K$. Since $\mathscr{D}_{W_i}$ is open in $\mathscr{D}_K$ and $\varphi - \psi_i \in \mathscr{D}_K \cap W_i$, it follows that $\varphi - \varphi_i \in (1-\delta_i)W_i$ for $\delta_i > 0$ (it is like if $x \in B_\epsilon(x) \subset W$, then

$x \in (1-\delta)B_{\epsilon/2}(x) \subset (1-\delta)W$) By the convexity of W_i, we get

$$\varphi - \varphi_i + \delta_i W_i \subset (1-\delta_i)W_i + \delta_i W_i = W_i.$$

So $\varphi + \delta_i W_i \subset \varphi_i + W_i \subset V_i$, $i = 1, 2$. Hence, $\varphi + (\delta_1 W_1) \cap (\delta_2 W_2) \subset V_1 \cap V_2$. This proves (a).

(b) Let $\varphi_1, \varphi_2 \in \mathscr{D}(\Omega)$ be distinct and

$$W = \{\varphi \in \mathscr{D}(\Omega) : \|\varphi\|_0 < \|\varphi_1 - \varphi_2\|_0\}.$$

Then $W \subset B$ and $\varphi_2 \in \varphi_1 + W$. Since φ_2 is arbitrary, it implies that $\{\varphi_1\}$ is closed set relative to τ. Notice that for every pair of $\psi_1, \psi_2 \in D(\Omega)$,

$$(\psi_1 + \tfrac{1}{2}W) + (\psi_2 + \tfrac{1}{2}W) = (\psi_1 + \psi_2) + W.$$

Hence, addition is continuous in $(D(\Omega), \tau)$.

Pick $\alpha_0 \in \mathbb{C}$ and $\varphi_0 \in D(\Omega)$. Then $\varphi_0 + \frac{1}{2}sW$ for some $s > 0$. Let $|\alpha - \alpha_0| < \frac{1}{s}$ and $t = \frac{s}{2(1+(|\alpha_0|s)}$. Then for $\varphi \in \varphi_0 + tW$,

$$\begin{aligned}\alpha\varphi - \alpha_0\varphi_0 &= \alpha(\varphi - \varphi_0) + (\alpha - \alpha_0)\varphi_0 \\ &\in K/tW + \tfrac{1}{2}W \\ &\in \tfrac{1}{2}W + \tfrac{1}{2}W = W,\end{aligned}$$

since $|\alpha|t < (|\alpha| + \frac{1}{s})t = \frac{1}{2}$. Thus,

$$\alpha(\varphi_0 + tW) \subset \alpha_0\varphi_0 + |\alpha|tW \subset \alpha_0\varphi_0 + W.$$

Hence, scalar multiplication is continuous. From onward, by $D(\Omega)$ we mean $(D(\Omega), \tau)$.

□

Theorem 3.2.2. *(a) A convex balanced subset $V \in \mathscr{D}(\Omega)$ is open if and only if $V \in \mathcal{B}$.*

(b) The topology τ_K of $\mathscr{D}_K \subset \mathscr{D}(\Omega)$ coincides with the topology on $\mathscr{D}_K$ that is inherited from $\mathscr{D}(\Omega)$.

(c) If E is a bounded subset of $\mathscr{D}(\Omega)$, then $E \subset \mathscr{D}_K$ for some compact $K \subset \Omega$ and there exists $0 \leq M_N < \infty$ such that

$$\|\varphi\|_N \leq M_N, \forall \varphi \in E, \qquad N = 0, 1, 2, \ldots$$

(d) $D(\Omega)$ has the Heine-Borel property.

(e) $\{\varphi_i\}$ is a Cauchy sequence in $\mathscr{D}(\Omega)$, then $\{\varphi_i\} \in \mathscr{D}_K$ for some $K \subset \Omega$, K compact.

(f) If $\varphi_i \to 0$ in $\mathscr{D}(\Omega)$, then there exists compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_i \subset K$ for all i, and $D^\alpha \varphi_i \to 0$ uniformly for all α.

(g) In $\mathscr{D}(\Omega)$, every Cauchy sequence is convergent.

Proof. (a) Suppose $V \in \tau$. Claim $V \in \mathcal{B}$. Consider $\varphi \in \mathscr{D}_K \cap V$. By previous theorem, there exists $W \in \mathcal{B}$ such that $\varphi + W \subset V$.

$$\Rightarrow \varphi + (\mathscr{D}_K \cap W) \subset \mathscr{D}_K \cap V$$

Since $\mathscr{D}_K \cap W$ is open in $\mathscr{D}_K$, it implies $\mathscr{D}_K \cap V$ is open in $\mathscr{D}_K$ for each $V \in \tau$.

Conversely, if $V \in \mathcal{B}$, then $V \in \tau$, since $\mathcal{B} \subset \tau$.

(b) Let $V \in \tau$, then $\mathscr{D}_K \cap V \in \tau_K$ (by (a)). That is, $\tau \cap \mathscr{D}_K \in \tau_K$ for all $K \subset \Omega$.

Conversely, suppose $E \in \tau_K$ for some $K \subset \Omega$.

