# Lecture Notes on Hardy Spaces

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# Contents

## Preface

These notes were prepared for the course MA650 (Jan–May 2022) at IIT Guwahati. Their aim is to introduce Hardy spaces as a meeting point of complex analysis and harmonic analysis, and to develop, in a self-contained way, the structural results that make the theory so useful.

**Prerequisites.** A reader should be comfortable with the basics of complex analysis (holomorphic functions, Cauchy's integral theorem and formula, power series) and real analysis (Lebesgue integration on $\mathbb{R}$, $L^p$ spaces, and elementary Hilbert space theory). When we use a more advanced tool from functional analysis, it is stated explicitly and proved or referenced.

**How the notes are organized.** After preliminaries, we study shift-invariant subspaces of $L^2(\mathbb{T}, \mu)$ and the Beurling-type picture that underlies Hardy spaces. We then develop the canonical (inner–outer) factorization in $H^p(\mathbb{D})$, discuss Szegő-type theorems and the Nevanlinna/Smirnov classes, and finally transfer the theory to the upper half-plane $H^p(\mathbb{C}_+)$, where the Fourier transform and the Cauchy kernel provide a complementary viewpoint.

## Notation and conventions

- $\mathbb{D}$ denotes the unit disk, $\mathbb{T}$ the unit circle, and $m$ the normalized arc-length measure on $\mathbb{T}$.
- For $1 \le p \le \infty$, $L^p(\mathbb{T})$ means $L^p(\mathbb{T}, m)$ unless another measure is specified.
- For $f \in L^1(\mathbb{T})$, the Fourier coefficients are

$$\widehat{f}(n) = \int_{\mathbb{T}} \overline{z}^n f(z)\, dm(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int}\, dt, \qquad n \in \mathbb{Z}.$$

- We use the standard Hardy space notation $H^p(\mathbb{D})$ (analytic functions on $\mathbb{D}$ with $L^p$ boundary control) and $H^p(\mathbb{C}_+)$ for the upper half-plane model.

# Chapter 1

# Introduction

*Hardy spaces form a bridge between complex analysis and harmonic analysis. They encode the boundary behaviour of holomorphic functions on the unit disk and the upper half-plane, and they interact in a precise way with Fourier series, singular integrals, and shift operators. These notes develop the basic structural results (Beurling-type theorems, inner–outer factorization, and canonical factorization) and then use them to study problems of approximation and invariant subspaces.*

**Learning objectives.**

- Understand the definition of $H^p(\mathbb{D})$ through boundary values and Poisson extensions.
- See how Fourier analysis and the shift operator lead naturally to invariant subspaces and inner functions.
- Learn the role of inner–outer and canonical factorization in approximation and extremal problems.

Hardy introduced these spaces in 1915 in the context of power series and boundary growth. Over the subsequent decades, the subject was developed by many authors—notably the Riesz brothers, Szegő, Kolmogorov, Paley–Wiener, and later Beurling, Helson, and others—into a central toolkit of modern analysis. From the viewpoint of this course, the historical remark is mainly a guide: Hardy spaces are useful precisely because they package analytic information (holomorphy) together with quantitative boundary control (an $L^p$ condition).

## 1.1 What is a Hardy space?

For $0 < p \leq \infty$, the Hardy space $H^p(\mathbb{D})$ consists of holomorphic functions $f$ on $\mathbb{D}$ whose boundary values are controlled in $L^p(\mathbb{T})$. One convenient definition is via radial means:

$$\|f\|_{H^p} := \sup_{0<r<1} \Big( \int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta) \Big)^{1/p}, \qquad (0 < p < \infty),$$

with the usual modification for $p = \infty$. A key theorem (Fatou) states that such $f$ have non-tangential boundary limits $f^* \in L^p(\mathbb{T})$ and that $f$ can be recovered from $f^*$ by the Poisson integral. Thus $H^p(\mathbb{D})$ may be viewed as a closed subspace of $L^p(\mathbb{T})$ consisting of functions whose negative Fourier coefficients vanish.

## 1.2 Invariant subspaces and inner functions

On $L^2(\mathbb{T})$, multiplication by $z$ is an isometry (the *shift operator*). The closed subspaces invariant under this shift are governed by Beurling's theorem: every nontrivial closed subspace $E \subset H^2$ with $zE \subset E$ has the form $E = \Theta H^2$, where $\Theta$ is an *inner function* (analytic in $\mathbb{D}$ with unimodular boundary values a.e.). This result is one of the main structural pillars of the subject, and it explains why Hardy spaces are a natural playground for operator theory and functional analysis.

## 1.3 Organization of the notes

We begin with measure-theoretic preliminaries and the basic Fourier-analytic model of $H^2$. We then study shift-invariant subspaces of $L^2(\mathbb{T}, \mu)$ (Wiener, Wold–Kolmogorov, Helson) and derive first applications such as inner–outer factorization and Szegő-type extremal problems. Next we develop canonical factorization in $H^p(\mathbb{D})$, including Blaschke products, singular inner factors, and the Nevanlinna/Smirnov classes. Finally, we transfer the theory to the upper half-plane $H^p(\mathbb{C}_+)$, emphasizing the Fourier transform and the Cauchy kernel as complementary tools. Throughout, exercises and problem sets are included to help consolidate the ideas.

# Chapter 2

# Preliminaries and notation

These notes use standard notation from complex analysis, measure theory, and basic functional analysis. For the reader's convenience we fix conventions that will be used throughout.

## 2.1 The unit circle and normalized Lebesgue measure

We write

$$\mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}, \qquad \mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}.$$

The parametrization $z = e^{it}$, $t \in [0, 2\pi)$ identifies $\mathbb{T}$ with the quotient group $\mathbb{R}/(2\pi\mathbb{Z})$ via the homomorphism $t \mapsto e^{it}$. Accordingly, any function $f : \mathbb{T} \to \mathbb{C}$ may be viewed as a $2\pi$-periodic function on $\mathbb{R}$ by setting $f(t) := f(e^{it})$.

We denote by $m$ the *normalized arc-length measure* on $\mathbb{T}$, i.e.

$$\int_{\mathbb{T}} f\, dm \ = \ \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\, dt, \qquad f \in L^1(\mathbb{T}, m).$$

With this normalization $m(\mathbb{T}) = 1$ and $m$ is translation invariant:

$$\int_0^{2\pi} f(t - t_0)\, dt = \int_0^{2\pi} f(t)\, dt, \qquad t_0 \in [0, 2\pi).$$

## 2.2 Complex Borel measures and total variation

Let $\mathcal{B}(\mathbb{T})$ be the Borel $\sigma$-algebra of $\mathbb{T}$. A (finite) *complex Borel measure* on $\mathbb{T}$ is a countably additive map $\mu : \mathcal{B}(\mathbb{T}) \to \mathbb{C}$ with $\mu(\varnothing) = 0$ and

$$\mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \sum_{j=1}^{\infty} \mu(B_j) \quad \text{for every disjoint family } \{B_j\}_{j \geq 1} \subset \mathcal{B}(\mathbb{T}),$$

where the series is absolutely convergent. The Banach space of all finite complex Borel measures on $\mathbb{T}$ will be denoted by $\mathcal{M}(\mathbb{T})$.

The *total variation* of $\mu \in \mathcal{M}(\mathbb{T})$ is the positive measure $|\mu|$ defined by

$$|\mu|(\mathbb{T}) = \sup\Big\{\sum_{j=1}^{\infty} |\mu(B_j)| : \ \{B_j\}_{j\geq 1} \text{ disjoint and } \bigcup_{j\geq 1} B_j = \mathbb{T}\Big\}.$$

The quantity $\|\mu\| := |\mu|(\mathbb{T})$ is the total variation norm, and $(\mathcal{M}(\mathbb{T}), \|\cdot\|)$ is a Banach space.

**Exercise 2.2.1.** Show that the definition of $|\mu|(\mathbb{T})$ is unchanged if the supremum is taken only over *finite* Borel partitions of $\mathbb{T}$.

Every $\mu \in \mathcal{M}(\mathbb{T})$ defines a bounded linear functional on $C(\mathbb{T})$ by

$$T_\mu(f) := \int_{\mathbb{T}} f\, d\mu, \qquad f \in C(\mathbb{T}),$$

and $\|T_\mu\| = \|\mu\|$. Conversely, every bounded linear functional on $C(\mathbb{T})$ arises this way.

**Theorem 2.2.2** (Riesz representation theorem)**.** *For every bounded linear functional $T$ on $C(\mathbb{T})$ there exists a unique $\mu \in \mathcal{M}(\mathbb{T})$ such that $T(f) = \int_{\mathbb{T}} f\, d\mu$ for all $f \in C(\mathbb{T})$. Equivalently, $\mathcal{M}(\mathbb{T}) \cong C(\mathbb{T})^*$ isometrically.*

## 2.3 The weak-* topology on $\mathcal{M}(\mathbb{T})$

Via Theorem 2.2.2 we identify $\mathcal{M}(\mathbb{T})$ with the dual space $C(\mathbb{T})^*$. The corresponding weak-* topology on $\mathcal{M}(\mathbb{T})$ will be denoted by $w^*$.

A typical $w^*$-neighborhood of $\mu_0 \in \mathcal{M}(\mathbb{T})$ is of the form

$$U(\mu_0; f_1, \ldots, f_N; \varepsilon) := \Big\{\mu \in \mathcal{M}(\mathbb{T}) : \ |\langle \mu - \mu_0, f_k\rangle| < \varepsilon, \ k = 1, \ldots, N\Big\},$$

where $f_1, \ldots, f_N \in C(\mathbb{T})$, $\varepsilon > 0$, and $\langle \mu, f\rangle := \int_{\mathbb{T}} f\, d\mu$.

We record a basic duality fact that will be used repeatedly.

**Proposition 2.3.1.** *Let $E$ be a Banach space. A linear functional $\Phi : (E^*, w^*) \to \mathbb{C}$ is continuous if and only if there exists $x \in E$ such that $\Phi(f) = f(x)$ for all $f \in E^*$. Equivalently,*

$$(E^*, w^*)^* \cong E$$

*via the canonical embedding $x \mapsto (f \mapsto f(x))$.*

*Proof.* If $x \in E$, then $f \mapsto f(x)$ is $w^*$-continuous by definition of the weak-* topology.

Conversely, suppose $\Phi$ is $w^*$-continuous. Continuity at 0 means that there exist $x_1, \ldots, x_N \in E$ and $\varepsilon > 0$ such that

$$\big(|f(x_1)| + \cdots + |f(x_N)| < \varepsilon\big) \implies |\Phi(f)| < 1.$$

In particular, $\Phi(f) = 0$ whenever $f(x_k) = 0$ for all $k$, i.e. $\bigcap_{k=1}^N \ker(\mathrm{ev}_{x_k}) \subseteq \ker(\Phi)$. Therefore $\Phi$ factors through the finite-dimensional quotient $E^*/\bigcap_{k=1}^N \ker(\mathrm{ev}_{x_k})$, and hence can be written as a linear combination of the coordinate functionals $f \mapsto f(x_k)$. That is,

$$\Phi(f) = \sum_{k=1}^{N} c_k f(x_k) = f\Big(\sum_{k=1}^{N} c_k x_k\Big) \quad \text{for some } c_1, \dots, c_N \in \mathbb{C}.$$

Setting $x := \sum_{k=1}^N c_k x_k$ gives the desired representation $\Phi(f) = f(x)$. □

**Corollary 2.3.2.** *The dual of* $(\mathcal{M}(\mathbb{T}), w^*)$ *is canonically isomorphic to* $C(\mathbb{T})$.

# Chapter 3

# Invariant subspaces of $L^2(\mathbb{T}, \mu)$

In this section, consider shift-invariant subspaces of square integrable functions on $\mathbb{T}$. Let

$$L^2(\mathbb{T}, \mu) = \{f : \mathbb{T} \to \mathbb{C} \text{ is measurable and} \|f\|_2^2 = \int_{\mathbb{T}} |f|^2 d\mu < \infty\},$$

where $\mu$ is a finite positive Borel measure on $\mathbb{T}$.

For $f \in L^1(\mathbb{T}, m)$, we define the Fourier coefficients of $f$ by

$$\widehat{f}(n) = \int_{\mathbb{T}} \overline{z}^n f(z)\, dm(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int}\, dt, \qquad n \in \mathbb{Z}.$$

where $n \in \mathbb{Z}$, and the corresponding Fourier series is $f \sim \sum_{n=-\infty}^{\infty} e^{int} \hat{f}(n)$. Consider an operator $S$ on $L^2(\mathbb{T}, m)$ defined by

$$S(f)(z) = zf(z), \tag{3.0.1}$$

where $z \in \mathbb{T}$. Then $\widehat{(Sf)}(n) = \hat{f}(n-1)$. That is, the Fourier coefficients got a right-shift due to the action of $S$. The operator $S$ is known as the shift operator. The following question can be raised.

**Question 3.0.1.** What are the shift-invariant subspaces $E$ of $L^2(\mathbb{T}, \mu)$?

That is, when $zE \subseteq E$? We shall use the notation $\operatorname{clos} E$ for the closure of $E$, and $\bar{E}$, the complex conjugate of $E$. We always consider $E$ to be a closed subspace unless it is specified.

**Example 3.0.2.** When $f \in L^2(\mu)$, the space $E_f = \overline{\operatorname{span}}\{z^n f : n \geq 0\}$ is shift-invariant.

Further, what are $f \in L^2(\mu)$ such that $E_f = L^2(\mu)$? If so, we say $f$ is a cyclic vector. More generally, we consider identifying $f \in L^2(\mu)$ such that $zE_f = E_f$.

Let $E$ be a closed subspace of $L^2$. Typically, we discuss the characterization of the following two distinct cases.

We say $E$ is **simply invariant** (or **1-invariant**) if $zE \subset E$ and $zE \neq E$. On the other hand, when $zE = E$, we say $E$ is **doubly invariant** (or **2-invariant**). Note that $zE = E$ if and only if $\bar{z}E = E$ (since $z\bar{z} = |z|^2 = 1$). This means $zE \subseteq E$ and $\bar{z}E \subseteq E$, and hence $E$ is known as reducing space as well.

For a measurable set $\sigma \subset \mathbb{T}$, the space $E_\sigma = \chi_\sigma L^2(\mu) = \{\chi_\sigma f : f \in L^2(\mu)\} = \{f \in L^2(\mu) : f = 0 \ \mu\text{-a.e. on } \mathbb{T} \setminus \sigma\}$ satisfies $zE_\sigma = E_\sigma$.

**Question 3.0.3.** Does every reducing subspace look like $E_\sigma$?

**Theorem 3.0.4.** (Norbert Wiener) *Let $E \subset L^2(\mathbb{T},\mu)$. Then $zE = E$ if and only if there exists a unique (up to set of measure zero) measurable set $\sigma \subset \mathbb{T}$ such that $E = \chi_\sigma L^2(\mu)$.*

*Proof.* Suppose $zE = E$. Let $P_E$ be the orthogonal projection of $L^2(\mu)$ onto $E$. Set $\chi = P_E 1$ (the evaluation of $P_E$ at the constant function 1). Then $\chi \in E$ and $1 - \chi = (I - P_E)1 \in E^\perp$. But $z^n E \subseteq E$, implies $z^n\chi \in E$ and hence $z^n\chi \perp 1 - \chi$, $\forall\, n \in \mathbb{Z}$. That is,

$$\int_{\mathbb{T}} z^n \chi(1-\bar{\chi})d\mu = 0, \ \forall\, n \in \mathbb{Z}. \tag{3.0.2}$$

Let $g = \chi(1-\bar{\chi})$, then $d\nu = g d\mu$ is a finite complex Borel measure because of $\chi \in L^1(\mu)$. Thus by (3.0.2), $T_\nu : L^2(\mu) \to \mathbb{C}$ defined by $T_\nu(f) = \int_{\mathbb{T}} f d\nu$ satisfies $T_\nu(z^n) = 0$. Since trigonometric polynomials are dense in $C(\mathbb{T})$, it follows that $T_\nu(C(\mathbb{T})) = \{0\}$. By Riesz representation theorem, $T_\nu = 0$ and hence $\nu = 0$. (Note that $\|T_\nu\| = \|\nu\|$). That is, $g = \chi(1-\bar{\chi}) = 0$. This implies that $\chi = |\chi|^2$. Thus, $\chi$ takes values either 0 or 1. Let $\sigma = \{t \in \mathbb{T} : \chi(t) = 1\}$. Then $\sigma$ is measurable. For simplicity, let $\mathbb{P}$ denotes the space of all trigonometric polynomials on $\mathbb{T}$. Since $\chi \in E$, we get $z^n\chi \in E$ and hence $\chi\mathbb{P} \subset E$. This implies $\operatorname{clos}(\chi\mathbb{P}) \subseteq E$. On the other hand, $\operatorname{clos}(\chi\mathbb{P}) = \chi L^2(\mu)$, as we know $\operatorname{clos}\mathbb{P} = L^2(\mu)$. Thus, $\chi L^2(\mu) \subseteq E$. Therefore, it remains to show that $\chi L^2(\mu) = E$.

