DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA650: Advanced Course on Hardy spaces — MidSem
Instructor: Rajesh Srivastava — March 1, 2022
Time duration: two hour — Maximum Marks: 25

N.B. Answer without proper justification will attract zero mark.

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1. (a) If Θ is an inner function in $H^2(\mathbb{T}, m)$ such that $\Theta H^2(m) = H^2(m)$, does it imply that Θ is constant a.e. m? **[1]**

   (b) If $f, g \in H^2(\mathbb{T}, m)$ are two non-zeoro functions such that $\hat{g}(0) = 0$. Does it imply that $\widehat{(fg)}(0) = 0$? **[1]**

   (c) Let $f \in H^2(\mathbb{T})$ be such that $\frac{1}{f} \in H^\infty(\mathbb{T})$. Does it imply that $\frac{1}{f} \in E_f$? **[1]**

   (d) For $f \in H^\infty(\mathbb{D})$, let $f_{(r)}(z) = f(rz)$ if $|z| < \frac{1}{r}$ and $0 \leq r < 1$. Does it imply $\lim\limits_{r \to 1} \|f_{(r)}\|_\infty = \|f\|_{H^\infty(\mathbb{D})}$? **[1]**

2. Let $\{\Theta_i \in H^2 : i \in I\}$ be a family of inner functions. Show that span $\{\Theta_i H^2 : i \in I\} = \Theta H^2$, where $\Theta = \gcd\{\Theta_i : i \in I\}$. **[3]**

3. Show that a polynomial $p(z)$ on $\mathbb{C}$ is an outer function in $H^2(\mathbb{T})$ if and only if zero set $Z(p) \subset \{z \in \mathbb{C} : |z| \geq 1\}$. **[3]**

4. For $w \in L^1(\mathbb{T})$, define $E_w = \overline{\text{span}}\{z^n w : n \geq 0\}|_{L^1(\mathbb{T})}$. Does there exists $w \in L^1(\mathbb{T})$ such that $\bar{z} \in E_w$? Find all possible $w \in L^1(\mathbb{T})$ such that $\bar{z} \in E_w$. **[3]**

5. Let $M(\mathbb{T})$ denote the space of all complex Borel measure on $\mathbb{T}$. Let $W = \{\mu \in M(\mathbb{T}) : \hat{\mu}(k) = 0 \text{ if } k < 0\}$. Suppose $\mu_n \in W$ converges to $\mu \in M(\mathbb{T})$ in the week* topology of $M(\mathbb{T})$. Show that there exists $h \in H^1(\mathbb{T})$ such that $\hat{\mu}(k) = \hat{h}(k)$, $k \in \mathbb{Z}$. **[3]**

6. Let $f, g \in H^2(\mathbb{T}, m)$. Show that $fg \in H^1(\mathbb{T}, m)$. Does the same conclusion hold if $f \in L^2(\mathbb{T}, m)$? **[3]**

7. Using the identification of $H^1(\mathbb{D})$ with $H^1(\mathbb{T})$, show that convergence in $H^1(\mathbb{T})$ implies uniform convergence on every disc in $\mathbb{D}$. **[3]**

8. Let $f \in H^\infty(\mathbb{D})$. Show that $f_{(r)}$ converges to $\tilde{f}$ in the week* topology of $L^\infty(\mathbb{T})$. **[3]**

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