Claim. $E = \mathscr{D}_K \cap V$ for some $V \in \tau$. Let $\varphi \in E$, then there exists N and $\delta > 0$ such that

$$\{\psi \in \mathscr{D}_K : \|\psi - \varphi\|_N < \delta\} \subset E$$

or

$$\{\psi \in \mathscr{D}_K : \|\psi\|_N < \delta\} \subset E - \varphi$$

Let $W_\varphi = \{\psi \in \mathscr{D}_K : \|\psi\|_N < \delta\}$, then $W_\varphi \cap \mathscr{D}_K \in \tau_K$ (an open ball in $\mathscr{D}_K$). Hence $W_\varphi \in \mathcal{B}$, and

$$\mathscr{D}_K \cap (\varphi + W_\varphi) = \varphi + W_\varphi \cap \mathscr{D}_K \subset \varphi + E - \varphi = E$$

Let $V = \bigcup_{\varphi \in E} (\varphi + W_\varphi)$, then

$$\begin{aligned} E &= \bigcup_{\varphi \in E} (\varphi + W_\varphi) \cap \mathscr{D}_K \\ &= \text{ union of all balls around } \varphi \in E \\ &= V \cap \mathscr{D}_K. \end{aligned}$$

(c) Let E be a bounded set in $\mathscr{D}(\Omega)$. Suppose $E \notin \mathscr{D}_K$ for any K. Then there exists $\varphi_m \in E$ and a sequence $\{x_m\} \in \Omega$ having no limit point such that $\varphi_m(x_m) \neq 0$, $m = 1, 2, \ldots$

Let

$$W = \left\{ \varphi \in \mathscr{D}(\Omega) : |\varphi(x_m)| < \tfrac{1}{m} \varphi_m(x_m), m = 1, 2, \ldots \right\}$$

Since each K contains only finitely many x_m,

$$W \cap \mathscr{D}_K = \left\{ \varphi \in \mathscr{D}_K : |\varphi(x_m)| < \tfrac{1}{m} \varphi_m(x_m) \right\}$$

is open in $\mathscr{D}_K$. For this, let $\varphi \in W \cap \mathscr{D}_K$. Then $|\varphi(x_m)| < \frac{1}{m}|\varphi_m(x_m)|, m = 1, 2, \ldots l$ Let

$$p(\varphi) = \sup_{1 \le m \le l} |\varphi(x_m)| < C_l, \quad \text{where } C_l = \max_{1 \le m \le l} \frac{1}{m}|\varphi_m(x_m)|$$

Since p is continuous, it follows that $W \cap \mathscr{D}_K$ is open in $\mathscr{D}_K$. Thus $W \in \mathcal{B}$. Since $\varphi_m \notin mW$ for any m, it follows that E is not bounded.

Thus every bounded set $E \subset \mathscr{D}(\Omega)$ must lie in some $\mathscr{D}_K$. By (b), E is bounded in $\mathscr{D}_K$. This implies

$$\sup\{\|\psi\|_N : \psi \in E\} \le M_N < \infty, \quad N = 0, 1, 2, \ldots$$

(d) It follows from (c), since $\mathscr{D}_K$ has the Heine-Borel property. If E is a closed and bounded set in $\mathscr{D}(\Omega)$, then E is closed and bounded in $\mathscr{D}_K$, hence compact. Thus, E is compact in $\mathscr{D}(\Omega)$.

(e) If $\{\varphi_i\}$ is a Cauchy Sequence in $\mathscr{D}(\Omega)$, then it is bounded and hence $\varphi_i \in \mathscr{D}_K$ for some K. By (b), $\{\varphi_i\}$ is Cauchy Sequence relative to $\mathscr{D}_K$.

(f) It is just restatement of (e).

Finally, (g) follows from (b), (e) and completeness of $\mathscr{D}_K$ (i.e., D_K is a Fréchet space).

□

Theorem 3.2.3. *Let Λ be a linear map from $\mathscr{D}(\Omega)$ to a locally convex space Y. Then the following are equivalent:*

(i) Λ is continuous.

(ii) Λ is bounded.

(iii) If $\varphi_i \to 0$ in $\mathscr{D}(\Omega)$, then $\Lambda\varphi_i \to 0$ in Y.

(iv) For all $K \subset \Omega$, the restriction $\Lambda : \mathscr{D}_K \to Y$ is continuous.

Proof. $(i) \implies (ii)$: Known.

$(ii) \implies (iii)$: Suppose Λ is bounded and $\varphi_i \to 0$ in $\mathscr{D}(\Omega)$. Then $\varphi_i \to 0$ in some $\mathscr{D}_K$, and hence $\Lambda/\mathscr{D}_K$ is bounded. Therefore, $\Lambda : \mathscr{D}_K \to Y$ is continuous, and thus $\Lambda\varphi_i \to 0$ in Y.

$(iii) \implies (iv)$: Suppose $\{\varphi_i\} \subset \mathscr{D}_K$ and $\varphi_i \to 0$ in $\mathscr{D}_K$. Then by (b) of the previous theorem, $\varphi_i \to 0$ in $\mathscr{D}(\Omega)$. By (iii), $\Lambda\varphi_i \to 0$ in Y.

$(iv) \implies (i)$: Let U be a convex balanced neighborhood of 0 in Y, and write $V = \Lambda^{-1}(U)$. Then V is a convex, also balanced set in $\mathscr{D}(\Omega)$. By (a) of the previous theorem, $V \in \tau$ if and only if $\mathscr{D}_K \cap V \subset \tau_K$ for each $K \subset \Omega$. By (iv), $\mathscr{D}_K \cap V \in \tau_K$, hence $V \in \tau$. Hence Λ is continuous.

□

Definition 3.2.4. A linear functional Λ on $\mathscr{D}(\Omega)$ which is continuous in the topology τ of $\mathscr{D}(\Omega)$ is called **distribution**.

The space of all distributions is denoted by $\mathscr{D}'(\Omega)$.

Theorem 3.2.5. *Let Λ be a linear functional on $(\mathscr{D}(\Omega), \tau)$. Then the following are equivalent:*

(i) $\Lambda \in \mathscr{D}'(\Omega)$.

(ii) For each compact set $K \subset \Omega$, there exist $N \in \mathbb{N}$ and $C > 0$ such that

$$|\Lambda\psi| \leq C\|\psi\|_N \quad \text{for all } \psi \in \mathscr{D}_K \tag{$*$}$$

This result is nothing but equivalence of (i) and (iv) in the previous theorem.