For this, let $f \in E$ and $f \perp z^n\chi$, $\forall\, n \in \mathbb{Z}$ (since $\operatorname{clos}(\chi\mathbb{P}) = \chi L^2(\mu)$). Since $z^n f \in E$ and $1 - \chi \perp z^n f$, $\forall\, n \in \mathbb{Z}$. It follows that

$$\int_{\mathbb{T}} f\bar{\chi}\bar{z}^n d\mu = \int_{\mathbb{T}} z^n f(1-\bar{\chi})d\mu = 0 \tag{3.0.3}$$

$\forall\, n \in \mathbb{Z}$. Thus, (3.0.3) is satisfied by every polynomial $p \in \mathbb{P}$, and hence for every function $g \in C(\mathbb{T})$ in place of $p$. By Theorem 2.2.2, we get $f\bar{\chi} = f(1-\bar{\chi}) = 0$ a.e. $\mu$. This implies that $f = 0$ a.e. $\mu$. Thus $\chi L^2(\mathbb{T}) = E$. $\square$

## 3.1 Simply invariant subspaces of $L^2(\mu)$

Let $\mathcal{B} = \{z^n\}_{n\in\mathbb{Z}}$. Notice that the Fourier series of $f \in L^2(\mathbb{T}, m)$ with respect to the orthonormal basis $\mathcal{B}$ is $f \sim \sum \hat{f}(n)z^n$, where $\hat{f}(n) = \int_{\mathbb{T}} f\bar{z}^n dm$. This implies that $L^2(\mathbb{T}, m)$ can be identified with $l^2(\mathbb{Z})$. Since $\widehat{(z^k f)}(n) = \hat{f}(n-k)$, multiplication operator $f \mapsto zf$ acts as a right-shift operator on $l^2(\mathbb{Z})$. And hence it is legitimate to consider the space

$$H^2 = \overline{\text{span}}\{z^n : n \geq 0\} = \{f \in L^2(m) : \hat{f}(n) = 0, n < 0\},$$

known as **Hardy space.** The space $H^2$ is a simply invariant subspace of $L^2(m)$, and plays a prominent role in complex and harmonic analysis $H^2$.

The following theorem says that all the simply invariant subspaces have a somewhat similar structure.

**Theorem 3.1.1.** (A. Beurling, H. Helson) *Let $E$ be a closed subspace of $L^2(\mathbb{T})$ and $zE \subset E$, $zE \neq E$. Then there exists a unique $\Theta$ (up to constant of modulus 1) with $|\Theta| = 1$ a.e. $m$ on $\mathbb{T}$ such that $E = \Theta H^2$.*

*Notice that $f \mapsto \Theta f$ is an isometry on $L^2(m)$, and hence $\Theta H^2$ is closed.*

*Proof.* Since $zE \subsetneq E\,(zE \neq E)$, we consider the orthogonal complement of $zE$ in $E$, and denote it by $E \ominus zE = (zE)^{\perp}$. Then $E \ominus zE$ is non-trivial, and consider $\Theta \in E \ominus zE$ with $\|\Theta\|_2 = 1$. Notice that $\Theta \in E$ and $\Theta \perp zE$. Hence $z^n\Theta \in zE$, $\forall\, n \geq 1$ and $\Theta \perp z^n\Theta$, $\forall\, n \geq 1$.

$$\int_0^{2\pi} \bar{\Theta}\Theta z^n dm = \int_0^{2\pi} |\Theta|^2 z^n dm = 0,\ \forall\, n \geq 1.$$

By taking complex conjugate, we have

$$\int_0^{2\pi} |\Theta|^2 \bar{z}^n dm = 0,\ \forall\, n \geq 1.$$

This implies that $\widehat{(|\Theta|^2)}(n) = 0$, $\forall\, n \in \mathbb{Z} \setminus \{0\}$. By the uniqueness of Fourier series, it follows that $|\Theta|^2 = c\,$(constant) a.e. $m$, and we get $1 = \int\limits_0^{2\pi} |\Theta|^2 dm = c$. Thus, $|\Theta| = 1$ a.e. $m$. Clearly, $f \mapsto \Theta f$ is an isometry. Note that $\Theta \in E$. Hence $z^n\Theta \in E$, $\forall\, n \geq 0$, implies linear span of $\{z^n : n \geq 0\}$ has the same property. Let $\mathbb{P}_+ = \text{span}\{z^n : n \geq 0\}$. Then $\Theta\,\mathbb{P}_+ \subset E$ and $\text{clos}\,(\Theta\,\mathbb{P}_+) = \Theta\ \text{clos}(\mathbb{P}_+) = \Theta H^2$. Thus, $\Theta H^2 \subseteq E$. It only remains to show that $\Theta H^2$ coincides with $E$.

Let $f \in E$ and $f \perp \Theta H^2$. We claim that $f = 0$. Since $f \perp \Theta H^2$, we get $f \perp \Theta z^n$, $\forall\, n \geq 0$. Also, $f \in E$ implies $z^n f \in zE$, $\forall\, n \geq 1$ and hence $z^n f \perp \Theta$, $\forall\, n \geq 1$ since $\Theta \perp zE$. Thus,

$$\int_{\mathbb{T}} f\bar{\Theta}\bar{z}^n dm = 0,\ \forall\, n \geq 0 \text{ and } \int_{\mathbb{T}} z^n f\bar{\Theta} dm = 0,\ \forall\, n \geq 1.$$

That is, $\widehat{(f\bar{\Theta})}(n) = 0$, $\forall\, n \in \mathbb{Z}$. This implies $f\bar{\Theta} = 0$ a.e. $m$. Since $|\Theta| = 1$ a.e., we get $f = 0$ a.e. $m$.

Uniqueness: Let $\Theta_1 H^2 = \Theta_2 H^2$ and $|\Theta_1| = |\Theta_2| = 1$ a.e. on $\mathbb{T}$. Then $\Theta_1\bar{\Theta}_2 H^2 = H^2$ and we get $\Theta_1\bar{\Theta}_2 \in H^2$. Also, by symmetry $\Theta_2\bar{\Theta}_1 \in H^2$, or $\Theta_1\bar{\Theta}_2 \in \bar{H}^2$. But $\overline{H^2 \cap \bar{H}^2} =$ constant. (Hint: If $f \in H^2$, then $\hat{f}(n) = 0$, $n < 0$ and $\bar{f} \in H^2$, then $(\widehat{\bar{f}})(n) = \overline{\hat{f}(-n)} = 0$, $n < 0$. This means $\hat{f}(n) = 0, \forall\, n \in \mathbb{Z} \setminus \{0\}$.) Hence $\Theta_1\bar{\Theta}_2 = c$. Since $|\Theta_1||\bar{\Theta}_2| = 1$, we have $\Theta_1 = c\,\bar{\Theta}_2$, where $|c| = 1$. ☐

**Corollary 3.1.2.** (Beurling theorem) *Let $E \neq \{0\}, E \subset H^2$ and $zE \subset E$. Then there exists $\Theta \in H^2$ with $|\Theta| = 1$ a.e. on $\mathbb{T}$ such that $E = \Theta H^2$.*

*Proof.* It is impossible that $\bar{z}E \subset E$. On the contrary, suppose this could be the case. Then for $f \in E$ with $f \neq 0$, there exists $n \in \mathbb{N}$ such that $\hat{f}(n) \neq 0$. By assumption, $\bar{z}^{n+1}f \in E$. However, $(\widehat{\bar{z}^{n+1}f})(-1) = \hat{f}(n) \neq 0$ implies $\bar{z}^{n+1}f \notin H^2$ leads to a contradiction. This means $E$ is simply invariant, and in view of Theorem 3.1.1 (Beurling-Helson), it follows that $E = \Theta H^2$ and $\Theta \in H^2$ by definition of $H^2$. ☐

**Definition 3.1.3.** A function $\Theta \in H^2$, with $|\Theta| = 1$ a.e. is called **inner** function.

## 3.2 Uniqueness theorem in $H^2$

**Theorem 3.2.1.** *If $f \in H^2$ and $f = 0$ on a set of positive measure, then $f = 0$ a.e. on $\mathbb{T}$.*

*Proof.* For $f \neq 0, E_f = \overline{\text{span}}\{z^n f : n \geq 0\} \subset H^2$ and $zE_f \subset E_f = \Theta H^2$, where $\Theta$ is an inner function. Let $\sigma = \{z \in \mathbb{T} : f(z) = 0\}$, Then $m(\sigma) > 0$. Let us verify that $g|_\sigma = 0, \forall\, g \in E_f$. Since $g \in E_f$, there exists sequence $p_n \in \mathbb{P}_+$ (the space of all polynomials) such that $p_n f \to g$ in $L^2(m)$. Hence

$$0 \leq \int_\sigma |g|^2 dm = \int_\sigma |g - p_n f|^2 \leq ||g - p_n f||_2^2 \to 0 \text{ as } n \to \infty.$$

Implies $g|_\sigma = 0$ a.e. $m$. In particular, for $g = \Theta$, $\Theta|_\sigma = 0$, which is a contradiction. ☐

## 3.3 Invariant subspaces of $L^2(\mu)$

(Absolutely continuous and singular subspaces)

Let $\mu$ be a finite Borel measure on $\mathbb{T}$, and $E \subset L^2(\mu)$ with $zE \subset E$. We consider invariant subspaces of $L^2(\mu)$ which are based on Lebesgue decomposition of $\mu$. A measure $\nu$ is called **absolutely continuous** with respect to $m$ if $m(B) = 0$ implies $\nu(B) = 0$, where $B \in \mathcal{B}$ and we write $\nu \ll m$. By Radon-Nikodym theorem, there exists a positive integrable function $w$ such that $d\nu = wdm$. That is,

$$\int_\mathbb{T} f d\nu = \int_\mathbb{T} f w dm$$

for each Borel measurable function $f$ on $\mathbb{T}$.

A measure $\nu$ is called **singular** with respect to $m$ if it is concentrated on a set $C$ of Lebesgue measure zero. That is, $\nu \perp m$ if $\nu(B) = m(B \cap C)$ for every $B \in \mathcal{B}(\mathbb{T})$. Let $\mu$ be a finite and positive Borel measure on $\mathbb{T}$, then by Lebesgue decomposition,

$$\mu = \mu_a + \mu_s, \text{ where } \mu_a \ll m \text{ and } \mu_s \perp m.$$

So, if $f \in L^2(\mu)$, then

$$\int_{\mathbb{T}} |f|^2 d\mu = \int_{\mathbb{T}} |f|^2 d\mu_a + \int_{\mathbb{T}} |f|^2 d\mu_s$$

By this, we can construct an orthogonal decomposition of $f$. Let $\sigma$ be the concentration set for $\mu_s$. Then

$$L^2(\mu_s) \subset L^2(\mu) \text{ and } L^2(\mu_a) \subset L^2(\mu) \text{ and } L^2(\mu_s) \perp L^2(\mu_a). \tag{3.3.1}$$

Now, $f = f\chi_{\mathbb{T}\setminus\sigma} + f\chi_\sigma = f_a + f_s$. This means

$$L^2(\mu) = L^2(\mu_a) \oplus L^2(\mu_s). \tag{3.3.2}$$

The subspaces $L^2(\mu_a)$ and $L^2(\mu_s)$ are invariant subspaces and are known as absolutely continuous and singular spaces, respectively.

We need the following results in order to prove the main result about invariant subspaces of $L^2(\mu)$.

**Lemma 3.3.1.** *Let $\mu$ be a finite complex Borel measure on $\mathbb{T}$.*

*(i) If $\widehat{(d\mu)}(n) = \int\limits_{\mathbb{T}} e^{-int} d\mu(t) = 0$ for all $n \in \mathbb{Z}$, then $\mu = 0$.*

*(ii) If $\widehat{(d\mu)}(n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$, then $d\mu = cdm$.*

*Proof.* (i) Let $f \in C^2(\mathbb{T})$, then $f$ is Borel measurable and we have

$$\begin{aligned} T_\mu(f) &= \int_{\mathbb{T}} f(t) d\mu(t) \\ &= \int_{\mathbb{T}} \Big( \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{int} \Big) d\mu(t) \\ &= \sum_{n\in\mathbb{Z}} \hat{f}(n) \int_{\mathbb{T}} e^{int} d\mu(t) \text{ (by Fubini's Theorem)} \\ &= 0 \text{ (by assumption).} \end{aligned}$$

Hence $T_\mu(f) = 0$ for all $f \in C^2(\mathbb{T})$. Since $C^2(\mathbb{T})$ is dense in $C(\mathbb{T})$, by Theorem 2.2.2, we get $\mu = 0$.

(ii) From the given condition and similar to the proof of case $(i)$, we can write

$$\int_{\mathbb{T}} f(t) d\mu(t) = \hat{f}(0) \int_{\mathbb{T}} d\mu = \mu(\mathbb{T}) \int_{\mathbb{T}} f(t) dt.$$

Thus $d\mu = \mu(\mathbb{T}) dm$, where $dm = dt$. ∎

Let $T : H \to H$ be an isometry (or $T \in \mathrm{iso}(H)$) on the Hilbert space $H$. A subspace $D$ of $H$ is called **wandering** if $T^m D \perp T^n D$ for $m \neq n\,(m, n \geq 0)$.

**Lemma 3.3.2.** (H. Wold, A. Kolmogorov) *Suppose $T \in iso(H)$ and $TE \subset E$. Let $D = E \ominus TE$. Then $D$ is a wandering subspace of $T$, and $E = \Big(\sum_{n\geq 0} \oplus T^n D\Big) \oplus \Big(\bigcap_{n\geq 0} T^n E\Big) = E_0 \oplus E_\infty$, where $T|_{E_\infty}$ is unitary, and $T|_{E_0}$ is completely non-unitary (i.e. if $E' \subset E_0$ and $TE' \subset E'$ implies $T|_{E'}$ is not unitary).*

**Theorem 3.3.3.** (H. Helson 1964) *Let $d\mu = wdm + d\mu_s$ be the Lebesgue decomposition of a positive finite Borel measure $\mu$ and let $E \subset L^2(\mu)$ be simply invariant. Then there exists $\sigma \subseteq \mathbb{T}$ with $m(\sigma) = 0$ and a measurable function $\Theta$ such that*

$$E = E_0 \oplus E_\infty = \Theta H^2 \oplus \chi_\sigma L^2(\mu_s), \text{ where}$$

$$\Theta H^2 \subset L^2(\mu_a),\ \chi_\sigma L^2(\mu_s) \subset L^2(\mu_s) \text{ and}$$

$$|\Theta|^2 w \equiv 1. \tag{3.3.3}$$

*Conversely, if $\sigma$ is measurable and $\Theta$ verified (3.3.3), then $\Theta H^2 \oplus \chi_\sigma L^2(\mu_s)$ is simply invariant.*

*Proof.* Set $D = E \ominus zE = (zE)^\perp \neq \{0\}$ and let $E = \Big(\sum_{n\geq 0} z^n D\Big) \oplus \Big(\bigcap_{n\geq 0} z^n E\Big) = E_0 \oplus E_\infty$ be the Wold-Kolmogorov decomposition of $E$. Let $\Theta \in D$ with $\|\Theta\|_2 = 1$, then $\Theta \in E$ and $\Theta \perp zE$. This implies $z^n\Theta \in zE$, $\forall\, n \geq 1$, and hence $z^n\Theta \perp \Theta\,\forall\, n \geq 1$. That is,

$$\int_{\mathbb{T}} (z^n\Theta)\bar{\Theta}d\mu = \int_{\mathbb{T}} |\Theta|^2 z^n d\mu = 0,\ \forall\, n \geq 1.$$

And by conjugation

$$\int_{\mathbb{T}} |\Theta|^2 \bar{z}^n d\mu = 0,\ \forall\, n \geq 1.$$

Thus $(\widehat{|\Theta|^2 d\mu})(n) = 0$, $\forall\, n \in \mathbb{Z} \setminus \{0\}$. By Lemma 3.3.1 (ii), we get $|\Theta|^2 d\mu = cdm$. But, $1 = \int_{\mathbb{T}} |\Theta|^2 d\mu = c\int_{\mathbb{T}} dm = c$. Thus,

$$\begin{aligned} dm &= |\Theta|^2 d\mu \\ &= |\Theta|^2 d\mu_a + |\Theta|^2 d\mu_s \\ &= |\Theta|^2 wdm + |\Theta|^2 d\mu_s. \end{aligned} \tag{3.3.4}$$

Implies $|\Theta|^2 = 0$ a.e. $\mu_s$ on $\mathbb{T}$ (because $m$ has no singular part) and $dm = |\Theta|^2 wdm$ implies $|\Theta|^2 w = 1$ a.e. $m$. By Wold-Kolmogorov Lemma 3.3.2, restriction $z|_{E_\infty}$ is unitary, $zE_\infty \subseteq E = E_\infty \oplus E_0$, and $z|_{E_0}$ is non-unitary on every section of $E_0$, etc. Thus, we conclude that $zE_\infty = E_\infty$. By Wiener theorem, $E_\infty = \chi_\sigma L^2(\mu)$ for some $\sigma \subset \mathbb{T}$. As $\Theta \in D \subset E_0 \perp E_\infty$, implies $\Theta \perp \chi_\sigma L^2(\mu)$. In particular, this implies $\int_\sigma \Theta\bar{\Theta}d\mu = \int_\sigma |\Theta|^2 d\mu = 0$. Hence $\Theta|_\sigma = 0$ a.e. $\mu$. But

$\Theta \neq 0$ a.e. $m$ implies $m(\sigma) = 0$ (since $dm = |\Theta|^2 d\mu$). Thus, in view of (3.3.2) we obtain

$$E_\infty = \chi_\sigma L^2(\mu) = \chi_\sigma L^2(\mu_s) \subset L^2(\mu_s).$$

We have already shown that $D \subset L^2(\mu_a)$, because $D \subset E_0 \perp E_\infty = L^2(\mu_s)$ implies $D \subset L^2(\mu_a)$. Therefore, $E_0 = \sum_{n\geq 0} \oplus z^n D \subset L^2(\mu_a)$. Also, $\overline{\text{span}}\{z^n\Theta : n \geq 0\} \subset E_0$, since $\Theta \in E_0$. We claim that $E_0 = \overline{\text{span}}\{z^n\Theta : n \geq 0\}$.

