DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA650: Advanced Course on Hardy spaces	EndSem
Instructor: Rajesh Srivastava	May 5, 2022
Time duration: three hour	Maximum Marks: 35

N.B. Answer without proper justification will attract zero mark.

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1. (a) It is possible that Blaschke product can be an outer function? **[1]**

   (b) Does generalized Jensen's inequality hold for H^p for $0 < p < 1$? **[1]**

   (c) Is it possible that an inner function can be uniform limit of Blaschke products having distinct zero? **[1]**

   (d) If $f \in \mathrm{Nev}(\mathbb{D})$ and f is outer on $\mathbb{D}$, does it imply that f is outer on $\frac{1}{2}\mathbb{D}$? **[1]**

   (e) If $u \in L^\infty(\mathbb{T})$ is real-valued, does it imply that its Hilbert transform $\tilde{u} \in L^\infty(\mathbb{T})$? **[1]**

2. Let $f \in \mathrm{Hol}(\mathbb{D})$. If there exist a non-negative harmonic function g on $\mathbb{D}$ such that $|f(z)| \leq g(z)$ for all $z \in \mathbb{D}$, then show that $f \in H^1(\mathbb{T})$. **[2]**

3. Show that $\left\{g \in L^\infty(\mathbb{T}) : \int_{\mathbb{T}} gf dm = 0 \text{ for all } f \in H_0^1\right\} = H^\infty$. **[3]**

4. Show that $\dfrac{1}{\lambda - z}$ is an outer function on $\mathbb{D}$ for $|\lambda| > 1$. **[3]**

5. Let $p, q, r \geq 1$. Let $f \in H^p(\mathbb{D})$. For any $g \in H^q$, suppose that $g/f \in H^r$ whenever $g/f \in L^r(\mathbb{T})$. Show that f is outer. **[4]**

6. Let σ be a subset of positive Lebesgue measure in $\mathbb{T}$. Define $f_n = \left[n\chi_\sigma + \frac{1}{n}\chi_{\mathbb{T}\setminus\sigma}\right]$ for $n \geq 2$. Show that $\frac{1}{n} < |f_n(z)| < n$ for $z \in \mathbb{D}$. and $|f_n|(\mathbb{T}) \subset \{\frac{1}{n}, n\}$. **[4]**

7. Let $E = \overline{\mathrm{span}}\{z^m f_k : f_k \in L^2(\mathbb{T}),\, m \geq 0,\, 1 \leq k \leq n\}$. Show that if $zE \neq E$, then $\theta\frac{f_j}{f_k} \in \mathrm{Nev}(\mathbb{D})$ for all j, k, where θ is an inner function. **[4]**

8. If $f \in H^1(\mathbb{C}_+)$ and $f \not\equiv 0$, then show that $\int\limits_{\mathbb{R}} \frac{|\log|f(x)||}{1+x^2} dx < \infty$. **[3]**

9. Let $f \in \mathrm{Hol}(\mathbb{D})$ and $f \not\equiv 0$ and $f = f_1/f_2$, where $f, f_2 \in H^1$. Show that there exist $g_1, g_2 \in H^\infty$ such that $f = g_1/g_2$. **[4]**

10. Show that $H^2(\mathbb{T}) = \overline{\mathrm{span}}_{L^2(\mathbb{T})}\left\{\frac{1}{1-\bar{\lambda}z} : |\lambda| < 1\right\}$. **[3]**

END