Note that if N in $(*)$ is independent of the choice of K, then the minimum of such N's is called the **order of the distribution** Λ. If no such N exists, then we say Λ has ∞ order.

Remark 3.2.6. Since each $\mathscr{D}_K$ is closed, it is obvious that $\mathscr{D}_K$ has no interior in $\mathscr{D}(\Omega)$. Since there exists a countable sequence of compact sets in Ω such that $\Omega = \bigcup_{i=1}^{\infty} K_i, \quad K_i \subset K_{i+1}$ we get

$$\mathscr{D}(\Omega) = \bigcup_{i=1}^{\infty} \mathscr{D}_{K_i}$$

Since Cauchy sequence in $\mathscr{D}(\Omega)$ does converges in $\mathscr{D}(\Omega)$, by the Baire Category Theorem, $\mathscr{D}(\Omega)$ cannot be metrizable.

Example 3.2.7. Let $f \in L^{loc}(\mathbb{R}^n)$, then

$$\Lambda_f(\varphi) = \int f\varphi, \quad \varphi \in \mathcal{D}(\mathbb{R}^n)$$

defines a distribution on $\mathcal{D}(\mathbb{R}^n)$. However, every distribution cannot be generated by a function in this way.

For example, Dirac distribution δ_0 cannot be produced by any $f \in L_{loc}(\mathbb{R}^n)$.

On contrary, suppose, there exists $f(\neq 0) \in L_{loc}(\mathbb{R}^n)$ such that $\delta_0(\varphi) = \int f\varphi$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Consider $\varphi_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ such that support of $\varphi_\varepsilon \subseteq B_\varepsilon(0)$, $0 \leq \varphi_\varepsilon \leq 1$, $\varphi_\varepsilon = 1$ on $B_{\varepsilon/2}(0)$. Then

$$\delta_0(\varphi_\varepsilon) = \int f\varphi_\varepsilon$$

$$\implies 1 = \varphi_\epsilon(0) = \int_{B_\varepsilon(0)} f\varphi_\varepsilon \leq \int_{B_\varepsilon(0)} |f| \to 0 \text{ as } \varepsilon \to 0.$$

However, every distribution is weakly assigned to some derivative of a continuous function. We see it later. Notice that

$$|\delta_0(\varphi)| = |\varphi(0)| \leq \|\varphi\|_\infty = \|\varphi\|_0, \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n)$$

Hence, δ_0 is a distribution of order 0.

Example: Let μ be a Radon measure on Ω. Then

$$\Lambda(\varphi) = \int \varphi(x)\, d\mu(x)$$

defines a distribution and

$$|\Lambda(\varphi)| \leq \|\varphi\|_\infty \mu(K),\ \varphi \in \mathcal{D}_K,\ \text{and for every choice of } K,\ \text{compact in } \Omega.$$

Hence, $\Lambda = \Lambda_\mu$ is a distribution of order 0. Later, we see that every distribution of order zero is given by a Radon measure.

3.3 Local Equality of Distribution

Let $\Lambda_i \in \mathcal{D}'(\Omega),\ i = 1, 2,$ and let $O \subset \Omega$ be open. Then we say $\Lambda_1 = \Lambda_2$ in O if

$$\Lambda_1\varphi = \Lambda_2\varphi,\ \forall \varphi \in \mathcal{D}(O).$$

For example, if $f \in L^{loc}(\mathbb{R})$ and $\varphi \in \mathcal{D}(O)$, then $\Lambda_f = 0$ if and only if $f = 0$ almost everywhere on O.

Similarly, if μ is a Radon measure, then $\Lambda_\mu = 0$ if $\mu(B) = 0$, for all $B \in \mathcal{B}(O)$, the Borel σ-algebra on O.

Therefore, distribution can be discussed locally, and that leads to ways to describe distributions globally, if its behavior is known locally.

For this, we need to describe "partition of unity".

Theorem 3.3.1. *Let $\mathcal{A} = \{O_i; i \in I\}$ be an open cover of Ω. Then, there exists a sequence $\{\psi_i\}_{i\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ with $\psi_i \geq 0$ such that*

(i) each ψ_i has support in some $O_i \in \mathcal{A}$,

(ii) $\sum_{i\in\mathbb{N}} \psi_i(x) = 1,\ \forall x \in \Omega$,

(iii) for each compact set $K \subset \Omega$, $\exists m \in \mathbb{N}$ and an open set $O \supset K$ such that

$$\psi_1(x) + \ldots + \psi_m(x) = 1, \quad \forall x \in O.$$

The collection $\{\psi_i\}$ is called a locally finite partition of unity in Ω subordinate to the cover $\mathcal{A}$ of Ω.

Remarks: From (ii) and (iii), it follows that each point $x \in \Omega$ has an open neighborhood that intersects the supports of only finitely many ψ_i.

Proof. Let $S = \{p_1, p_2, \ldots\}$ be a countable dense set in Ω.
For $r_i \in \mathbb{Q}$, write $B_i = \overline{B}_{r_i}(p_i)$, a closed ball that is contained in some $O_i \in \mathcal{A}$. Let $V_i = B_{r_i/2}(p_i)$. Then, $\Omega = \bigcup_i V_i$; since $\overline{S} = \Omega$, we can construct $\varphi_i \in \mathcal{D}(\Omega)$ such that $0 \le \varphi_i \le 1$, $\varphi_i = 1$ on V_i, $\varphi_i = 0$ outside B_i.

Define $\psi_1 = \varphi_1$, and inductively write

$$(1)\ \psi_{i+1} = (1-\varphi_1)\cdots(1-\varphi_i)\varphi_{i+1},\ i \ge 1.$$

Then $\psi_i = 0$ outside B_i. This proves (i).