On the contrary, suppose there exists $f \in E_0 \ominus \overline{\text{span}}\{z^n\Theta : n \geq 0\}$. Then $f \perp z^n\Theta$, $\forall\, n \geq 0$. Recall that $\Theta \perp zE$. But $f \in E$, implies $z^n f \in E$ and hence $z^n f \perp \Theta$, $\forall\, n \geq 1$. Thus,

$$\int f\overline{z^n\Theta}d\mu = 0\,\forall\, n \geq 0 \;\text{ and }\; \int z^n f\bar{\Theta}d\mu = 0,\,\forall\, n \geq 1.$$

That is $(\widehat{f\bar{\Theta}d\mu})(n) = 0\,\forall\, n \in \mathbb{Z}$. By Lemma 3.3.1(i), it implies that $f\bar{\Theta}d\mu = 0$. Since $\bar{\Theta} \neq 0$ a.e. $m$ and $f \in E_0 \subset L^2(\mu_a)$, it follows that $f \equiv 0$. Now, by Parseval identity, it is easy to verify that

$$\overline{\text{span}}\{z^n\Theta : n \geq 0\} = \Big\{ \sum_{n\geq 0} a_n z^n \Theta : \sum_{n\geq 0} |a_n|^2 < \infty \Big\}.$$

(Notice that $\{z^n\Theta\}_{n\geq 0}$ is an orthonormal set in $L^2(\mu_a)$, since $d\mu_a = w\,dm$ and $|\Theta|^2 w \equiv 1$.) Further, it is easy to see that

$$E_0 = \Theta\Big\{ \sum_{n\geq 0} a_n z^n : \sum_{n\geq 0} |a_n|^2 < \infty \Big\} = \Theta H^2.$$

Indeed, $f \mapsto \Theta f$ is an isometry from $L^2(\mathbb{T}, dm)$ onto $L^2(d\mu_a) = L^2(wdm)$. That is,

$$\int_{\mathbb{T}} |f|^2 dm = \int_{\mathbb{T}} |\Theta f|^2 d\mu_a.$$

□

# Chapter 4

# First Applications

We have seen that there is one to one correspondence between simply invariant subspace of $L^2(\mu)$ with the set of measurable unimodular functions (inner functions) due to Helson's theorem. This congruence opens many possibilities to apply Hilbert space geometry and operator theory to $L^2(\mu)$ and vice-versa. Here we discuss inner-outer decomposition of the Hardy class functions, Szegö infimum, and Riesz brother's theorem for "analytic measure". That is, for which positive measure $\mu$ on $\mathbb{T}$, the "analytic half" $\mathbb{P}_+ = \text{span}\{z^n : n \geq 0\}$ is dense in $L^2(\mathbb{T}, \mu)$.

## 4.1 Some consequences of Helson's theorem

Let $\mu$ be a positive Borel measure on $\mathbb{T}$ with $d\mu = wdm + d\mu_s$. Notice that if $zE \subset E \subset L^2(\mu)$, then $E = E_a \oplus E_s$, where $zE_a \subset E_a \subset L^2(\mu_a)$, because $E = \Theta H^2 \oplus \chi_\sigma L^2(\mu_s)$, where $\Theta H^2 \subset L^2(\mu_a)$ and $\chi_\sigma L^2(\mu_s) \subset L^2(\mu_s)$.

**(a)** If $\mu = \mu_s$, then $zE \subset E \subset L^2(\mu_s)$, implies $zE = E$, because, by Helson's theorem 3.3.3, we already have $E = \chi_\sigma L^2(\mu_s)$, which is 2-invariant.

**(b)** Show that for $d\mu = d\mu_a = w\, dm$, the followings are equivalent:

(i) There exists $E$ such that $zE \subsetneq E \subset L^2(\mu_a)$.

(ii) There exists $\Theta$ such that $|\Theta|^2 w = 1$ a.e. $m$.

(iii) $w > 0$ almost everywhere $m$.

(iv) $m$ is absolutely continuous with respect to $\mu_a$.

**(c)** If $d\mu = d\mu_a = w\, dm$ and $zE \subsetneq E \subset L^2(\mu_a)$, then $E = \Theta H^2$ with $|\Theta|^2 w \equiv 1$ a.e. $m$.

## 4.2 Reducing subspaces

Let $f \in L^2(\mu)$ and $d\mu = wdm + d\mu_s$. We look for sufficient conditions that ensure that $E_f$ is reducing. If there exists measurable set $e \subset \mathbb{T}$ such that $m(e) > 0$ and $f|_e = 0$. Then

$E_f$ is a reducing subspace, and there exists $\sigma \subset \mathbb{T} \setminus e$ such that $E_f = \chi_\sigma L^2(\mu)$. In fact, $\sigma = \{z \in \mathbb{T} : f(z) \neq 0\}$. On the contrary, suppose $zE_f \subsetneq E_f$. Then by Theorem 3.3.3 we get $E_f = \Theta H^2 \oplus \chi L^2(\mu_s)$, and hence $f \in E_f$ implies $f = f_a + f_s$, where $f_a = \Theta h$, $h \neq 0$ a.e. $m$ (by Theorem 3.2.1, since $h \in H^2$). This implies $f_a \neq 0$ a.e. $m$, which is impossible because $f|_e = 0$ and $m(e) > 0$ implies $f_a|_e = 0$ with $m(e) > 0$. Thus, $E_f = zE_f = \chi_\sigma L^2(\mu)$ for $\sigma \subset \mathbb{T}$ (by Wiener theorem). Notice that $E_f = \overline{\text{span}}\{z^n \chi_{\mathbb{T}\setminus e} f : n \geq 0\} = \chi_{\mathbb{T}\setminus e} E_f = \chi_\sigma L^2(\mu)$ and $1 \in L^2(\mu)$, implies $\sigma \subset \mathbb{T} \setminus e$. Indeed $\sigma = \{z \in \mathbb{T} : f(z) \neq 0\}$, which is defined up to a set of $\mu$ measure zero.

## 4.3 The problem of weighted polynomial approximation

We know that the space of trigonometric polynomials $\mathbb{P} = \text{span}\{z^n : n \in \mathbb{Z}\}$ is dense in $L^p(\mu)$ for every positive and finite measure $\mu$ and $1 \leq p < \infty$. Let $\mathbb{P}_+ = \text{span}\{z^n : n \geq 0\}$. One of the main problems is describing the closure of $\mathbb{P}_+$ in $L^2(\mu)$. Denote $H^2(\mu) = \text{clos}\, \mathbb{P}_+|_{L^2(\mu)}$. The most important part of this problem is to distinguish between the completeness case $H^2(\mu) = L^2(\mu)$, from the incompleteness case $H^2(\mu) \subsetneq L^2(\mu)$.

**Corollary 4.3.1.** $H^2(\mu) = H^2(\mu_a) \oplus L^2(\mu_s)$.

*Proof.* $H^2(\mu) = \overline{\text{span}}\{z^n : n \geq 0\}$. By Helson decomposition $H^2(\mu) = E_a \oplus E_s$ with $E_a \subset L^2(\mu_a)$ and $E_s \subset L^2(\mu_s)$. Since we know that $zE_s = E_s$, by Wiener theorem, $E_s = \chi_\sigma L^2(\mu_s)$ with $m(\sigma) = 0$. Since $1 \in H^2(\mu)$, we have $1 = 1_a + 1_s$ with $1_s \neq 0$ a.e. $\mu_s$. But $1_s \in E_s = \chi_\sigma L^2(\mu_s)$ implies $\chi_\sigma L^2(\mu_s) = L^2(\mu_s)$, i.e., $E_s = L^2(\mu_s)$.

Further, $(\mathbb{P}_+)_a \subset E_a$ implies $\text{clos}\,(\mathbb{P}_+)_a = H^2(\mu_a) \subseteq E_a$. But, for $f \in E_a \subset H^2(\mu)$ implies there exists $p_n \in \mathbb{P}_+$ such that $\|f - p_n\|_{L^2(\mu)} \to 0$. Since $\|f - p_n\|^2_{L^2(\mu)} = \|f - p_n\|^2_{L^2(\mu_a)} + \|f - p_n\|^2_{L^2(\mu_s)} = \|f - p_n\|^2_{L^2(\mu_a)} + \|p_n\|^2_{L^2(\mu_s)}$ (since $f = 0$ $\mu_s$-a.e.) and $\|f - p_n\|^2_{L^2(\mu_a)} \leq \|f - p_n\|^2_{L^2(\mu_a)} + \|p_n\|^2_{L^2(\mu_s)} = \|f - p_n\|^2_{L^2(\mu)} \to 0$ we get $f \in H^2(\mu_a)$.

□

*Remark* 4.3.2. Note that for $H^2(\mu_a)$, the closure of $\mathbb{P}_+$ in $L^2(\mu_a)$ has two possibilities:

(i) $zH^2(\mu_a) = H^2(\mu_a)$ and hence by Wiener theorem $H^2(\mu_a) = \chi_\sigma L^2(\mu_a) = L^2(\mu_a)$, because $1_a \in H^2(\mu_a)$ implies that there does not exist $\sigma \subset \mathbb{T}$ such that $m(\mathbb{T} \setminus \sigma) > 0$.

(ii) $zH^2(\mu_a) \subsetneq H^2(\mu_a)(\subset L^2(\mu_a))$, and hence $H^2(\mu_a) = \Theta H^2$ with $|\Theta|^2 w \equiv 1$.

The following results help to distinguish the above two cases.

**Lemma 4.3.3.** *$H^2(\mu)$ is reducing (and hence $H^2(\mu) = L^2(\mu)$) if and only if $\bar{z} \in H^2(\mu)$.*

*Proof.* If $H^2(\mu)$ is reducing, then $\bar{z} \in H^2(\mu)$ is trivial. Suppose $\bar{z} \in H^2(\mu)$, then exists $p_n \in \mathbb{P}_+$ such that $\|\bar{z} - p_n\|_{L^2(\mu)} \to 0$. Let $q \in \mathbb{P}_+$. Then

$$\int_{\mathbb{T}} |\bar{z}q - qp_n|^2 d\mu \leq \|q\|^2_\infty \int_{\mathbb{T}} |\bar{z} - p_n|^2 \to 0 \text{ as } n \to \infty.$$

This implies $\bar{z}\,\mathbb{P}_+ \subset H^2(\mu)$, or $\mathbb{P}_+ \subset zH^2(\mu)$ (closed). Hence $H^2(\mu) \subseteq zH^2(\mu)$, i.e. $\bar{z}H^2(\mu) \subseteq H^2(\mu)$. But $zH^2(\mu) \subset H^2(\mu)$ implies $zH^2(\mu) = H^2(\mu)$. Now, it is clear from Wiener theorem and theorem 3.2.1 that $H^2(\mu) = \chi_\sigma L^2(\mu) = L^2(\mu)$. □

**Corollary 4.3.4.** $H^2(\mu) = L^2(\mu)$ *if and only if* $\mathrm{dist}(1, H_0^2(\mu)) = 0$, *where* $H_0^2(\mu)$ *is the closure of* $\mathrm{span}\{z^n : n \geq 1\}$ *in* $L^2(\mu)$.

*Proof.* Let $H^2(\mu) = L^2(\mu)$, then $\bar{z} \in H^2(\mu)$, implies $\mathrm{dist}\,(1, H_0^2(\mu)) = \mathrm{dist}\,(\bar{z}, H^2(\mu)) = 0$. On the other hand, if $\mathrm{dist}\,(1, H_0^2(\mu)) = 0$, then $\bar{z} \in H^2(\mu)$, and hence $H^2(\mu) = L^2(\mu)$. □

Note that the quantity

$$dist^2\,(1, H_0^2(\mu)) = \inf_{p \in \mathbb{P}_+^0} \int_{\mathbb{T}} |1-p|^2 d\mu$$

is known **Szegö infimum**, where $\mathbb{P}_+^0 = \mathrm{span}\{z^n : n \geq 1\}$.

It can be seen that $\mathrm{dist}(1, H_0^2(\mu))$ depends only on the absolute part of the measure $\mu$. Let $d\mu = wdm + d\mu_s$ be the lebesgue decomposition of $\mu$. As similar to Corollary 4.3.1, it can be seen that $H_0^2(\mu) = H_0^2(\mu_a) \oplus L^2(\mu_s)$. We also use the fact that if $M_1$ and $M_2$ are subspaces of a Hilbert space $H$ such that $M_1 \perp M_2$, then $P_{M_1 \oplus M_2} = P_{M_1} + P_{M_2}$ for $M_1 \perp M_2$. Thus, we can write

$$\begin{aligned} dist^2(1, H_0^2(\mu)) &= \|P_{H_0^2(\mu)} \perp 1\|_{L^2(\mu)}^2 \\ &= \|(P_{H_0^2(\mu_a)} \oplus P_{L^2(\mu_s)}) \perp (1_a + 1_s)\|_{L^2(\mu)} \\ &= \|P_{H_0^2(\mu_a)} \perp 1_a\|_{L^2(\mu_a)}^2 (\text{ since } 1_s \in L^2(\mu_s)) \\ &= \inf_{p \in \mathbb{P}_+^0} \int_{\mathbb{T}} |1-p|^2 wdm. \end{aligned}$$

The evaluation of Szegö infimum is intimately related to the multiplicative structure of $H^2$.

## 4.4 The inner-outer factorization

Recall that a function $f \in H^2$ is called **inner** if $|f| = 1$ a.e. on $\mathbb{T}$. On the other hand, $f \in H^2$ is called **outer** if $E_f = H^2$.

**Theorem 4.4.1.** (V. Smirnov, 1928) *Let* $f \in H^2$ *and* $f \not\equiv 0$, *then there exists an inner function* $f_{inn} \in H^2$ *and an outer function* $f_{out} \in H^2$ *such that* $f = f_{inn} f_{out}$. *Moreover, this factorization is unique and* $E_f = f_{inn} H^2$.

*Proof.* Note that $E_f \subset H^2$, $E_f \neq \{0\}$, and $E_f$ is not reducing, else $\bar{z} \in H^2$. Here, $E_f = \overline{\mathrm{span}}\{z^n f : n \geq 0\} \subset H^2$. By Theorem 3.1.1, we have $E_f = \Theta H^2$, where $|\Theta| = 1$ a.e. $m$. Let

$f_{inn} = \Theta$, then $f = \Theta g$, where $g \in H^2$. We claim $E_g = H^2$. Let $h \in H^2$. Since $E_f = \Theta H^2$ and $\Theta h \in \Theta H^2$, there exists $p_n \in \mathbb{P}_+$ such that $p_n \Theta g = p_n f \to \Theta h$ in $L^2$. But, multiplication by an inner function is an isometry, we get

$$\|p_n g - h\|_2 = \|\Theta(p_n g - h)\|_2 \to 0.$$

Hence, $E_g = H^2$. Here $g = f_{out}$ is desired outer function.

Uniqueness: Take $f = f_1 f_2$, where $f_1$ is inner and $f_2$ is outer. As $f_1$ is inner, $h \mapsto f_1 h$ is an isometry, and hence as $E_{f_2} = H^2$, we get

$$f_{inn} H^2 = E_f = \overline{\text{span}}\{z^n f_1 f_2 : n \geq 0\} = f_1 \overline{\text{span}}\{z^n f_2 : n \geq 0\} = f_1 H^2.$$

By the uniqueness of the representing inner function of the simply invariant space $E_f$ (cf. Theorem 3.1.1 and Corollary 3.1.2), we get $f_{inn} = \lambda f_1$ with $|\lambda| = 1$, and $\lambda f_1 f_{out} = f_1 f_2$ implies $f_{out} = \bar{\lambda} f_2$. □

## 4.5 Arithmetic of inner functions

**Definition 4.5.1.** Let $\Theta_1, \Theta_2$ be two inner functions in $H^2$. We say $\Theta_1$ divides $\Theta_2$ if $\frac{\Theta_2}{\Theta_1} \in H^2$.

Equivalently, $\Theta_1$ divides $\Theta_2$ if and only if $\Theta_1 H^2 \supset \Theta_2 H^2$. For this, if $\Theta_2 = \Theta\Theta_1$, then $\Theta$ is necessarily inner, and $\Theta_2 H^2 = \Theta_1 \Theta H^2 \subset \Theta_1 H^2$, since $\Theta H^2 \subset H^2$. On the other hand, if $\Theta_1 H^2 \supset \Theta_2 H^2$, then we get $\Theta_2 \in \Theta_1 H^2$ implies $\Theta = \frac{\Theta_2}{\Theta_1} \in H^2$.