The relation

$$(2)\ \psi_1 + \ldots + \psi_i = 1 - (1-\varphi_1)\cdots(1-\varphi_i)$$

is trivially true if $i = 1$. Suppose (2) is true for some i, then by adding (2) at (i) we get (2) is true for $i+1$. Since $\varphi_i = 1$ in V_i, from (2), it follows that

$$\psi_1(x) + \ldots + \psi_m(x) = 1, \quad \forall x \in V_1 \cup \ldots \cup V_m = O.$$

Since for any $x \in \Omega$, there exists m such that $x \in V_1 \cup \ldots \cup V_m$, this proves (ii). Moreover, if K, compact in Ω, then $K \subset \bigcup_{i=1}^m V_i$ for some m. This proves (iii).

Now, suppose $\Lambda_1, \Lambda_2 \in \mathcal{D}'(\Omega)$ and for each $x \in \Omega$, there exists O_x open in Ω such that

$$\Lambda_1(\varphi) = \Lambda_2(\varphi),\ \forall \varphi \in \mathcal{D}(\Omega_x).$$

Then there exists a partition of unity $\{\psi_i, B_i\}_{i=1}^\infty$ such that

$$\sum_{i=1}^\infty \psi_i(x) = 1, \quad \forall x \in \Omega.$$

Let $\varphi \in \mathcal{D}(\Omega)$, then $\varphi = \sum_{i=1}^\infty \psi_i \varphi$. The summation in RHS makes sense, since support of φ intersects support of only finitely many ψ_i. Thus,

$$\Lambda_1(\varphi) = \sum \Lambda_1(\psi_i\varphi) = \sum \Lambda_2(\psi_i\varphi) = \Lambda_2(\varphi),$$

since $\psi\varphi \in \mathcal{D}(B_i) \subset \mathcal{D}(O_{x_i})$, for some $x_i \in \Omega$. Hence, $\Lambda_1 = \Lambda_2$ in $\mathcal{D}(\Omega)$. □

Theorem 3.3.2. *Let $\mathcal{A}$ be an open cover of Ω, and for each $O \in \mathcal{A}$, there exists $\Lambda_0 \in \mathcal{D}'(O)$ such that*

$$\Lambda_0' = \Lambda_0'' \quad \forall O' \cap O'' \neq \emptyset.$$

Then there exists unique $\tilde{\Lambda} \in \mathcal{D}'(\Omega)$ *such that*

$$\tilde{\Lambda}\varphi = \Lambda_0 \textit{ in } O, \quad \forall O \in \mathcal{A}.$$

Proof: Let $\{\psi_i, B_i\}_{i=1}^N$ be a partition of unity subordinate to $\mathcal{A}$. Let $\varphi \in \mathcal{D}(\mathbb{R})$, then

$$\varphi = \sum_{i=1}^{N} \psi_i \varphi \quad \text{(finite sum for each } \varphi)$$

Define

$$\tilde{\Lambda}\varphi = \sum \Lambda_{B_i}(\psi_i \varphi).$$

Then $\tilde{\Lambda}$ is linear. To show that $\tilde{\Lambda}$ is continuous on $\mathcal{D}(\mathbb{R})$,
let $\varphi_j \to 0$ in $\mathcal{D}(\mathbb{R})$. Then $\operatorname{supp} \varphi_j \subset K$, K' for some K compact in $\mathbb{R}$.

$$\implies \text{supp } \psi_i \varphi_j \subset K \cap \overline{B}_i \subset \overline{B}_i,$$

$$\implies \psi_i \varphi_j \to 0 \text{ in } \mathcal{D}(B_i) \quad \text{(by Leibniz rule)}$$

Hence, $\tilde{\Lambda}\varphi_j \to 0$ in $\mathbb{C}$ in $\mathcal{D}'(\Omega)$ (the weak* topology of $\mathcal{D}(\mathbb{R})$). Thus, $\tilde{\Lambda} \in \mathcal{D}'(\not\leq)$.

Let $\varphi \in \mathcal{D}(O)$, $O \in \mathcal{A}$. Then

$$\psi_i \varphi \in \mathcal{D}(B_i \cap O) \quad \forall i,$$

and

$$\Lambda_{B_i}(\psi_i \varphi) = \Lambda_0(\psi_i \varphi) \quad \text{(by hypothesis)}$$

Thus,

$$\tilde{\Lambda}\varphi = \sum \Lambda_0(\psi_i \varphi) = \Lambda_0(\varphi).$$

Suppose $\overline{\Lambda}$ be any other distribution such that

$$\Lambda_{O'} = \Lambda_{O''} \quad \text{ if } O' \cap O'' \neq \emptyset.$$

Then for each B_i, there exists $O_i \in \mathcal{A}$ such that $B_i \subset O_i$

$$\overline{\Lambda}_{B_i} = \Lambda_{O_i} = \tilde{\Lambda}_{B_i}.$$

For $\varphi \in \mathcal{D}(\not\leq)$, $\varphi = \sum \psi_i \varphi$, $\operatorname{supp} \psi_i \subset B_i$.

$$\overline{\Lambda}(\varphi) = \sum \overline{\Lambda}(\psi_i \varphi) = \sum \tilde{\Lambda}_{B_i}(\psi_i \varphi) = \sum \tilde{\Lambda}(\varphi)$$

$$\implies \overline{\Lambda} = \tilde{\Lambda}$$

Theorem 3.3.3. *A distribution $\Lambda \in \mathcal{D}'(\Omega)$ is of order 0 if and only if there exists a Radon measure μ (possibly complex-valued) such that $\Lambda = \Lambda_\mu$.*

Proof. If $\exists \mu$ a Radon measure. Then $\text{order}(\Lambda_\mu) = 0$.

Conversely, suppose $\text{order}(\Lambda) = 0$. Then there exists $0 < C < \infty$, such that $|\Lambda\varphi| \le C\,||\varphi||_\infty$, $\forall \varphi \in C_c^\infty(\Omega)$. Consider $\{\psi_i, B_i\}_{i=1}^\infty$, a partition of unity. Then $\operatorname{supp}\psi_i \subset B_i$, $\cup B_i = \Omega$. Then Λ is continuous on each $\mathcal{D}(B_i)$ and hence it can be extended to $C(B_i)$. By Riesz representation theorem, there exists a complex-valued Radon measure μ_i on B_i such that

$$\Lambda(\varphi) = \int \varphi d\mu_i, \quad \forall \varphi \in C(B_i).$$

In particular, for each $\varphi \in \mathcal{D}(B_i)$. Let φ belong to $\mathcal{D}(\nleq)$, then

$$\varphi = \sum \psi_i \varphi,$$

and

$$\Lambda(\varphi) = \sum \Lambda(\psi_i\varphi) = \sum \int \psi_i \varphi d\mu_i$$

$$\text{i.e.} \quad \Lambda\varphi = \int \varphi \left(\sum \psi_i d\mu_i\right) = \int \varphi d\mu,$$

where $\mu = \sum \psi_i d\mu_i$. □

3.4 Derivative of distribution

Notice that for $\varphi \in \mathcal{D}(\Omega)$ and $f \in C^\infty(\Omega)$,

$$\int_\Omega f\varphi' = f\varphi|_{\delta\Omega} - \int_\Omega f\varphi' = -\int_\Omega f\varphi',$$

since $\operatorname{supp}\varphi \subset K \subset \Omega$. This gives way to define the derivative of distribution $\Lambda \in \mathcal{D}'(\Omega)$ by

$$\Lambda'(\varphi) = -\Lambda(\varphi').$$

or,

$$\partial^\alpha \Lambda(\varphi) = (-1)^{|\alpha|}\Lambda(\partial^\alpha \varphi).$$

Hence, $D^\alpha\Lambda$ is a linear map. Since $\Lambda \in \mathcal{D}'(\Omega)$, for compact set $K \subset \Omega$, $\exists 0 < C < \infty$ and $N \in \mathbb{N}$ such that

$$|\Lambda\varphi| \le C||\varphi||_N, \quad \forall \varphi \in \mathcal{D}_K.$$

Then

$$|D^\alpha \Lambda(\varphi)| = |(-1)^{|\alpha|}\Lambda(D^\alpha\varphi)| \le C||\varphi||_{N+|\alpha|} \text{ for all } \varphi \in \mathcal{D}_K.$$

Thus, $\partial^\alpha \Lambda \in \mathcal{D}'(\Omega)$. We infer that every distribution in $\mathcal{D}'$ is infinitely differentiable in the weak sense. Since

$$D^\alpha D^\beta \varphi = D^{\alpha+\beta}\varphi = D^\beta D^\alpha \varphi,$$

it follows that

$$D^\alpha D^\beta \Lambda = D^\beta D^\alpha \Lambda.$$

Example 3.4.1. Let $f \in L^1_{\text{loc}}(\mathbb{R})$. Then show that

$$D^\alpha f \in \mathcal{D}'(\Omega) \quad \text{and} \quad D^\alpha \Lambda_f(\varphi) = (-1)^{|\alpha|}(D^\alpha \varphi).$$

Does distributional derivative of a function is same as its usual derivative?

i.e., whether

$$\int D^\alpha f \varphi = (-1)^{|\alpha|} \int f D^\alpha \varphi?$$

If $f \in C^\infty(\mathbb{R})$, then

$$\int D^\alpha f \varphi = (-1)^{|\alpha|} \int f D^\alpha \varphi,$$

by “integration by parts". However, this is not true in general.

Example 3.4.2. Let $\Omega = (-2, 2)$, consider f is the Cantor function on $[0, 1]$. Then $f \in L^1(-2, 2)$ and $f' = 0$ almost everywhere.

$$\int f'\varphi = 0 \neq -\int f\varphi'$$

Example 3.4.3. If f is absolutely continuous on each $[a, b] \subset \mathbb{R}$, then $\Lambda'_f = \Lambda_{f'}$. That is,

$$\int f'\varphi = -\int f\varphi'.$$

(Note that “integration by parts" holds for absolutely continuous and integrable functions

3.5 Multiplication by a function

Let $\Lambda \in \mathcal{D}'(\Omega)$, and $f \in C^\infty(\Omega)$. Then

1. $(f\Lambda)(\varphi) = \Lambda(f\varphi)$ defines a linear functional on $\mathcal{D}(\Omega)$.
2. $D^\alpha(f\varphi) = \sum_{\beta \leq \alpha} c_{\alpha,\beta} D^{\alpha-\beta} f D^\beta f \cdot D\beta\varphi$ (By Leibniz formula)

Since $\Lambda \in \mathcal{D}'(\Omega)$, for each compact set K in Ω, there exists $0 < C < \infty$ and $N \in \mathbb{Z}_+$ such that

$$|\Lambda\varphi| \leq C\|\varphi\|_N, \quad \forall \varphi \in \mathcal{D}_K.$$

By (2), there exists $C' = C'(f, K, N)$ such that

$$\|f\varphi\| \leq C'\|\varphi\|_N, \quad \forall\varphi \in \mathcal{D}_K$$

Hence,

$$|f\Lambda(\varphi)| \leq CC'\|\varphi\|_N, \forall\varphi \in \ \mathcal{D}_K.$$

Thus, $f\Lambda \in \mathcal{D}'(\Omega)$.