We deduce the following two elementary properties:

**Theorem 4.5.2.** *Let* $\Theta = \gcd\{\Theta_1, \Theta_2\}$, *the greatest common divisor of* $\Theta_1$ *and* $\Theta_2$. *Then*

*(i) span* $\{\Theta_1 H^2, \Theta_2 H^2\} = \Theta H^2$

*(ii)* $\Theta_1 H^2 \cap \Theta_2 H^2 = \tilde{\Theta} H^2$, *where* $\tilde{\Theta} = \text{lcm}\{\Theta_1, \Theta_2\}$.

*Proof.* (i) $\Theta_k H^2 \subset \text{span}\{\Theta_1 H^2, \Theta_2 H^2\} = \Theta H^2\,;\ k = 1, 2$ for some inner function $\Theta$ (by Beurling's theorem) implies $\Theta$ divides $\Theta_k\,;\ k = 1, 2$. Let $\Theta'$ be another divisor of $\Theta_k : k = 1, 2$. Then $\Theta' H^2 \supset \Theta_k H^2$, and hence $\Theta' H^2 \supset \text{span}\{\Theta_k H^2; k = 1, 2\} = \Theta H^2$. This implies $\Theta'$ divides $\Theta$ and thus $\Theta = \gcd\{\Theta_k;\, k = 1, 2\}$. The proof of (ii) is similar to (i). □

**Definition 4.5.3.** Let $\{\Theta_i : i \in I\}$ be a family of inner functions.

(i) $\Theta = \gcd\{\Theta_i : i \in I\}$ if $\Theta$ divides each $\Theta_i$, and $\Theta$ is divisible by every other inner function that divides each $\Theta_i$.

(ii) $\Theta = \text{lcm}\{\Theta_i : i \in I\}$ if each $\Theta_i$ divides $\Theta$ and $\Theta$ divides every other inner function that is divisible by each $\Theta_i$

Convention: In case the gcd or the lcm does not exist, we write $\gcd\{\Theta_i : i \in I\} = 1$ and $\operatorname{lcm}\{\Theta_i : i \in I\} = 0$.

**Corollary 4.5.4.** $\operatorname{span}\{\Theta_i \in H^2 : i \in I\} = \Theta H^2$, *where* $\Theta = \gcd\{\Theta_i : i \in I\}$ *and* $\cap\Theta_i H^2 = \tilde{\Theta}H^2$, *where* $\tilde{\Theta} = \operatorname{lcm}\{\Theta_i : i \in I\}$.

**Corollary 4.5.5.** *Let $F$ be a proper subset of $H^2$. Then* $\overline{\operatorname{span}}\{z^n F : n \geq 0\} = \Theta H^2$, *where* $\Theta = \gcd\{f_{inn} : f \in F \setminus \{0\}\}$, *and $f_{inn}$ stands for inner factor of $f$.*

*Proof.* We have $\overline{\operatorname{span}}\{z^n F : n \geq 0\} = \overline{\operatorname{span}}\{f_{inn}H^2 : f \in F \setminus \{0\}\}$. (By Smirnov's theorem). By applying Corollary 4.5.4 we get the required. □

## 4.6 Characterization of outer functions

**Theorem 4.6.1.** *(Integral Maximum Principle) Let $f \in H^2$. Then the followings are equivalent:*

*(i) $f$ is outer*

*(ii) $f$ is a divisor of the space $H^2$, i.e. if $g \in H^2$ and $\frac{g}{f} \in L^2$, then $\frac{g}{f} \in H^2$.*

*Proof.* (ii) $\Longrightarrow$ (i): Let $f = f_{inn}f_{out}$ be an inner-outer factorization of $f$. Then $\bar{f}_{inn} = \frac{1}{f_{inn}} = \frac{f_{out}}{f} \in L^2$ because of $f_{inn} \in H^2 \subset L^2$. By (ii), we get $\bar{f}_{inn} \in H^2$. But $f_{inn} \in H^2$ implies $\bar{f}_{inn} = \lambda\,(\text{constant})$ with $|\lambda| = 1$. Hence $f = \bar{\lambda} f_{out}$.
(i) $\Longrightarrow$ (ii): Given $f$ is outer, we have $E_f = H^2$. Since $1 \in H^2$, there exists $p_n \in \mathbb{P}_+$ such that $p_n f \to 1$ in $L^2$. Let $g \in H^2$ and $h = \frac{g}{f} \in L^2$. Then

$$\int_{\mathbb{T}} |p_n g - h| = \int_{\mathbb{T}} |p_n f - 1||h| \leq \|p_n f - 1\|_2 \|h\|_2 \to 0 \text{ as } n \to \infty. \tag{4.6.1}$$

But $p_n g \in H^2$, implies $\widehat{(p_n g)}(k) = 0$ if $k < 0$. Since $\varphi \mapsto \hat{\varphi}(k)$ is continuous linear functional on $L^1(\mathbb{T})$ for each $k$, by (4.6.1) we get $(\hat{h})(k) = 0$, $\forall\, k < 0$. Thus $h \in H^2$. □

**Corollary 4.6.2.** *If two outer functions $f_1$ and $f_2$ verify $|f_1| = |f_2|$ a.e. on $\mathbb{T}$, then $f_1 = \lambda f_2$ where $|\lambda| = 1$.*

*Proof.* Since $f_2$ is outer, $f_1 \in H^2$, and $|\frac{f_1}{f_2}| = 1 \in L^2$, by Theorem 4.6.1, we get $\frac{f_1}{f_2} \in H^2$. In the similar way $\overline{\frac{f_1}{f_2}} = \frac{f_2}{f_1} \in H^2$ implies $\frac{f_1}{f_2} = \lambda\,(\text{constant})$ and hence $f_1 = \lambda f_2$ with $|\lambda| = 1$. Thus, an outer function is completely defined by its modulus. □

**Corollary 4.6.3.** *Let $w \geq 0$, $w \in L^1(\mathbb{T})$. If there exists $f \in H^2$ such that $|f|^2 = w$ a.e. $\mathbb{T}$, then there exists a unique outer function $f_0 \in H^2$ such that $|f_0|^2 = w$ a.e. $\mathbb{T}$.*

(Hint: By Smirnov theorem, $f = f_{inn}f_{out}$ etc.)

**Corollary 4.6.4.** *If $f \in H^2(\mathbb{T})$ is simultaneously inner and outer then $f$ is constant.*

*Proof.* Since $f \in H^2(\mathbb{T})$ is inner $|f| = 1$ and hence $1/f = \overline{f} \in H^2(\mathbb{T})$ by the Theorem 4.6.1. Since $f, \overline{f} \in H^2(\mathbb{T})$ hence $f$ is constant. □

## 4.7 Szegö infimum and Riesz Brother's theorem

Here we consider two theorems in two different settings by using the fact that in an orthogonal complement of the analytic polynomials $\mathbb{P}_+$ the absolute component of a measure is only important.

**Theorem 4.7.1.** (Szegö and Kolmogorov) *Let $\mu$ be a finite Borel measure on $\mathbb{T}$ with Lebesgue decomposition $d\mu = wdm + d\mu_s$, where $w \in L^1_+(\mathbb{T})$. Then*

*(i) either there does not exist any $f \in H^2$ such that $|f|^2 = w$ a.e. $m$, then*

$$\inf_{p\in\mathbb{P}^0_+} \int_{\mathbb{T}} |1-p|^2 d\mu = 0.$$

*(ii) or there exists (unique) $f \in H^2$ such that $|f|^2 = w$ a.e. $m$, and $f$ is outer, then*

$$\inf_{p\in\mathbb{P}^0_+} \int_{\mathbb{T}} |1-p|^2 d\mu = |\hat{f}(0)|^2.$$

*Proof.* (ii)We know that the Szegö infimum $I$ will satisfy

$$\begin{aligned} I^2 = \operatorname{dist}^2(1, H^2_0(\mu)) &= \operatorname{dist}^2(1, H^2_0(\mu_a)) \\ &= \inf_{p\in\mathbb{P}^0_+} \int_{\mathbb{T}} |1-p|^2 wdm. \end{aligned}$$

Given that $|f|^2 = w$ a.e. $m$, and $f$ is outer. Hence

$$I^2 = \inf_{p\in\mathbb{P}^0_+} \int_{\mathbb{T}} |f - pf|^2 dm.$$

As $f$ is an outer function, we can verify that $\overline{\operatorname{span}}\{z^n f : n \geq 1\} = zH^2$. Hence $I = \operatorname{dist}_{H^2}(f, zH^2)$. Note that $f = \sum_{n\geq 0} \hat{f}(n)z^n = \hat{f}(0) + g$, where $g \in zH^2$. Since $\hat{f}(0) \perp zH^2$, it follows that $I = \operatorname{dist}_{H^2}(\hat{f}(0), zH^2) = |\hat{f}(0)|$.

(i). Now, we consider the invariant space $E_a = H^2_0(\mu_a)$. If $zE_a \neq E_a$, then there exists $\Theta$ such that $E_a = \Theta H^2$ with $|\Theta|^2 w \equiv 1$. But $z \in E_a$ and hence $z = \Theta f$ for some $f \in H^2$. This implies that $|f|^2 = \frac{1}{|\Theta|^2} = w$ (since $|z| = 1$), and this leads to case (ii). Hence, case (i) is possible only if $zE_a = E_a$. But, then $E_a = L^2(\mu_a)$ by Remark 4.3.2(i). Hence $\operatorname{dist}(1, H^2_0(\mu)) = 0$, since $1 \in L^2(\mu_a) = H^2_0(\mu_a)$. □

The above Theorem (Szegö and Kolmogorov) leads to the problem of computing $|\hat{f}(0)|^2$ in terms of $w$. In order to do this, we have to consider $H^2$ as a space of analytic functions on the unit disc, which we do later.

Riesz Brother's result is an important consequence of Helson's theorem. For that, we need to recall an important result related to the Radon-Nikodym derivative.

Let $|\mu|$ be the total variation measure of a complex-valued Borel measure $\mu$ on $\mathbb{T}$, i.e.

$$|\mu|(\sigma) = \sup\Big\{ \sum_{i\in I} |\mu(\sigma_i)| : \{\sigma_i\}_{i\in I} \text{ is a partition of } \sigma \text{ in } \mathcal{B}(\mathbb{T})\Big\}.$$

Suppose $\mu$ is absolutely continuous with respect to a positive measure $\lambda$ on $\mathcal{B}(\mathbb{T})$. Then there exists $\varphi \in L^1(\lambda)$ (the Radon-Nikodym derivative of $\mu$ with respect to $\lambda$) such that

$$|\mu|(\sigma) = \int_\sigma |\varphi| d\lambda.$$

**Theorem 4.7.2.** (Riesz Brother's, 1916) *Let $\mu$ be a complex-valued Borel measure on $\mathbb{T}$ such that*

$$\int_{\mathbb{T}} z^n d\mu = 0, \, \forall\, n \geq 1.$$

*Then $\mu \ll m$ and $d\mu = h\, dm$, where $h \in H^1 = \{f \in L^1(\mathbb{T}) : \hat{f}(k) = 0, \, k < 0\}$.*

Note that, a measure $\mu$ that satisfies $\int_{\mathbb{T}} \bar{z}^n d\mu = 0$ for $n < 0$ will be called **analytic measure.**

*Proof.* It is clear that $\mu \ll |\mu|$. Let $g \in L^1(|\mu|)$ be the corresponding Radon-Nikodym derivative of $\mu$ with respect to $|\mu|$. We claim that $|g| = 1$ a.e. $\mu$. For $\delta > 0$, set $\sigma = \{t : |g(t)| < 1 - \delta\}$. Then $|\mu|(\sigma) = \int_\sigma |g| d|\mu| \leq (1-\delta)|\mu|(\sigma)$. Implies $|\mu|(\sigma) = 0$. Similarly, the case $\sigma' = \{t : |g(t)| > 1 - \delta\}$. This proves the claim. As a consequence of the Corollary 4.3.1, we get

$$H_0^2(|\mu|) = H^2(|\mu|_a) \oplus L^2(|\mu|_s). \tag{4.7.1}$$

But $|g| = 1$ a.e. $|\mu|$ implies $\bar{g} \in L^2(|\mu|)$, and

$$\langle z^n, \bar{g}\rangle_{L^2(|\mu|)} = \int_{\mathbb{T}} z^n g d|\mu| = \int_{\mathbb{T}} z^n d\mu = 0, \, n \geq 1.$$

In other words, $\bar{g} \perp z^n$, $n \geq 1$ in the Hilbert space $L^2(|\mu|)$, and hence $\bar{g} \perp H_0^2(|\mu|)$. In view of (4.7.1), we obtain $\bar{g} \perp H_0^2(|\mu|_s)$. Now, by construction, $|g| = 1$ a.e. $|\mu|$, which implies $|g| = 1$ a.e. $|\mu|_s$. This is impossible ( since $\bar{g} \perp H_0^2(|\mu|_s)$ ), unless $|\mu|_s = 0$. Finally, $\mu \ll |\mu|$ implies

$$\mu(\sigma) = \int_\sigma g d|\mu| = \int_\sigma g d|\mu|_a = \int_\sigma g w dm$$

for each $\sigma \in \mathcal{B}(\mathbb{T})$. That is $\mu \ll m$ with Radon-Nikodym derivative $h = gw \in L^1(\mathbb{T})$, and

$$\hat{h}(k) = \int_{\mathbb{T}} \bar{z}^k h dm = \int_{\mathbb{T}} \bar{z}^k gw dm = \int_{\mathbb{T}} \bar{z}^k d\mu = 0 \text{ if } k \leq -1.$$

Hence $h \in H^1$. □

**Question 4.7.3.** *

For $g \in L^1(\mathbb{T})$, define $g_f = \overline{\text{span}}\{z^n g : n \geq 0\}|_{L^1(\mathbb{T})}$. Characterize all possible $g \in L^1(\mathbb{T})$ such that $\inf\limits_{p \in P_+^0} \|1 - p\, g\|_1 = 0$.

## 4.8 Exercises

**Example 4.8.1.** $b_\lambda = \frac{\lambda - z}{1 - \bar{\lambda} z}$ where $\lambda \in \mathbb{D}$ is an inner.

*Proof.* $b_\lambda = \lambda - z \sum_{n \geq 0} \overline{\lambda}^n z^n (|z| = 1)$ and clearly $\widehat{b}_\lambda(k) = 0$ for $k < 0$, and $\sum_{k \geq 0} |\widehat{b}_\lambda(k)|^2 < \infty$; hence $b_\lambda \in H^2(\mathbb{T})$. Moreover, for $|z| = 1$ we have $|\lambda - z| = |\overline{\lambda} - \overline{z}| = |1 - \overline{\lambda} z|$, thus $|b_\lambda(z)| = 1$. □

**Example 4.8.2.** $f = \Pi_{k=1}^N b_{\lambda_k}$ is an inner.

*Proof.* For $f, g \in H^\infty$ we have $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$ hence $H^\infty . H^\infty \subset H^\infty$, a product of inner function is inner. □

**Example 4.8.3.** $S_{\zeta,\alpha} = \exp(\frac{-a(\zeta + z)}{\zeta - z})$ where $a > 0, \zeta \in \mathbb{T}$.

*Proof.* As Re $\left(\frac{\zeta + z}{\zeta - z}\right) = \frac{1 - |z|^2}{|\zeta - z|^2} \geq 0$ for any $\zeta \in \mathbb{T}, |z| \leq 1, z \neq \mathbb{T}$, we obtain that $|S_{\zeta,a}| = 1$ on $\mathbb{T}$. Moreover for every $n > 0$ we have $\widehat{S}_{\zeta,a}(-n) = \int_{\mathbb{T}} z^n S_{\zeta,a}(z) dm = \lim_{r \to 1} \int_{\mathbb{T}} f_r(z) dm = 0$ where $f(z) = z^n S_{\zeta,a}(z)$ and $f_r(z) = f(rz), 0 \leq r < 1$ ($\widehat{f_r}(0) = 0$ since $f_r$ is analytic in $|z| < 1/r$ and $f_r(0) = 0$.) □

**Example 4.8.4.** $f = \prod_{k=1}^N S_{\zeta_k, a_k}$ where $a_k > 0$ $\zeta_k \in \mathbb{T}$.

*Proof.* See the proof of (ii). □

Examples related to the outer functions you will get in Chapter 6, Subsection 6.2.

**Exercise 4.8.5.** For every $f \in L^2$ prove that $f \cdot H^\infty(\mathbb{T}) \subset E_f = \overline{\text{span}}\{f, zf, z^2 f, \dots\}$.