3.6 Sequence of Distributions

Since the topology of $\mathcal{D}(\Omega)$ provides a weak*-topology on $\mathcal{D}'(\Omega)$, that makes $\mathcal{D}'(\Omega)$ a locally convex topological vector space, the convergence in $\mathcal{D}'(\Omega)$ is understood by point evaluation. That is, $\{\Lambda_i\}_{i=1}^{\infty} \in \mathcal{D}'(\Omega)$ is said to converge to Λ if

$$\Lambda_i(\varphi) \to \Lambda(\varphi), \ \forall\varphi \in \mathcal{D}(\Omega)$$

In particular, if $f_i \in L^1_{loc}\infty(\mathbb{R}^n)$, then $f_i \to \Lambda$ in $\mathcal{D}'(\mathbb{R}^n)$ if

$$\lim f_i\varphi = \lambda\varphi, \ \forall\varphi \in \mathcal{D}(\mathbb{R}^n).$$

Theorem 3.6.1. *Let $\Lambda_i \in \mathcal{D}'(\Omega)$ and $\Lambda(\varphi) = \lim \Lambda_i(\varphi)$ exists for each $\varphi \in \mathcal{D}(\Omega)$. Then $\Lambda \in \mathcal{D}'(\Omega)$ and $D^\alpha\Lambda_i \to D^\alpha\Lambda$ in $\mathcal{D}(\Omega)$.*

Proof. Since $\Lambda\varphi = \lim \Lambda_i\varphi, \ \forall\varphi \in \mathcal{D}(\Omega)$, it implies that

$$\Lambda(\varphi) = \lim \Lambda_i(\varphi), \ \forall\varphi \in \mathcal{D}_K$$

As $\mathcal{D}_K$ is a Fréchet space, by Banach-Steinhaus Theorem, $\Lambda/\mathcal{D}_K$ is continuous for each $K \subset \Omega$. Hence, Λ is constant on $\mathcal{D}(\Omega)$.

Now,

$$\begin{aligned} D^\alpha(\Lambda)(\varphi) &= (-1)^{|\alpha|}\Lambda(D^\alpha\varphi) \\ &= (-1)^{|\alpha|} \lim \Lambda_i(D^\alpha\varphi) \\ &= \lim D^\alpha\Lambda_i(\varphi) \end{aligned}$$

□

Theorem 3.6.2. *If $\Lambda_i \to \Lambda$ in $\mathcal{D}'(\Omega)$ and $g_i \to g$ in $C^\infty(\Omega)$, then $g_i\Lambda_i \to g\Lambda$ in $\mathcal{D}'(\Omega)$.*

Proof. Note that $g_i \to g$ in $C^\infty(\Omega)$ means the Fréchet space topology of $C^\infty(\Omega)$.

(i.e., topology generated by $p_N(f) = \sup_{|\alpha|\leq N, x\in K_N} |D^\alpha f(x)|$, where $\Omega = \bigcup K_N,\ K_N \subset K_{N+1}$ with local base

$$V_N = \{f \in C^\infty(\Omega) : p_N(f) < 1/N, \}\ N = 1, 2, \ldots$$

Now, for fixed $\varphi \in \mathcal{D}(\Omega)$, define a bilinear form $\mathcal{B}(g, \Lambda) = g\Lambda(\varphi) = \Lambda(g\varphi)$. Then $\mathcal{B}$ is coordinatewise continuous, and by Theorem 2.17 (Rudin FA, Page 52), and the fact that $C^\infty(\Omega)$ is a Fréchet space, $\mathcal{D}'(\Omega)$ and $\mathbb{C}$ are topological vector spaces, it follows that

$$\mathcal{B}(g_i, \Lambda_i) \to \mathcal{B}(g, \Lambda) \text{ as } i \to \infty$$

Hence,

$$(g_i\Lambda_i)(\varphi) \to (g\Lambda)(\varphi),\ \forall \varphi \in \mathcal{D}(\Omega).$$

□

3.7 Support of a Distribution

Let U be an open set in Ω and $\Lambda \in \mathcal{D}'(\Omega)$. We say that Λ is zero in O if

$$\Lambda(\varphi) = 0, \quad \forall \varphi \in \mathcal{D}(O)$$

Let $W = \bigcup\{O \subset \Omega : \Lambda|_O = 0\}$. Then $\Lambda|_W = 0$. The complement of W is called the support of Λ. Note that O forms an open cover of W.

There exists a partition of unity $\{\psi_i\}$ in W such that $\operatorname{supp}\psi_i \subset O_i$ for some O_i such that $\Lambda|_{O_i} = 0$, and

$$\varphi = \sum_{i=1}^{\infty} \varphi\psi_i, \quad \forall \varphi \in \mathcal{D}(W)$$

Hence,

$$\Lambda\varphi = \sum_{i=1}^{\infty} \Lambda(\varphi_i\varphi) = 0, \text{ that is, } \Lambda|_W = 0.$$

Theorem 3.7.1. *Let $\Lambda \in \mathcal{D}'(\Omega)$ and set $S_\Lambda = \operatorname{supp}\Lambda$.*

(a) If $\operatorname{supp}\varphi \cap S_\Lambda = \varnothing$ for some $\varphi \in \mathcal{D}(\Omega)$, then $\Lambda\varphi = 0$ (by definition of support).

(b) If $S_\Lambda = \varnothing$, then $\Lambda = 0$ (i.e., $W = \Omega$).

(c) If $\psi \in C^\infty(\Omega)$ and $\psi = 1$ on an open set $V \supset S_\Lambda$, then $\psi\Lambda = \Lambda$.