*Proof.* Clearly $f\mathcal{P}_a \subset E_f$, where $\mathcal{P}_a$ is the space of analytic polynomials. It only remains to show $(f\mathcal{P}_a)^\perp \subset (fH^\infty)^\perp$ (orthogonal complement in $L^2$). Let $g \in (f\mathcal{P}_a)^\perp$, i.e. $\int_{\mathbb{T}} \overline{g} f p dm = 0$ for any polynomial $p \in \mathcal{P}_a$. Thus for any $h \in H^\infty$, $\int_{\mathcal{T}} \overline{g} f h dm = 0$ because $\overline{g} f \in L^1$ and $h$ is a weak limit $\sigma(L^\infty, L^1)$ of its Fejer's polynomials. () □

**Example 4.8.6.** If $f \in H^2(\mathbb{T})$ such that $1/f \in H^\infty(\mathbb{T})$, then $f$ is an outer.

*Proof.* By the exercise 4.8.5, $1 = f \cdot 1/f \in E_f$ hence $E_f = H^2(\mathbb{T})$. □

**Exercise 4.8.7.** Let $f, g \in L^2(\mathbb{T})$ (thus $fg \in L^1(\mathbb{T})$). Show that for every $n \in \mathbb{Z}, \overline{fg}(n) = \sum_{k\in\mathbb{Z}} \overline{g}(k)\overline{f}(n-k)$; the series converges absolutely.

*Proof.* By Cauchy Schwarz's inequality $\|f(g-g')\| \leq \|f\|_2\|g-g'\|_2$, the multiplication $M_g f = fg$ is continuous $L^2(\mathbb{T}) \to L^1(\mathbb{T})$. Moreover the Fourier series $g = \sum_{k\in\mathbb{Z}} \widehat{g}(k)z^k$ converges for the norm of $L^2(\mathbb{T})$. Hence $fg = \sum_{k\in\mathbb{Z}} \widehat{g}(k)z^k f$ converges in $L^1(\mathbb{T})$, wich implies $\widehat{fg}(n) = \sum_{k\in\mathbb{Z}} \widehat{g}(k)\widehat{(z^k f)}(n)$. The calculation follows from $\widehat{(z^k f)}(n) = \widehat{f}(n-k)$. □

**Exercise 4.8.8.** Let $f = f_{in} f_{out} \in H^2(\mathbb{T})$. Show that $\sup\{|\widehat{g}(0)| : g \in H^2(\mathbb{T}), |g| \leq |f| \text{ a.e. on } \mathbb{T}\} = |\widehat{f}_{out}(0)|$

*Proof.* From the previous exercise $\widehat{\varphi\psi}(0) = \widehat{\varphi}(0)\widehat{\psi}(0)$ for all $\varphi, \psi \in H^2(\mathbb{T})$. Moreover for every inner function $h$, we have $|\widehat{h}(0)| \leq \|h\|_1 = 1$. Given $g \in H^2(\mathbb{T},)|g| \leq |f|$, which implies $|\widehat{g}(0) = |\widehat{g}_{in}(0)\widehat{g}_{out}(0)| \leq |\widehat{g}_{out}(0)|$. Then by Theorem 4.7.1

$$|\widehat{g}(0)|^2 \leq |\widehat{g}_{out}(0)|^2 = \inf_{p\in\mathcal{P}_a} \int_{\mathbb{T}} |1-p|^2|g|^2 dm \leq \inf_{p\in\mathcal{P}_a} \int_{\mathbb{T}} |1-p|^2|f|^2 dm = |\widehat{f}_{out}(0)|^2$$

□

# Chapter 5

# Canonical factorization in $H^p(\mathbb{D})$

In this section, we discuss the canonical factorization of functions in $H^p$- spaces on the open unit disc as a product of three factors, namely a Blaschke product, a singular inner function, and an outer function in its Schwarz-Herglotz representation. This will help us analyze the questions raised earlier. In particular, Szegö infimum etc.

**Definition 5.0.1.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathrm{Hol}(\mathbb{D})$ denotes the space of analytic functions on $\mathbb{D}$. For $p > 0$, set

$$H^p(\mathbb{D}) = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{H^p}^p = \sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{it})|^p dt < \infty \right\},$$

and $H^\infty(\mathbb{D}) = \{f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)| < \infty\}$. Here $dt$ is the normalized measure on $\mathbb{T}$.

For $p \ge 1$, set $L^p = L^p[0, 2\pi] = (L^p[0, 2\pi], dt)$ and $H^p = \{f \in L^p : \hat{f}(k) = 0,\ k < 0\}$.

The space $H^p(\mathbb{D})$ and $H^p$ are called **Hardy spaces of the disc** and **Hardy space** respectively. Later on we canonically identify these two spaces as same.

### 5.0.1 Properties of $H^p$ spaces

(i) $H^p(\mathbb{D})$ is a linear space.

(ii) $f \longmapsto \|f\|_{H^p}$ is a norm if $p \ge 1$.

(iii) $H^p(\mathbb{D}) \subset H^q(\mathbb{D})$ if $p > q$.

(iv) For $p = 2$, let $f \in \mathrm{Hol}(\mathbb{D})$, and

$$f(z) = \sum_{n \ge 0} \hat{f}(n) z^n,\ \hat{f}(n) \in \mathbb{C}.$$

By Parseval's identity

$$\int_0^{2\pi} |f(re^{it})|^2 dt = \sum_{n\geq 0} |\hat{f}(n)|^2 r^{2n},\ 0 \leq r < 1$$

and we have

$$\sup_{0\leq r<1} \int_0^{2\pi} |f(re^{it})|^2 dt = \sum_{n\geq 0} |\hat{f}(n)|^2.$$

Thus for $f \in \mathrm{Hol}(\mathbb{D})$, we have $f \in H^2(\mathbb{D})$ if and only if $\sum_{n\geq 0} |\hat{f}(n)|^2 < \infty$.

(v) If $1 \leq p \leq \infty$, $H^p$ is a Banach space, and $0 < p < 1, H^p$ is a complete metric space [12](p. 37). If $p < 1$, then $\|.\|_p$ is not a true norm, in fact $H^p$ is not normable. However the expression $d(f, g) = \|f - g\|_p^p$ defines a metric on $H^p$ if $p < 1$.

**Example 5.0.2.** The function f(z)=$\frac{1}{1-z}$ is analytic on $\mathbb{D}$ but is not in $H^2(\mathbb{D})$.

*Proof.* Since $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, the coefficients of $f$ are not square-summable. □

For $f \in H^\infty, \|f\|^2 = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) \leq \|f\|_\infty^2 < \infty \implies f \in H^2$, hence $H^\infty \subset H^2$.

**Example 5.0.3.** The inclusion $H^\infty(\mathbb{D}) \subset H^2(\mathbb{D})$ is strict since the function f(z)=log $\frac{1}{1-z}$ is an unbounded analytic function on $\mathbb{D}$ but it is member of $H^2(\mathbb{D})$, because it has a Taylor series:

$$\log \frac{1}{1-z} = \sum_{n\geq 1} \frac{z^n}{n}$$

has square summable coefficients.

## 5.1 A Revisit to Fourier Series

The functions in $L^p[0, 2\pi]$ can be thought of as functions on $(0, 2\pi)$, which can be extended periodically to real line $\mathbb{R}$.

**Lemma 5.1.1.** *Let $f \in L^1[0, 2\pi]$, $g \in L^p[0, 2\pi]$, $1 \leq p \leq \infty$. Then*

*(i) for almost every $x \in (0, 2\pi)$, $y \longmapsto f(x - y)g(y)$ is integrable on $(0, 2\pi)$.*

*(ii) $f * g(x) = \int_0^{2\pi} f(x - y)g(y)dy$ is well defined and belongs to $L^p[0, 2\pi]$.*

*(iii) $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.*

*Proof.* Note that $(x, y) \longmapsto f(x - y)g(y)$ is measurable, and by Fubini's theorem $|f * g(x)| \leq \int |f(x - y)||g(y)|dy < \infty$ a.e. $x$. By Minkowski integral inequality,

$$\Big|\Big| \int f(x - y)g(y)dy \Big|\Big|_p \leq \int \|f(x - y)g(y)\|_p dy = \|g\|_p \|f_1\|.$$

Further, if $f \in L^1(0, 2\pi)$ and $\hat{f}(n) = \int_0^{2\pi} f(t)e^{-int}dt$, then $\widehat{(f * g)}(n) = \hat{f}(n)\hat{g}(n)$, whenever $g \in L^p$ and $1 \le p \le \infty$ (using Fubini's theorem). □

### 5.1.1 Approximation identity (or good kernel)

(i) If a family $(E_\alpha) \subset L^1$ satisfies

(a) $\sup_\alpha \|E_\alpha\|_1 < \infty$

(b) $\lim_\alpha \hat{E_\alpha}(n) = 1$,

then $\lim_\alpha \|f - f * E_\alpha\|_p = 0$ for $f \in L^p (1 \le p < \infty)$. This is still true for $p = \infty$, if $f \in C(\mathbb{T})$ (called **approximate identity of $L^p$**.)

(ii) If $(E_\alpha) \subset L^1$ satisfies

(a) $\sup_\alpha \|E_\alpha\|_1 < \infty$

(b) $\lim_\alpha \int_0^{2\pi} E_\alpha dx = 1$

(c) $\lim_\alpha \sup_{\delta < |x| < \pi} |E_\alpha(x)| = 0, \forall\, \delta > 0$.

then conditions of (a) and (b) of (i) is satisfied and we get $\lim_\alpha \|f - f * E_\alpha\|_p = 0$.

### 5.1.2 Dirichlet, Fejer and Poisson Kernels

(i) Dirichlet kernel

$$D_m = \sum_{k=-m}^{m} e^{ikt} = \frac{\sin(m + \frac{1}{2})t}{\sin(t/2)}.$$

(ii) Fejer kernel

$$\Phi_n(t) = \frac{1}{n+1} \sum_{m=0}^{n} D_m = \sum_{k=-n}^{n} \Big(1 - \frac{|k|}{n+1}\Big) e^{ikt} = \frac{1}{n+1} \Big(\frac{\sin \frac{n+1}{2} t}{\sin(t/2)}\Big)^2.$$

(iii) Poisson kernel

$$P_r(t) = P(re^{it}) = \frac{1 - r^2}{|1 - re^{it}|^2} = \sum_{k \in \mathbb{Z}} r^{|k|} e^{ikt},\ 0 \le r < 1.$$

**Result:** If $f \in L^1$, then

1. $f * D_m(t) = \sum_{k=-m}^{m} \hat{f}(k) e^{ikt} = S_m(f; t)$ (Partial Fourier series sums of $f$)

2. $f * \Phi_n(t) = \sum \hat{f}(j) \Big(1 - \frac{|j|}{n+1}\Big) e^{ijt} = \frac{1}{n+1} \sum_{m=0}^{n} S_m(f; t)$ (Arithmetic mean of partial sum of Fourier series of $f$)

3. $f * P_r(t) = \sum_{k\in\mathbb{Z}} \hat{f}(k) r^{|k|} e^{ikt}$, $0 \le r < 1$.

4. $(\Phi_n)_{n\ge1}$ and $(P_r)_{0\le r<1}$ are good kernels, and $\|P_r\|_1 = \|\Phi_n\|_1 = 1$.

5. $P_r * P_{r'} = P_{rr'}$ for $0 \le r, r' < 1$ (semi group property).

**Corollary 5.1.2.** *If $f \in L^p$, $1 \le p < \infty$, then $\lim_{n\to\infty} \|f - f * \Phi_n\|_p = 0$. Hence trigonometric polynomials are dense in $L^p$. (Hint: This follows from the property of the good kernel.)*

The same is true for $p = \infty$, if $f \in C(\mathbb{T})$.

**Corollary 5.1.3.** *If $f \in L^1$, $\hat{f}(n) = 0$, $\forall\, n \in \mathbb{Z}$, then $f = 0$.*

**Notations:** For $f \in L^1$, set $f_r = f * P_r$, $0 \le r < 1$.
For $f \in \mathrm{Hol}(\mathbb{D})$, we set $f_{(r)}(z) = f(rz)$, if $|z| < \frac{1}{r}$, $0 \le r < 1$. Clearly $f_{(r)}$ is analytic in bigger domain: $|z| < \frac{1}{r} < 1 + \epsilon$.

**Corollary 5.1.4.** *If $0 \le r < \rho < 1$ and $f \in L^p$, $1 \le p < \infty$, then $\lim_{r\to1} \|f_r - f\|_p = 0$. Moreover, $\|f_r\|_p \le \|f_\rho\|_p \le \|f\|_p$(using maximum modulus principle).*

*If $f \in \mathrm{Hol}(\mathbb{D})$, then $\|f_{(r)}\|_p \le \|f_{(\rho)}\|_p$ and the limit (possible infinite) $\lim_{r\to1} \|f_{(r)}\|_p \le \infty$, exists. In fact, $\lim_{r\to1} \|f_{(r)}\|_p = \|f\|_{H^p(\mathbb{D})}$ if $f \in H^p(\mathbb{D})$. (It follows due to $P_r$ is a good kernel.)*

## 5.2 Identification of $H^p(\mathbb{D})$ with $H^p(\mathbb{T})$

**Theorem 5.2.1.** *Let $1 \le p \le \infty$.*

*(i) If $f \in H^p(\mathbb{D})$, then $\lim_{r\to1} f_{(r)} = \tilde{f}$ exists in $L^p(\mathbb{T})$ and $\tilde{f} \in H^p(\mathbb{T})$. (For $p = \infty$, the limit holds in the weak* topology of $L^\infty(\mathbb{T})$ i.e. in $\sigma(L^\infty, L^1)$.)*

*(ii) $f \longmapsto \tilde{f}$ is an isometry.*

*(iii) $f$ and $\tilde{f}$ are related by $f_{(r)} = (\tilde{f})_r = \tilde{f} * P_r$.*

*The function $\tilde{f}$ is called the* ***boundary limit*** *of function $f$.*

*Proof.* Let $f = \sum_{n=0}^{\infty} a_n z^n \in H^p(\mathbb{D})$, then

$$M = \sup_{0\le r<1} \|f_{(r)}\|_p < \infty. \tag{5.2.1}$$

(i) For $1 < p < \infty$, by Banach Alaoglu theorem (5.2.1) implies that $(f_{(r)})_{0\le r<1}$ is weakly relatively compact in $L^p(\mathbb{T})$. Since $L^p = (L^{p'})^*$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $f_{(r)} \in L^p$; $M = \sup_{0\le r<1} \|\Lambda f_{(r)}\| < \infty$, where $\Lambda f_{(r)} \in (L^{p'})^*$. This gives a limit point $\tilde{f} \in L^p(\mathbb{T})$ of $(f_{(r_k)})_{r_k\to1}$ in the weak topology of $L^p$. We claim that the convergence takes place in $L^p$. As the functional $\phi \longmapsto \hat{\phi}(n)$

is continuous on $L^p$ ( since $|\widehat{\varphi}(n)| \leq \|\varphi\|_{L^p}$ ) for $\epsilon > 0$, $0 < r < 1$, $\exists\, r_k$ with $r < r_k < 1$ such that $|\hat{f}_{(r)}(n) - \hat{\tilde{f}}(n)| < \epsilon$. Note that

$$\|f_{(r)} - \tilde{f}\|_p \leq \|f_{(r)} - f_{(r_k)}\|_p + \|f_{(r_k)} - \tilde{f}\|_p \to 0 \text{ as } r \to 1,$$

if we suppose $f_{(r_k)} \to \tilde{f}$ in $L^p$. But then as $r \to 1$, $\hat{f}_{(r)}(n) = a_n r^n \to a_n$, $n \in \mathbb{Z}$ with $a_n = 0$ if $n < 0$. Hence $a_n = \widehat{(\tilde{f})}(n)$, which implies $\tilde{f} \in H^p(\mathbb{T})$.

We deduce that $\tilde{f}$ does not depends on $(r_k)_{k \geq 1}$ and for $\xi \in \mathbb{T}$,

$$(\tilde{f} * P_r)(\xi) = \sum a_n r^k \xi^n = \sum \widehat{(\tilde{f})}(n) r^{|n|} \xi^n = f_{(r)}(\xi). \tag{5.2.2}$$

Now, by property of good kernel $P_r$ we get

$$\|f_{(r)} - \tilde{f}\|_p = \|(\tilde{f})_r - \tilde{f}\|_p \to 0 \text{ as} r \to 1.$$

That is $f_{(r)} \to \tilde{f}$ in $L^p$.

For $p = \infty$, the similar reasoning gives the convergence $f_{(r)} = (\tilde{f})_r \to \tilde{f}$ in weak* topology of $L^\infty$.

Case $p = 1$ : The space $L^1(\mathbb{T})$ can be regarded as a subspace of $\mathcal{M}(\mathbb{T})$, the space of all complex measures on $\mathbb{T}$. As $\mathcal{M}(\mathbb{T}) = C(\mathbb{T})^*$, by Banach Alaoglu theorem, the balls of $\mathcal{M}(\mathbb{T})$ are weak* relatively compact.

We again get the existence of limit $\tilde{f} \in \mathcal{M}(\mathbb{T})$ as $\lim\limits_{r \to 1} f_{(r)} = \tilde{f}$, but this is weak* limit in $\mathcal{M}(\mathbb{T})$. That is, $\int f_{(r)} g \to \int \tilde{f} g$, $g \in C(\mathbb{T})$. As before take $g(t) = e^{-int}$, then $\widehat{(\tilde{f})}(n) = \hat{\mu}(n) = \lim\limits_{r \to 1} \hat{f}_{(r)}(n)$, $n \in \mathbb{Z}$, and hence $\hat{\mu}(n) = 0$ if $n < 0$. By Riesz Brother's theorem we get $\mu << m$, and the corresponding Radon Nikodym derivative of $\mu$ with respect to $m$ is equal to $\tilde{f} \in H^1$. Using the same argument as in the beginning of the proof, we get $\widehat{(\tilde{f})}(n) = a_n$, $n \geq 0$, $f_r = (\tilde{f})_r$. Hence

$$\lim_{r \to 1} \|\tilde{f} - f_{(r)}\|_1 = \|\tilde{f} - (\tilde{f})_r\|_1 \to 0$$

because $f_r \to f$ in $L^p$ for $1 \leq p < \infty$ by Corollary 5.1.4.

(ii) Let us first consider the case $p < \infty$. Since $\tilde{f} = \lim\limits_{r \to 1} f_{(r)}$, we get using Corollary 5.1.4,

$$\|\tilde{f}\|_p = \lim_{r \to 1} \|f_{(r)}\|_p = \|f\|_{H^p(\mathbb{D})}.$$

For $p = \infty$, observe that as $\tilde{f}$ is weak* limit of $f_{(r)}$, we get

$$\|\tilde{f}\|_\infty \leq \liminf_{r \to 1} \|f_{(r)}\|_\infty = \|f\|_{H^\infty(\mathbb{D})}.$$

On the other hand $f_{(r)} = \tilde{f} * P_r$, we get

$$\limsup_{r\to 1} \|f_{(r)}\|_\infty \le \|\tilde{f}\|_\infty.$$

Hence, we conclude that $\|f\|_{H^\infty(\mathbb{D})} = \|\tilde{f}\|_{H^\infty(\mathbb{T})} = \|\tilde{f}\|_\infty$.

(iii) has been given in (5.2.2).

□

**Convention:** Thus in view of Theorem 5.2.1, for $p \ge 1$ we can identify $f \in H^p(\mathbb{D})$ and its boundary limit $\tilde{f}$ by

$$f_{(r)} = f_r = f * P_r \text{ and } f = \sum_{n\ge 0} \hat{f}(n) z^n.$$

Now $\hat{f}(n)$ represents Fourier coefficient of $\tilde{f}$ at $n$ and Taylor's coefficient as well. Note that if $f \in H^p(\mathbb{D})$ then $f(0) = \widehat{f}(0)$ always.

**Corollary 5.2.2.** *For every $\xi \in \mathbb{D}$, the point wise evaluation map $\varphi_\xi : H^1(\mathbb{D}) \to \mathbb{C}$, defined by $\varphi_\xi(f) = f(\xi), f \in H^1(\mathbb{D})$, is a continuous linear functional on $H^1$ (and hence on $H^p$, $1 \le p < \infty$).*

*Proof.* Let $\tilde{f}$ be the boundary limit of $f \in H^1(\mathbb{D})$. Write $\xi = re^{it}$, $0 \le r < 1$. Then

$$\tilde{f} * P_r(e^{it}) = \sum \hat{\tilde{f}}(n) e^{int} r^{|n|} = \sum a_n e^{int} r^n = f_{(r)}(e^{int}) = f(re^{int}) = f(\xi).$$

Thus $|f(\xi)| \le \|\tilde{f}\|_1 \|P_r\|_\infty \le \|\tilde{f}\|_1 \dfrac{1+|\xi|}{1-|\xi|}$. □

*Remark* 5.2.3. If $\tilde{f}_n \to \tilde{f}$ in $H^p, 1 \le p < \infty$, then $f_n \to f$ uniformity on compact sets in $\mathbb{D}$.

*Proof.* For $|\Lambda| \le r < 1$, $|f_n(\lambda) - f(\lambda)| \le \|\tilde{f}_n - \tilde{f}\| \frac{1+|\lambda|}{1-|\lambda|} = \|\tilde{f}_n - \tilde{f}\| \frac{1+|r|}{1-|r|} \to 0$ as $n \to \infty$, since $\|\tilde{f}_n - \tilde{f}\| \to 0$. Any arbitrary compact set $K \subseteq \{|\lambda| \le r\}$, hence $f_n \to f$ uniformly on $K$. □

## 5.3 Jensen's formula and Jensen's inequality

**Lemma 5.3.1.** *Let $f \in H^1$ with $\hat{f}(0) \ne 0$ (because $f(0) = \hat{f}(0)$) and let $\lambda_n$ be the sequence of zeroes of $f$ in $\mathbb{D}$ counted with multiplicity. Then*

$$\log|f(0)| + \sum_{n\ge 1} \log \frac{1}{|\lambda_n|} \le \int_{\mathbb{T}} \log|f(t)| dm(t).$$

*In particular*

$$\log|f(0)| \le \int_{\mathbb{T}} \log|f(t)| dm(t).$$

*If* $f \in \mathrm{Hol}(\mathbb{D}_{1+\epsilon})$, *then*

$$\log|f(0)| + \sum_{n\geq 1} \log\frac{1}{|\lambda_n|} = \int_{\mathbb{T}} \log|f(t)|dm(t).$$

*Proof.* First we consider $f \in \mathrm{Hol}(\mathbb{D}_{1+\epsilon})$. Let us assume that $Z(f) \cap \mathbb{T} = \emptyset$, i.e. $f$ has no zeroes on $\mathbb{T}$. Then $Z(f) \cap \mathbb{D}$=finite=$\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Set $B(z) = \prod_{j=1}^{n} \frac{|\lambda_j|}{\lambda_j}\frac{(\lambda_j - z)}{(1-\bar{\lambda}_j z)}$. For $B_\lambda(z) = \frac{|\lambda|}{\lambda}\frac{(\lambda-z)}{(1-\bar{\lambda}z)}$, it is easy to see that

$$|B_\lambda(z)|^2 = 1 - \frac{(1-|\lambda|^2)(1-|z|^2)}{|1-\bar{\lambda}z|^2}.$$

Thus we set $|B| = 1$ on $\mathbb{T}$, and $f/B$ is a zero free holomorphic function on $\mathbb{D}_{1+\delta}$ for some $\delta > 0$. Hence, $\log|f/B|$ is a harmonic function on $\mathbb{D}_{1+\delta}$ and allow to apply MVT (because $\log g(z) = \log|g(z)| + i\arg(g(z))$, if $g(z) \neq 0$) and we get

$$\log|(f/B)(0)| = \int_{\mathbb{T}} \log|f/B|dm = \int_{\mathbb{T}} \log|f|dm.$$

As $\log|(f/B)(0)| = \log|(f)(0)| + \sum_{j=1}^{\infty} \log|\lambda_j|^{-1}$, we get the desired formula.

For $f$ having zero on $\mathbb{T}$, we consider $f_r$, $0 \leq r < 1$, where $f_r(z) = f(rz)$. Note that $f_r$ is analytic in $|z| < 1/r < 1+\epsilon$. Choose $r$ such that $f_r$ has no zero on $\mathbb{T}$. If for all $r$ $f_r$ has zeros on $\mathbb{T}$, then $f$ has uncountably many zeroes on $\mathbb{T}$ hence zero set has a limit point in $T$ and $f$ is identically zero. (Note that if $\lambda$ is a zero of $f$ if and only if $\lambda/r$ is a zero of $f_r$.) For such an $r$, apply the previous case:

$$\log|f(0)| + \sum_{|\lambda_n|\leq r} \log\frac{r}{|\lambda_n|} = \int_{\mathbb{T}} \log|f_r|dm(t) \tag{5.3.1}$$

Now $f$ is analytic in $\mathbb{D}_{1+\epsilon}$, so $f$ has finite number of zeros on $\mathbb{T}$. Let $Z(f) \cap \mathbb{T} = \{\xi_i : i = 1, 2, \dots, k\}$. Hence $f = pg$ with $p = \Pi_{i=1}^{k}(z - \xi_i)$ and $g$ is a holomorfic functions such that $g$ and $\frac{1}{g}$ are bounded on $\mathbb{T}$. However for every $r$, $0 < r < 1$ and $z \in \mathbb{D}$

$$\begin{aligned} |\xi_i - z| \leq |\xi_i - rz| + |z(1-r)| \leq |\xi_i - rz| + |1-r| \leq 2|\xi_i - rz| \leq 2 \\ \implies \frac{1}{2}|\xi_i - z| \leq |\xi_i - rz| \leq 2 \end{aligned} \tag{5.3.2}$$

We will calculate for one zero $\xi_i \in \mathbb{T}$. $f_r(\xi) := f(r\xi) = |r\xi - \xi_j|^n g(r\xi) \implies \log|f(r\xi)| = n\log|r\xi - \xi_j| + \log|g(r\xi)|$

Now from (5.3.2)

$$\begin{aligned} &\frac{1}{2}|\xi_j - \xi| \le |r\xi - \xi_j| \\ \implies &\frac{1}{|r\xi - \xi_j|} \le \frac{2}{|\xi_j - \xi|} \end{aligned}$$

$$\begin{aligned} \log|f(r\xi)| &= -\left(n\log\frac{1}{|r\xi - \xi_j|} + \log|g(r\xi)|\right) \\ &\le \frac{-2n}{\log|\xi_j - \xi|} + \log|g(r\xi)| \\ &= 2n\log|\xi - \xi_j| + \log|g(r\xi)| := h(\xi) \text{ say} \end{aligned}$$

To apply DCT and take $\lim\limits_{r\to 1}$ inside the integration in (5.3.1), we need to show: $\int_{\mathbb{T}} |h(\xi)| d\xi < \infty$. This holds since $\int_{\mathbb{T}} \log|\xi - \xi_j| d\xi$ is integrable (in fact it is zero, See [7] P. 307, Lemma 15.17).

**The general case:** Let $f \in H^1$ and $f(0) \neq 0$. In order to pass limit in (5.3.1), note that $|\log x - \log y| \le C_\epsilon |x - y|$, if $x, y > \epsilon$. Hence

$$|\log(|f_r| + \epsilon) - \log(|f| + \epsilon)| \le C_\epsilon ||f_r| - |f|| \text{ on } \mathbb{T} \text{ and}$$

$$\log(|f_r| + \epsilon) \to \log(|f| + \epsilon) \text{ in } L^1(\mathbb{T}) \text{ as } r \to 1.$$

But from (5.3.1)

$$\log|f(0)| + \sum_{|\lambda_n| \le r} \log\frac{r}{|\lambda_n|} = \int_{\mathbb{T}} \log|f_r| dm \le \int_{\mathbb{T}} \log(|f_r| + \epsilon) dm(t). \tag{5.3.3}$$

As LHS in (5.3.3) is increasing in $r$ and RHS is convergent, we obtain

$$\log|f(0)| + \sum_{n\ge 1} \log\frac{1}{|\lambda_n|} \le \int_{\mathbb{T}} \log(|f| + \epsilon) dm$$

for each $\epsilon > 0$. This completes the proof.

Since $|\lambda_n| < 1$ for all $n \in \mathbb{N}$ hence the "in particular" case follows. □

**Corollary 5.3.2.** *(Generalized Jensen's inequality)*

*Let $g \in H^1$, $g \not\equiv 0$, and $|\xi| < 1$. Then*

$$\log|g(\xi)| \le \int \frac{1 - |\xi|^2}{|\xi - t|^2} \log|g(t)| dm(t). \tag{5.3.4}$$

Indeed, to begin with, we may assume that $g \in \text{Hol}(\mathbb{D}_{1+\epsilon})$. Apply the previous result to the function

$$f(z) = g\Big(\frac{\xi - z}{1 - \bar{\xi} z}\Big),$$

and remark that Jacobian of this change of variable is $\frac{1-|\xi|^2}{|\xi - z|}$. (Hint: Put $s = \frac{\xi - t}{1 - \bar{\xi} t}$ etc.)

*Remark* 5.3.3. (Confrontation of two Jensen inequalities) Curiously, Jensen's inequality of Lemma 5.3.1 and Corollary 5.3.2 for the holomorphic functions is, in a way, the opposite of the fundamental inequality of convexity in real analysis, which also bears the name of Johan Jensen. In fact, the Jensen convexity inequality states that:

$$\varphi \int_{\mathbb{T}} g dm \leq \int_{\mathbb{T}} \varphi g dm$$

for any real integrable function $g$ and any convex function $\varphi(\varphi'' > 0)$. Setting $g = \log|f|$ and $\varphi(x) = e^x$ we obtain the following:

$$\int_{\mathbb{T}} \log|f| dm \leq \log \int_{\mathbb{T}} |f| dm = \log \widehat{|f|}(0)$$

## 5.4 The boundary uniqueness theorem

**Corollary 5.4.1.** *If $g \in H^1, g \not\equiv 0$, then $\log|g| \in L^1(\mathbb{T})$. In particular, if $g \in H^1$ and $m\{t \in \mathbb{T} : g(t) = 0\} > 0$, then $g \equiv 0$.*

*Proof.* Indeed, $g \in H^1$ may be expanded in its Taylor's series (when realized on disc $\mathbb{D}$) as $g = \sum_{k \geq n} \hat{g}(k) z^k$, where $\hat{g}(n) \neq 0$, and $n \geq 0$ is the multiplicity of the zero at $z = 0$. By applying Jensen's inequality to function $f = g/z^n$, we get

$$\int_{\mathbb{T}} \log|g| dm = \int_{\mathbb{T}} \log|f| dm > -\infty.$$

Since, $\log x < x$ if $x > 0$, we also have

$$\int_{\mathbb{T}} \log|g| dm \leq \int_{\mathbb{T}} |g| dm < \infty.$$

Hence $\log|g| \in L^1(\mathbb{T})$. It is clear that if $m\{t \in \mathbb{T} :\ g(t) = 0\} > 0$, then $\int_{\mathbb{T}} \log|g| dm = -\infty$, which is possible only if $g \equiv 0$. □

*Remark* 5.4.2. The corollary is true for all $p > 0$. Proof for this using the MVT for harmonic function is done in the proof of Theorem 5.7.6.

*Remark* 5.4.3. Recall that we have seen the second statement of the above corollary for $f \in H^2$ using a completely different approach.

## 5.5 Blaschke Product

**Lemma 5.5.1.** (*Blaschke condition, interior uniqueness theorem*) *Suppose* $f \in \text{Hol}(\mathbb{D})$, $f \not\equiv 0$, *and let* $(\lambda_n)_{n\geq 1}$ *be the zero sequence of* $f$ *in* $\mathbb{D}$, *where each zero is repeated according to its multiplicity. Suppose that*

$$\liminf_{r\to 1} \int_{\mathbb{T}} \log |f_r| dm < \infty,$$

*then* $\sum_{n\geq 1}(1-|\lambda_n|) < \infty$. *In particular, this holds whenever* $f \in H^p(\mathbb{D})$, $p > 0$.

*Remark* 5.5.2. The condition $\sum\limits_{n\geq 1}(1-|\lambda_n|) < \infty$ is called Blaschke condition.

*Proof.* Without loss of generality, we can assume that $f(0) \neq 0$. But then Jensen's formula gives

$$\sum_{n\geq 1} \log \frac{1}{|\lambda_n|} = \liminf_{r\to 1} \sum_{|\lambda_n|\leq r} \log \frac{r}{|\lambda_n|} < \infty$$

As $|\lambda_n| \to 1$, we have $\log\left(\frac{1}{|\lambda_n|}\right) \sim (1-|\lambda_n|)$, and hence the desired conclusion followed. The $H^p(\mathbb{D})$ case is a consequence of the obvious estimate $\log x < C_p x^p$ for $x > 0$, $p > 0$, because

$$\liminf_{r\to 1} \int_{\mathbb{T}} \log |f_r| \leq \liminf_{r\to 1} \int_{\mathbb{T}} C_p |f_r|^p < \infty.$$

□

For $\lambda \in \mathbb{D}$, we define Blaschke factor by

$$b_\lambda(z) = \frac{|\lambda|}{\lambda} \frac{(\lambda - z)}{(1-\bar{\lambda} z)}.$$

(i) If we assume the normalization $b_\lambda\left(-\frac{\lambda}{|\lambda|}\right) = 1$, then for $\lambda = 0$, we can define $b_0(z) = z$.
(ii) Zero set $Z(b_\lambda) = \{\lambda\}$, $b_\lambda \in \text{Hol}(\mathbb{C} \setminus \{\frac{1}{\bar{\lambda}}\})$, $|b_\lambda| \leq 1$ on $\mathbb{D}$ and $|b_\lambda| = 1$ on $\mathbb{T}$.

**Lemma 5.5.3.** *(Blaschke, 1915) If* $(\lambda_n)_{n\geq 1} \in \mathbb{D}$ *satisfies the Blaschke condition* $\sum\limits_{n\geq 1}(1-|\lambda_n|) < \infty$, *then the infinite product*

$$B = \prod_{n\geq 1} b_{\lambda_n} = \lim_{r\to 1} \prod_{|\lambda_n|<r} b_{\lambda_n}$$

*converges uniformly on compact subsets of* $\mathbb{D}$ *and even on compact subsets of* $\mathbb{C} \setminus \text{clos}\{\frac{1}{\bar{\lambda}_n}\}_{n\geq 1}$. *Moreover,* $|B| \leq 1$ *in* $\mathbb{D}$, $|B| = 1$ *a.e. on* $\mathbb{T}$, *and* $Z(B) = (\lambda_n)_{n\geq 1}$ (*counting multiplicity*).