(d) If S_Λ is a compact set, then Λ is of finite order. In fact, there exists $0 < C < \infty$ and some $N \in \mathbb{N}U\{0\}$ such that

$$|\Lambda\varphi| \leq C\|\varphi\|_N, \quad \forall \varphi \in \mathcal{D}(\Omega)$$

Further, Λ extends uniquely to a continuous linear functional on $C^\infty(\Omega)$.

Proof. Proofs of (a) & (b) are trivial.

(c) If $\psi = 1$ on $V \supset S_\Lambda$, then

$$\operatorname{supp}(\varphi - \psi\varphi) \cap S_\Lambda = \emptyset, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Hence by (a), $\Lambda(\varphi - \psi\varphi) = 0$. That is,

$$\Lambda\varphi = \psi\Lambda\varphi, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

(d) If S_Λ is compact, then we can always find $\psi \in C_c^\infty(\Omega)$ such that $\psi = 1$ on $V \supset S_\Lambda$, for some open set $V \subset \Omega$. Let $\operatorname{supp}\psi = K$. Then from (c),

$$\Lambda(\varphi) = \psi\Lambda(\varphi), \quad \text{if } \varphi \in \mathcal{D}(\Omega).$$

Since $\Lambda \in \mathscr{D}'(\Omega)$, there exists $C_1 > 0$ such that

$$|\Lambda\varphi| \leq C_1\|\varphi\|_N, \quad \forall \varphi \in \mathscr{D}_K$$

for some $N \in \mathbb{N} \cup \{0\} = \mathbb{Z}^+$ (say). Further, by Leibniz's rule, it follows that there exists $C_2 > 0$ such that

$$\|\psi\varphi\|_N \leq C_2\|\varphi\|_N,$$

(i.e. $\operatorname{supp}\varphi = K$ cpt). Since $\Lambda\varphi = \Lambda(\psi\varphi)$ if $\varphi \in \mathscr{D}(\Omega)$, define

$$\Lambda f = \Lambda(\psi f) \quad \text{for } f \in C^\infty(\Omega).$$

Now if $f_i \to 0$ in $C^\infty(\Omega)$, then $D^\alpha f_i \to 0$ on uniformly on each compact set $K \subset \Omega$. Once again, by Leibniz's formula, it follows that

$$\psi f_i \to 0 \quad \text{in } \mathcal{D}(\Omega).$$

$$\implies \Lambda(\psi f_i) \to 0 \text{ in } \mathcal{D}'(\Omega).$$

That is, $\Lambda f_i \to 0$ in the toplogy of $\mathcal{D}'(\Omega)$. Notice that if $f \in C^\infty(\Omega)$ and $K_0 \subset \Omega$ is compact, then there exists $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi = f$ on K_0. (By Urysohn's lemma, there exists $\psi \in \mathcal{D}(\Omega)$ such that $\psi = 1$ on K_0, and hence $\varphi = f\psi = f$ on K_0). It follows that $\mathcal{D}(\Omega)$ is dense in $C^\infty(\Omega)$. (i.e. $\|\varphi - f\|_K = \|f\psi - f\|_K < \epsilon$). Hence, $\Lambda \in \mathcal{D}'(\Omega)$ has unique extension to $C^\infty(\Omega)$.

□

3.8 Exercise

1. (a) If Λ' is a compactly supported distribution, must it follow that Λ itself is compactly supported?

   (b) Is every compactly supported distribution necessarily of finite order?

   (c) Must the Fourier transform of every compactly supported function in $L^1(\mathbb{R})$ be real analytic?

   (d) Determine the distributional support of the function $\chi_{\mathbb{Q}}$, where $\mathbb{Q}$ denotes the set of rational numbers.

   (e) For $n \in \mathbb{N}$, let δ_n denote the Dirac delta distribution at n. Does $\delta_n \to 0$ in the weak* topology of $C_0(\mathbb{R})$ (the space of continuous functions vanishing at infinity)?

   (f) Determine the order of $\Lambda \in \mathcal{D}'(\mathbb{R})$ defined by

$$\Lambda(\varphi) = \int_{|x|>1} \log(x)\, \varphi(x)\, dx.$$

2. Suppose f is a continuous function on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} f\varphi = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Show that $f = 0$.

3. Let $\Lambda = \Lambda_f$, where f is a continuous function on $\mathbb{R}^n$. Show that $\operatorname{supp} \Lambda_f = \operatorname{supp} f$. Does the same statement remain valid for locally integrable functions?

4. Show that there exists $\psi \in \mathcal{D}(\mathbb{R})$ such that $\varphi = \psi^{(k)}$ if and only if

$$\int_{\mathbb{R}} p(x)\varphi(x)\, dx = 0$$

   for each polynomial p of degree at most $k - 1$.

5. If $\Lambda \in \mathcal{D}'(\mathbb{R})$ satisfies $\Lambda' = 0$, prove that $\Lambda = \Lambda_c$ for some constant c.

6. Show that every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ can be written as

$$\varphi = \psi' + c\varphi_0,$$

   where φ_0 is a fixed test function in $\mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi_0 \neq 0$.

7. Show that every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ can be written as

$$\varphi = x\psi + c\varphi_0,$$

   where φ_0 is a fixed test function in $\mathcal{D}(\mathbb{R})$ with $\varphi_0(0) \neq 0$. Deduce that if $\Lambda \in \mathcal{D}'(\mathbb{R})$ and $x\Lambda = 0$, then $\Lambda = c\delta_0$.

8. Determine all $f \in C^\infty(\mathbb{R})$ such that $f\delta_0' = 0$.

9. Show that if $\Lambda \in \mathcal{D}'(\mathbb{R})$ is compactly supported, then Λ' is also compactly supported.

10. Verify that
$$\langle \Lambda, \varphi \rangle = \sum_{n=1}^{\infty} \varphi^{(n)}(n)$$
defines a distribution on $\mathbb{R}$. Is Λ compactly supported?