*Proof.* Set $B^r = \prod_{|\lambda_n|<r} b_{\lambda_n}$. Then for $0 \le r < R < 1$, we have

$$\begin{aligned}\|B^R - B^r\|_2^2 &= 2 - 2\,\mathrm{Re}(B^R, B^r)\\ &= 2 - 2\,\mathrm{Re}\int B^R \bar{B}^r dm\\ &= 2 - 2\,\mathrm{Re}\int \frac{B^R}{B^r} dm \text{ (because } |B^r| = 1 \text{ on } \mathbb{T}).\end{aligned}$$

So by MVT for holomorphic function $\frac{B^R}{B^r}$ we get

$$\|B^R - B^r\|_2^2 = 2 - 2\,\mathrm{Re}\Big(\frac{B^R}{B^r}\Big)(0) = 2 - 2\prod_{r\le|\lambda_n|<R} |\lambda_n|.$$

By Blaschke condition $\sum_{n\ge1} \log|\lambda_n|^{-1} < \infty$, the product

$$\prod_{n\ge1} |\lambda_n|$$

converges, which implies $\lim_{r\to1} \prod_{r\le|\lambda_n|<R} |\lambda_n| = 1$. This shows that $(B^r)$ is a Cauchy sequence in $H^2 \subset L^2$ for every $r = r_k \to 1$. So we deduce the existence of $B = \lim_{r\to1} B^r$. Moreover, $|B| = 1$ a.e. on $\mathbb{T}$ because $|B^r| = 1$ on $\mathbb{T}$, and $B \in H^2$. As the point evaluation is continuous linear functional on $H^2$, the limit $\lim_{r\to1} B^r(\lambda) = B(\lambda)$ exists uniformly on compact subsets of $\mathbb{D}$, and hence $|B(\lambda)| \le 1$, $\lambda \in \mathbb{D}$. Using $\frac{B}{B^r} \to 1$ in $H^2$ (easy to see), we get $\frac{B}{B^r} \to 1$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$ and

$$\lim_{r\to1}\Big(\frac{B}{B^r}\Big)(\lambda) = 1. \tag{5.5.1}$$

This shows that $B(\lambda) = 0$, $|\lambda| < 1$ if and only if $\lambda = \lambda_n$ for some $n \ge 1$ (counting multiplicity). If $\lambda \ne \lambda_n$ and $B(\lambda) = 0$, then (5.5.1) will fail.

In order to prove convergence on compact subsets of $\mathbb{C}\setminus\mathrm{clos}\{\frac{1}{\bar\lambda_n}\}_{n\ge1}$, the following observation is enough.

$$|b_{\lambda_n} - 1| = \frac{(|1-|\lambda_n|)(\lambda_n + |\lambda_n|z)}{\lambda(1-\bar\lambda z)} \le \frac{(1-|\lambda_n|)(1+|z|)}{|\lambda_n||z - \frac{1}{\bar\lambda_n}|} \le c\frac{1-|\lambda|}{\mathrm{dist}(z,N)},$$

where $N = \mathrm{clos}\{\frac{1}{\bar\lambda_n} : n \ge 1\}$. □

**Corollary 5.5.4.** *(Frigyes Riesz, 1923) Let $f \in H^p(\mathbb{D})$, $p > 0$ with corresponding zero sequence $(\lambda_n)_{n\ge1}$. Then there exists $g \in H^p(\mathbb{D})$ with $g(\xi) \ne 0$, $\forall\, \xi \in \mathbb{D}$ such that $f = Bg$ and $\|f\|_p = \|g\|_p$ on $L^p(\mathbb{T})$.*

This may be thought as the Blaschke filtering of the holomorphic functions.

*Proof.* Take $B^r = \prod_{|\lambda_n|<r} b_{\lambda_n}$, $0 < r < 1$. Clearly, $\frac{f}{B^r} \in \mathrm{Hol}(\mathbb{D})$ and for $\rho \to 1$, we get $|B^r(\rho\xi)| \to 1$ uniformly on $\mathbb{T}$. Hence,

$$\Big|\Big|\frac{f}{B^r}\Big|\Big|_p^p = \lim_{\rho\to 1}\int_{\mathbb{T}}\Big|\frac{f}{B^r}(\rho\xi)\Big|^p dm(\xi) = \|f\|_p^p \tag{5.5.2}$$

And thus by definition of $H^p(\mathbb{D})$,

$$\Big(\int_{\mathbb{T}}\Big|\frac{f}{B^r}(\rho\xi)\Big|^p dm(\xi)\Big)^{\frac{1}{p}} \le \|f\|_p \text{ for every } 0 \le \rho < 1.$$

Fix $\rho$, set $g = \frac{f}{B}$, and letting $r \to 1$, we obtain

$$\Big(\int_{\mathbb{T}}\Big|g(\rho\xi)\Big|^p dm(\xi)\Big)^{\frac{1}{p}} \le \|f\|_p,$$

and hence $\|g\|_p \le \|f\|_p$. The other inequality follows from $g = \frac{f}{B}$. □

**Note:** In the proof of equation (5.5.2) we use the fact if $f_\rho \to f$ in $H^p$-norm and $g_\rho \to 1$ uniformly as $\rho \to 1$ then $f_\rho g_\rho \to f$ in $H^p$-norm. To prove this use: $|f_\rho g_\rho - f| = |f_\rho g_\rho - f_\rho + f_\rho - f|$ and to apply the DCT use Minkowski's inequalities and $g_\rho$ is uniformly bounded by $M$.

**Question 5.5.5.** * Is it possible to replace $\log|\cdot|$ in Jensen's inequality with some suitable increasing function?

*Remark* 5.5.6. It is useful to introduce the notion of the zero divisor (or multiplicity function) of a holomorphic function. For $f \in \mathrm{Hol}(\Omega)$, $\Omega \subset \mathbb{C}$, $f \not\equiv 0$, $\lambda \in \Omega$, set

$$d_f(\lambda) = \begin{cases} 0 & \text{if } f(\lambda) \neq 0 \\ m & \text{if } f(\lambda) = \cdots = f^{(m-1)}(\lambda) = 0 \text{ and } f^m(\lambda) \neq 0. \end{cases}$$

The value of $d_f(\lambda)$ is called zero multiplicity of $\lambda$. We can redefine the Blaschke condition. The zero divisor of $f \in \mathrm{Hol}(\mathbb{D})$ verifies the Blaschke condition if and only if

$$\sum_{\lambda\in\mathbb{D}} d_f(\lambda)(1-|\lambda|) < \infty.$$

The corresponding Blaschke product is given by

$$\prod_{\lambda\in\mathbb{D}} b_\lambda^{d_f(\lambda)} = \prod_{n\ge 1} b_{\lambda_n}^{d_f(\lambda_n)}.$$

**Corollary 5.5.7.** *Let $f \in H^p, p > 0$ then there exists $f_k \in H^p$; $k = 1, 2$ such that $f = f_1 + f_2$, $\|f_k\|_p \le \|f\|_p$, and $f_k(z) \neq 0$ for $z \in \mathbb{D}$*

*Proof.* If $f(z) \neq 0$, we may take $f_1 = f_2 = \frac{1}{2}f$. If $f$ has zeros, we have $f = Bg$, with $g \in H^p$ has no zeros. Thus $f(z) = [B(z) - 1]g(z) + g(z)$. □

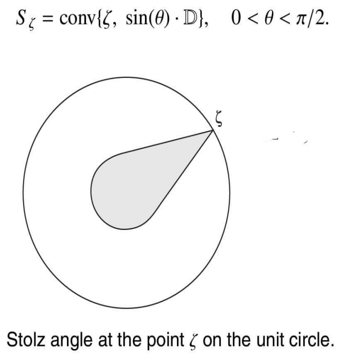


Figure 5.1: A Stolz angle at $\zeta \in \mathbb{T}$.

## 5.6 Non-tangential boundary limits and Fatou's Theorem

Recall that we have identified boundary limit $\tilde{f}$ of $f \in H^p(\mathbb{D})$ via

$$\lim_{r\to 1} \|f_r - \tilde{f}\|_p = 0,\ \tilde{f} \in H^p,\ 1 \le p < \infty.$$

We shall see another convergence of $f(z)$ to its boundary values, namely the so-called non-tangential convergence a.e. on $\mathbb{T}$ for $f \in H^p(\mathbb{D})$ with $0 < p \le \infty$.

Let $\mu$ be a complex valued Borel measure on $\mathbb{T}$ and $\mu \in \mathcal{M}(\mathbb{T})$. Let $d\mu = hdm + d\mu_s$, $h \in L^1(m)$ be Lebesgue decomposition of $\mu$ with respect to $m$. Then the derivative of $\mu$ with respect to $m$ exists at almost every point $\xi \in \mathbb{T}$, in the following sense.

$$\lim_{\Delta\to\xi,\,\xi\in\Delta} \frac{\mu(\Delta)}{m(\Delta)} = \frac{d\mu(\xi)}{dm}(= h(\xi)),$$

where $\Delta$ is an arc on $\mathbb{T}$ tending to $\xi$. Such a point will be called **Lebesgue point** of $\mu$.

**Definition 5.6.1.** A **Stolz angle at the point** $\zeta \in \mathbb{T}$ is the set

$$S_\zeta = \text{conv}\{\zeta, \sin(\theta)\mathbb{D} : 0 < \theta < \pi/2\}$$

where "conv" represents convex hull of sets.

A limit along a Stolz angle, $\lim\limits_{z\in S_\zeta, z\to\zeta} f(z)$ is called a **non-tangential limit at a point** $\zeta$.

Since the Poisson kernel satisfies $P(re^{i\theta}) = \dfrac{1-r^2}{|1-re^{i\theta}|^2}$, for $f \in L^p(\mathbb{T})\,(1 \le p < \infty)$, we have

$$\begin{aligned} P_r * f(e^{i\theta}) &= \int_{\mathbb{T}} \frac{1-r^2}{|1-re^{i(\theta-s)}|^2} f(e^{is}) dm(e^{is}) \\ &= \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} f(\zeta) dm(\zeta), \text{ put } (z = re^{i\theta},\, \zeta = e^{is}) \\ &= f * P(z)(\text{ write }). \end{aligned}$$

That is $P_r * f(e^{i\theta}) = f * P(z)$, where $z = re^{i\theta} \in \mathbb{D}$. Sometimes it is called the **Poisson integral of** $f$.

Now we see one of the most important result about non-tangential limit of the Poisson integral.

**Theorem 5.6.2.** *(P. Fatou's, 1996) Let $\mu \in \mathcal{M}(\mathbb{T})$ and $\zeta \in \mathbb{T}$ be a Lebesgue point of $\mu$, then the Poisson integral of $\mu$*

$$\mathscr{P}(z) = P * \mu(z) = \int \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta), z \in \mathbb{D}$$

*has a non-tangential limit at the point $\zeta$, which is equal to $\frac{d\mu}{dm}(\zeta)$ i.e.,*

$$\lim_{z \to \zeta, z \in S_\zeta} \mathscr{P}(z) = \frac{du}{dm}(\zeta) \text{ a.e. on } \mathbb{T}.$$

*In particular*

$$\lim_{r \to 1} \mathscr{P}(r\zeta) = \frac{d\mu}{dm}(\zeta) \text{ } m\text{-a.e. on } \mathbb{T}.$$

*Proof.* Since $P * m(z) = 1$ for every $z$ (see Rudin, Real and Complex analysis, 11.5, p. 233) the result is correct for $\mu = m$. With a replacement of $\mu$ if necessary by $\mu - cm (c \in \mathbb{C})$ and with the use of a rotation, it suffices to examine the case $\mu(\mathbb{T}) = \widehat{\mu}(0) = 0$ and $\zeta = 1$. Let $F$ be a premitive of $\mu$, i.e. a function on $[-\pi, \pi]$, left continuous and with a bounded variation, such that $\mu[e^{i\alpha}, e^{i\beta}) = F(\beta) - F(\alpha), F(-\pi) = F(\pi)$. As $F$ is defined upto a constant, we can assume $F(0) = 0$. Integration by parts in the integral

$$P * \mu(z) = \int_{-\pi}^{\pi} P(ze^{-is}) dF(s), z \in \mathbb{D}.$$

gives

$$P * \mu(z) = -\int_{-\pi}^{\pi} \frac{dP(z^{-is})}{ds} F(s) ds = \int_{-\pi}^{\pi} E_z(s) \frac{F(s)}{s} ds,$$

where $E_z(s) = -s\frac{dP(ze^{-is})}{ds}$. We denote $z = re^{i\theta}$ where $|\theta| \le \pi, 0 \le r < 1$, and calculate $E_z$ :

$$\begin{aligned} E_z(s) &= -s\frac{d}{ds}\frac{1-r^2}{|1-re^{i(\theta-s)}|^2} = -s\frac{d}{ds}\frac{1-r^2}{1+r^2-2r\cos(\theta-s)} = -\frac{(1-r^2)s\sin(\theta-s)}{|1-re^{i(\theta-s)}|^4} \\ &= -\frac{s\sin(\theta-s)}{(1-r^2)+4r\sin^2(\theta-s)/2}P(ze^{-is}). \end{aligned}$$

Let us show that the family $\{E_z : z \in S_1 \text{ stolz angle}\}$ satisfies the conditions (i)-(iii) for an approximate identity, given in 5.1.1 (ii).

(i) For every $z \in S_1$,

$$\|E_z\|_1 = \int_{-\pi}^{\pi} \left| s\frac{dP(ze^{-is})}{ds} \right| \frac{ds}{2\pi} \le A \int_{-\pi}^{\pi} P(z^{-is})\frac{ds}{2\pi} = A,$$

where, $A = \sup\left\{\frac{|s\sin(\theta-s)|}{(1-r^2)+4\sin^2(\theta-s)/2} : s \in [-\pi,\pi], z \in S_1\right\}$. It remains to show that $A < \infty$. Let $C > 0$ be such that $|\theta| \le C(1-r)$ for any $z = re^{i\theta} \in S_1$ (the existence of such a $C$ can be verified as an exercise).

(a) If $|s| \le 2C(1-r)$, then $\frac{|s\sin(\theta-s)|}{(1-r)^2+r\sin^2(\theta-s)/2} \le \frac{4C(1-r)|\sin(\theta-s)/2|}{(1-r^2)+4\sin^2(\theta-s)/2} \le C$.

(b) If $|s| > 2C(1-r)$ then $|s| > 2|\theta|$, and we have

$$\begin{aligned} \frac{|s\sin(\theta-s)|}{(1-r^2)+4\sin^2(\theta-s)/2} &\le \frac{|s|.(|s|+|\theta|)}{4\sin^2(\theta-s)/2} \le \frac{|s|(|s|+|\theta|)}{4(|\theta-s|/\pi)^2} \\ &\le \frac{|s|(|s|+|s/2|)}{4(|s|-|\theta|/\pi)^2} \\ &\le \frac{|s|^2.(3/2)}{4(|s|-|\theta|/\pi)^2} = (3/2)\pi^2. \end{aligned}$$

Therefore $A \le \max(C, 3\frac{\pi^2}{2})$.

(ii) Integration by parts gives:

$$\lim_{z\to 1, z\in S_1} \int_{-\pi}^{\pi} E_z(s)\frac{ds}{2\pi} = \lim_{z\to 1, z\in S_1}(1-P(-z)) = 1.$$

(Since $P$ is the real part of an analytic function it is harmonic hence continuous ,then take the limit inside and $P(-1) = 0$)

(iii) Let $\delta \le |s| \le \pi$. Then for $z \in S_1$ sufficiently close to 1 we have: $|\theta| < C(1-r) < \delta/2$ and hence

$$|E_z(s)| = |\frac{(1-r^2)s\sin(\theta-s)}{|1-re^{i(\theta-s)}|^4}| \le \frac{(1-r^2)\pi}{|1-re^{i\delta/2}|^4},$$

which tends to 0 as $z \to 1, z \in S_1$

These properties of $E_z$ and the evident relation:

$$\lim_{s\to 0}\frac{F(s)}{s}=\frac{1}{2\pi}\frac{d\mu}{dm}(1),$$

$\left(\frac{d\mu(1)}{dm}=\frac{1}{2\pi}\lim_{s\to 0}\frac{\mu(e^{i0},e^{is})}{s}=\frac{1}{2\pi}\lim_{s\to 0}\frac{F(s)-F(0)}{s}=\frac{1}{2\pi}\lim_{s\to 0}\frac{F(s)}{s}\right)$
as well as (ii) above, imply, when $z\to 1, z\in S_1$

$$\begin{aligned}P*\mu(z)-\frac{d\mu}{dm}(1)&=\int_{-\pi}^{\pi}E_z(s)\left(\frac{F(s)}{s}-\frac{1}{2\pi}\frac{d\mu}{dm}(1)\right)ds+o(1)\\&=\int_{-\delta}^{\delta}+\int_{\delta\le|s|\le\pi}+o(1),\end{aligned}$$

which tends to 0. Indeed by (i), for any $\epsilon>0$ there exists $\delta>0$ such that

$$\left|\int_{-\delta}^{\delta}\right|\le\max_{|s|\le\delta}\left|\frac{F(s)}{s}-\frac{1}{2\pi}\frac{d\mu}{dm}(1)\right|\int_{-\pi}^{\pi}|E_z(s)|ds<\epsilon 2\pi A,$$

and thus, given (iii) and above,

$$\overline{\lim}_{z\to 1,z\in S_1}\left|P*\mu(z)-\frac{d\mu}{dm}(1)\right|\le\epsilon 2\pi A$$

and the results follows. □

**Corollary 5.6.3.** *If $f\in H^p(\mathbb{D})$, $0<p\le\infty$, then the non-tangential boundary limits of $f$ exist a.e. on $\mathbb{T}$. That is,*

$$\lim_{z\to\xi,z\in S_\xi}f(z)=\tilde{f}(\xi)\ \text{for a.e. }\xi\in\mathbb{T}.$$

*The boundary function $\xi\mapsto\tilde{f}(\xi)$ is in $L^p(\mathbb{T})$, and for $p\ge 1$, $f(\xi)=\tilde{f}(\xi)$ a.e. on $\mathbb{T}$ ($\tilde{f}$ is defined in Theorem 5.2.1).*

*Proof.* For $p\ge 1$, the claim follows from Fatou's Theorem (5.6.2) and the Identification Theorem 5.2.1 (because radial limit exists).