11. Let $H = \chi_{(-\infty,0)}$ and let h_n be a sequence of differentiable functions such that $h_n \to H$ in $\mathcal{D}'(\mathbb{R})$. Show that $h_n' \to \delta_0$ in $\mathcal{D}'(\mathbb{R})$. Does the conclusion remain valid if $H = \chi_{(-\infty,0]}$?

12. Let $\Lambda_n \in \mathcal{D}'(\mathbb{R})$ be defined by
$$\langle \Lambda_n, \varphi \rangle = n\Big(\varphi\Big(\tfrac{1}{n}\Big) - \varphi\Big(-\tfrac{1}{n}\Big)\Big).$$
Determine $\lim \Lambda_n$.

13. For $a > 0$, define
$$\langle \Lambda_a, \varphi \rangle = \left(\int_{-\infty}^{-a} + \int_a^{\infty}\right) \frac{\varphi(x)}{|x|}\, dx + \int_{-a}^{a} \frac{\varphi(x) - \varphi(0)}{|x|}\, dx.$$
Show that Λ_a defines a distribution on $\mathcal{D}(\mathbb{R})$. Find $\lim_{a\to 0} \Lambda_a$ in $\mathcal{D}'(\mathbb{R})$ and compute its distributional derivative.

14. For $\Lambda \in \mathcal{D}'(\mathbb{R})$, define
$$\langle G, \varphi \rangle = \int_{\mathbb{R}} \langle \Lambda, \varphi_y \rangle\, dy,$$
where for $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we set $\varphi_y(x) = \varphi(x, y)$. Show that $G \in \mathcal{D}'(\mathbb{R}^2)$.

15. Let $\Lambda_i \in \mathcal{D}'(\mathbb{R})$ for $i = 1, 2$ be such that
$$\langle \Lambda_1, \varphi \rangle = 0 \quad \Longleftrightarrow \quad \langle \Lambda_2, \varphi \rangle = 0.$$
Show that $\Lambda_1 = c\Lambda_2$ for some constant c.

16. If $\Lambda \in \mathcal{D}'(\mathbb{R})$ satisfies $\Lambda^k = 0$, prove that Λ is a polynomial of degree at most $k - 1$.

17. Let $\Omega = (0, \infty)$. Define
$$\langle \Lambda, \varphi \rangle = \sum_{n=1}^{\infty} \varphi^{(n)}\Big(\tfrac{1}{n}\Big), \quad \varphi \in \mathcal{D}(\Omega).$$
Show that Λ is a distribution of infinite order, and prove that Λ cannot be extended to a distribution on $\mathbb{R}$.

18. If $\Lambda \in \mathcal{D}'(\mathbb{R})$ has order N, show that $\Lambda = f^{(N+2)}$ in $\mathcal{D}'(\mathbb{R})$ for some continuous function f. If $\Lambda = \delta_0$, what are the possible choices for f?

19. For $k \in \mathbb{N}$, define $f_k = k\chi_{(\frac{1}{k},\frac{2}{k})}$. Show that $f_k \to \delta_0$ in $\mathcal{D}'(\mathbb{R})$. Furthermore, show that although $f_k^2(x) \to 0$ pointwise, the sequence f_k^2 does not converge in the sense of distributions.

20. Define
$$f(x) = \begin{cases} x^2, & x < 1, \\ x^2 + 2x, & 1 \le x \le 2, \\ 2x, & x \ge 2. \end{cases}$$
Find the distributional derivative of f.

21. Define
$$f(t) = \begin{cases} e^{-t}, & t > 0, \\ -e^{t}, & t < 0. \end{cases}$$
Show that $f'' = 2\delta_0' + f$. Deduce that the Fourier transform of f is
$$\hat{f}(x) = -\frac{2ix}{1+x^2}.$$

22. If $H = \chi_{(-\infty,0)}$, show that
$$\begin{aligned} &(a) \quad H * \varphi(x) = \int_{-\infty}^{x} \varphi(t)\,dt, \\ &(b) \quad \delta_0' * H = \delta_0, \\ &(c) \quad 1 * \delta_0' = 0, \\ &(d) \quad 1 * (\delta_0' * H) = 1 * \delta_0 = 1, \\ &(e) \quad (1 * \delta_0') * H = 0. \end{aligned}$$

23. Let $\{x_k\}$ be a sequence of real numbers with $\lim |x_k| = \infty$. Show that $\delta_{(x - x_k)} \to 0$ in the sense of distributions.

24. Determine all $f, g \in C^\infty(\mathbb{R})$ such that $f\delta_0 + g\delta_0' = 0$.

25. Define
$$f(x) = \begin{cases} e^{-x}, & x \ge 0, \\ 1, & x < 0. \end{cases}$$
Show that the Fourier transform of f satisfies $(1 - ix)\hat{f} = \hat{H}$ in the sense of tempered distributions, where $H = \chi_{(-\infty,0)}$.

26. Find the distributional derivative of $f(x) = e^{x^2}\chi_{[0,1]}(x)$.

27. Suppose $f \in L^\infty(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} f(y)e^{-y^2}e^{2xy}\,dy = 0 \quad \forall x \in \mathbb{R}.$$

Prove that $f \equiv 0$.

28. Let Λ be a distribution on $\mathbb{R}$ such that $x^2\Lambda = 0$. Show that $\Lambda = c\delta_0 + d\delta_0'$ for some constants c, d.

29. For $n \in \mathbb{N}$, let $f_n = \chi_{[0,n)}$. Find $\lim_{n\to\infty} f_n'$ in the weak* topology of $\mathcal{D}(\mathbb{R})$.

30. Classify all continuous functions on $\mathbb{R}$ that define tempered distributions.

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