Note that for $f\in L^p(\mathbb{T})$ $(1\le p<\infty)$ and $d\mu=fdm$, we have

$$\begin{aligned}P*\mu(z)&=\int_{\mathbb{T}}\frac{1-|z|^2}{|\zeta-z|^2}f(\zeta)dm(\zeta)\\&=P_r*f(\xi)(\text{ let }z=r\xi)\\&=f_r(\xi)=f_{(r)}(\xi)=f(r\xi)\to\frac{d\mu}{dm}(\xi)=f(\xi)\text{ as }r\to 1(\text{Fatou's Theorem.})\end{aligned}$$

Now by identification Theorem 5.2.1 (i) $f_r\to\tilde{f}$ in $L^p$, as $r\to 1$. Since convergence in $L^p$, there exists a subsequence $(r_k)$ such that $P*\mu(\xi)\to\tilde{f}(\xi)$ as $r_k\to 1$ for a.e. $\xi\in\mathbb{T}$ ( since convergence in $L^p$ implies there exists a subsequence which is pt-wise a.e. convergence).

Hence $f(\xi) = \tilde{f}(\xi)$ for a.e. $\xi \in \mathbb{T}$.

■ For general case $p > 0$, we know that $f = Bg = B(g^{1/p})^p$, where $g \in H^p(\mathbb{D})$. This implies $g^{1/p} \in H^1(\mathbb{D})$. The result follows from the previous reasoning. □

**Notation:** From now onward, we identify the functions $f \in H^p(\mathbb{D})$ with their boundary values on $\mathbb{T}$, and write $H^p(\mathbb{D}) = H^p(\mathbb{T})$, $0 < p \leq \infty$, where $H^p(\mathbb{T})$ is the collection of boundary functions of $H^p(\mathbb{D})$.

## 5.7 The Riesz - Smirnov canonical factorization

Here we see the main result of the Hardy space theory - a parametric representation of $f \in H^p$ as a product of Blaschke product, a singular inner function, an outer (maximal) function. The last two functions are exponential of integral depending on the holomorphic Schwarz - Herglotz kernel $z \to \frac{\zeta+z}{\zeta-z}$, whose real part is the Poisson kernel.

**Theorem 5.7.1.** *Let $f \in L^p$, $0 < p \leq \infty$ be such that $\log|f| \in L^1$, and define*

$$[f](z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|f(\zeta)| dm(\zeta)\right),\ |z| < 1.$$

*Then*

*(i)* $[f] \in H^p(\mathbb{D})$ *and* $|[f]| = |f|$ *a.e. on* $\mathbb{T}$.

*(ii) If* $0 \not\equiv g \in H^q(\mathbb{D})$, $q \geq 1$, *and* $|g| \leq |f|$ *a.e. on* $\mathbb{T}$, *then* $|g| \leq |[f]|$ *on* $\mathbb{D}$ (*and hence* $g \in H^p(\mathbb{D})$).

*(iii)* $[\frac{f}{g}] = \frac{[f]}{[g]}$ *and* $[[f]] = [f]$.

*(iv)* $[f](z) \neq 0$ *in* $\mathbb{D}$ *and for any* $\alpha > 0$, $[|f|^\alpha] = [f]^\alpha$.

*Proof.* (i) For fixed $z$, $|\frac{\zeta+z}{\zeta-z}| < \infty^{*1}$ and $\log|f| \in L^1$ hence $[f](z)$ is well defined. Clearly, $[f]$ is a holomorphic function on $\mathbb{D}$. Recall that for a finite Borel measure $\mu$ and a convex function $\psi$, we have the Jensen-Young geometric mean inequality

$$\frac{\int \psi \circ F\, d\mu}{\int d\mu} \geq \psi\Big(\frac{\int F d\mu}{\int d\mu}\Big). \tag{5.7.1}$$

[ Proof Let $F : (\Omega, \mu) \to I \subset \mathbb{R}$ ($I$ is finite or infinite interval), set $\nu = \frac{\mu}{\int d\mu}$. Let $A = \{h : h(x) = ax + b;\ h \leq \psi \text{ on } I\}$. Then $h\big(\int F d\nu\big) = \int h \circ F d\nu \leq \int \psi \circ F d\nu$. We get the inequality since $\psi(x) = \sup\{h(x) : h \in A\}$. ] By apply inequality (5.7.1) to the Borel measure $d\mu = \frac{1-|z|^2}{|\zeta-z|^2}\, dm(\zeta)$, we get

$$|[f]|^p = \exp\Big(\int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} \log|f(\zeta)|^p dm(\zeta)\Big) \leq \int_{\mathbb{T}} |f(\zeta)|^p \frac{1 - |z|^2}{|\zeta - z|^2} dm(\zeta).$$

Set $z = re^{it}$. By Fubini's theorem, we get

$$\int_0^{2\pi} \Big|[f](re^{it})\Big|^p \frac{dt}{2\pi} \leq \int_{\mathbb{T}} |f(\zeta)|^p \Big(\int_0^{2\pi} \frac{1-|z|^2}{|\zeta - z|^2}\frac{dt}{2\pi}\Big) dm(\zeta) = ||f||_p^p.$$

Now, by Fatou's theorem and its corollary there, we have

$$\log|[f](\xi)| = \lim_{r\to 1} \log|[f](r\xi)| = \log|f(\xi)| \text{ a.e. } \xi \text{ on } \mathbb{T}.$$

The modifications in the case $p = \infty$ are obvious.

(ii) Given that $0 \not\equiv g \in H^q(\mathbb{D})$, $q \geq 1$, and $|g| \leq |f|$ a.e. on $\mathbb{T}$. This implies $\log|g| \in L^1$, and hence by generalized Jenson's inequality (5.3.4), we get

$$\begin{aligned}\log|g(z)| &\leq \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} \log|g(\zeta)| dm(\zeta) \\ &\leq \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} \log|f(\zeta)| dm(\zeta) \\ &= \log|[f](z)|.\end{aligned}$$

(iii) is a direct consequence of the definition.

(iv) It is a direct consequence of the definition. But here we only consider the fact $\log|f|^\alpha \in L^1$, whereas $f^\alpha \in L^p$ is not considered.

□

**Note:** *1[

$$\frac{\xi + z}{\xi - z} = 1 + 2\sum_{n=1}^{\infty} \frac{z^n}{\xi^n} \quad \text{since} \quad |\frac{z}{\xi}| < 1.$$

Since $z \in \mathbb{D}, z = r\xi, \xi \in \mathbb{T}$

$$\left|\frac{\xi + z}{\xi - z}\right| \leq 1 + 2\sum_{n=1}^{\infty} r^n = 1 + 2(\frac{1}{1-r} - 1) = \frac{1+r}{1-r} < \infty$$

Since $r$ fixed for fixed $z$.]

Note that from Theorem 5.7.1 to define $[f]$ the condition $\log|f| \in L^1$ is sufficient, but the extra condition $f \in L^p$ ensures that $[f] \in H^p(\mathbb{D})$.

The following result ensures the existence of enough harmonic functions as Poisson integrals of finite Borel measures.

**Theorem 5.7.2.** (*G. Herglotz, 1911*) *Let $u$ be a non-negative harmonic function on $\mathbb{D}$. Then* *there exists a unique finite Borel measure $\mu \geq 0$ such that $u = P * \mu$, that is*

$$u(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

*Proof.* By MVT we have for all $z$ in $\mathbb{D}$

$$u_r(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} u_r(\zeta) dm(\zeta) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu_r(\zeta),$$

where we have set $u_r(z) = u(rz)$, $0 \leq r < 1$, and $d\mu_r = u_r dm$. Then $\mu_r$ is a positive measure and $\mathrm{Var}(\mu_r) = \mu_r(\mathbb{T}) = u_r(0) = u(0) < \infty$. Thus the family $(u_r)_{0 \leq r < 1}$ is uniformly bounded in $\mathcal{M}(\mathbb{T})$, and has week* convergent subsequence $\mu_{r_n}$ that converges to $\mu \in \mathcal{M}(\mathbb{T})$. Recall that $\mathcal{M}(\mathbb{T})$ is dual of $C(\mathbb{T})^*$ with the duality $< f, \mu >= \int\limits_{\mathbb{T}} f d\mu$. Thus, if $f \in C(\mathbb{T})$, $f \geq 0$, then

$$\int_{\mathbb{T}} f d\mu = \lim_{n \to \infty} \int_{\mathbb{T}} f u_{r_n} dm \geq 0 \implies \mu \geq 0.$$

■Moreover, since $u$ is continuous on $\mathbb{D}$, for $z \in \mathbb{D}$, we have

$$u(z) = \lim_{n \to \infty} u(r_n z) = \lim_{n \to \infty} \int \frac{1-|z|^2}{|\zeta - z|^2} d\mu_{r_n}(\zeta) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

■Uniqueness of $\mu$: Note that $P * \mu(re^{it}) = \sum_{n \in \mathbb{Z}} r^{|n|} \hat{\mu}(n) e^{int}$. For any $\nu$ such that $P * \mu = P * \nu$ implies $\hat{\mu}(n) = \hat{\nu}(n)$. Hence $\mu = \nu$. □

**Theorem 5.7.3.** (*Singular inner function*)*: Let $S \in \mathrm{Hol}(\mathbb{D})$, then the following are equivalent:*

*(i) $|S(z)| \leq 1$ and $S(z) \neq 0$ on $\mathbb{D}$, $S(0) > 0$ and $|S(\xi)| = 1$ a.e. on $\mathbb{T}$.*

*(ii) there exists a unique finite Borel measure $\mu \geq 0$ on $\mathbb{T}$ with $\mu \perp m$ such that*

$$S(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)\right), \ z \in \mathbb{D}.$$

*Proof.* ($\Longleftarrow$) (ii) implies (i) is a corollary of Fatou's theorem (because of $S \in H^\infty(\mathbb{D})$ by (ii)). $|S(z)| = \exp\left(-\int_T \frac{1-|z|^2}{|\zeta-z|^2} d\mu(\zeta)\right)$, $z \in \mathbb{D}$ and $\mu \perp m$. By Fatou's theorem on $-\log|S(z)| = f(z)$, $\lim_{r \to 1} f(r\xi) = \frac{du}{dm}(\xi) = 0$ since $\mu \perp m$. $\lim_{r \to 1} \log|S(r\xi)| = 0 \implies \tilde{S}(\xi) = 1$ a.e. on $\mathbb{T}$. Also $|S(z)| = |S(re^{i\theta})| \leq |\tilde{S}(\xi)| = 1$.

($\Longrightarrow$)For (i) implies (ii), let $u = \log|S|^{-1}$, then by Herglotz theorem, there exists $\mu$ such that

$$\log|S(z)|^{-1} = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

Once again by Fatou's theorem (and $|S(\xi)| = 1$ a.e. on $\mathbb{T}$), we get

$$\frac{d\mu}{dm}(\xi) = \lim_{r\to 1} u(r\xi) = 0 \text{ a.e. on } \mathbb{T}.$$

Hence $\mu \perp m$.

■$|S(z)| = |S_\mu(z)|$ in $\mathbb{D}$. $S(z) = \lambda(z)S_\mu(z)$ with $|\lambda(z)| = 1$ for all $z \in \mathbb{D}$, but $S(0) > 0$ and $S_\mu(0) > 0$ which implies that $\lambda = 1$ and which further implies that $S = S_\mu$. □

**Definition 5.7.4.** A nonconstant inner function that has no zero in $\mathbb{D}$ is called a **singular inner function.** A function $S$ verifying (i) or (ii) of the preceding theorem is called a singular inner function.f The word "singular" is used because of the representation of such functions by singular measures.

*Notation* 5.7.5. $\log^+ x = \begin{cases} \log x, x \geq 1 \\ 0, 0 < x < 1 \end{cases}$ and $\log^- x = \begin{cases} -\log x, 0 < x \leq 1 \\ 0, x > 1 \end{cases}$

Then $\log = \log^+ - \log^-$; $|\log| = \log^+ + \log^-$ and $\log^+ x \leq x$ when $x > 0$. Also $|\log^+ x - \log^+ y| \leq |x - y|$ for $x, y > 0$.

**Theorem 5.7.6.** *(Smirnov, 1928: Canonical Factorization Theorem) Let $f \in H^p(\mathbb{D})$, $p > 0$. Then there exists a unique factorization $f = \lambda BS[f]$, where $\lambda \in \mathbb{C}$, $|\lambda| = 1, B$, $S$ and $[f]$ are defined earlier.*

*Proof.* First set

$$g = \tfrac{f}{B}.$$

We will show that any zero free function $g$ satisfies $\int_{\mathbb{T}} \log|g|dm > -\infty$. We may assume $g(0) = 1$. Since $g$ has no zeroes in $\mathbb{D}$, $\log|g(z)|$ is harmonic in $\mathbb{D}$. The MVT for the harmonic function says that any for any $r \in (0, 1)$

$$\begin{aligned} 0 = \log|g(0)| &= \int_{\mathbb{T}} \log|g(r\xi)|dm(\xi) \\ &= \int_{\mathbb{T}} \log^+ |g(r\xi)|dm(\xi) - \int_{\mathbb{T}} \log^- |g(r\xi)|dm(\xi) \end{aligned}$$

Thus $\int_{\mathbb{T}} \log^+ |g(r\xi)|dm(\xi) = \int_{\mathbb{T}} \log^- |g(r\xi)|dm(\xi) \leq \int_{\mathbb{T}} |g(r\xi)|dm(\xi)dm(\xi) \leq \|g\|$ (Cauchy Schwartz). Since $g \in H^p(\mathbb{D})$, $g$ along with the functions $\log^+ |g|$ and $\log^- |g|$ have radial limits a.e. on $\mathbb{T}$. By Fatou's lemma

$$\int_{\mathbb{T}} \log^- |g|dm \leq \lim_{r\to 1} \int_{\mathbb{T}} \log^- |g(r\xi)|dm(\xi) \leq \|g\|$$

which implies that $\log^{-1} |g|$ is integrable on $\mathbb{T}$. Simillary $\log^+ |g|$ and $\log g$ is integrable.

Then $|f| = |g|$ a.e. on $\mathbb{T}$, and hence $[g] = [f]$. Set $\lambda = \frac{g(0)}{[g](0)}$ and $S = \frac{g}{\lambda[g]}$. Then $f = Bg = B\lambda S[g] = \lambda BS[f]$. As $B$ and $[f]$ are uniquely defined for $f$, the uniqueness of factorization follows. □

Next, we consider the structure of the outer functions in $H^p$.

**Theorem 5.7.7.** (***Structure of outer function***) *Let $p, q, r \geq 1$ and $f \in H^p$. Then the following are equivalent.*

*(i) There exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$ such that $f = \lambda[f]$.*

*(ii) for all $z \in \mathbb{D}$, the generalized Jensen inequality is equality:*

$$\log|f(z)| = \int_{\mathbb{T}} P(z\bar{\xi}) \log|f(\xi)| dm(\xi). \tag{5.7.2}$$

*(iii) Identity (5.7.2) holds for at least one $z \in \mathbb{D}$.*

*(iv) If $g \in H^q$ and $\frac{g}{f} \in L^r$, then $\frac{g}{f} \in H^r$ (Integral Maximal principle).*

*If $p = 2$, then (i)-(iv) are equivalent to*

*(v) the function $f$ is outer in $H^2$(In the earlier sense i.e., $E_f = H^2$).*

*Proof.* (i) implies (ii) is followed from the definition of $[f]$. The implication (iii) goes to (ii) is trivial. For (iii) implies (i), suppose (5.7.2) holds for some $z_o \in \mathbb{D}$. By Riesz-Smirnov factorization theorem, we have $f = \lambda BS[f]$, and by (5.7.2), we get

$$|f(z_o)| = |\lambda B(z_o)S(z_o)[f](z_o)| \implies |B(z_o)S(z_o)| = 1 \implies |B(z_o)| = |S(z_o)| = 1.$$

By maximum principle, $B = S =$constant$= 1$ in $\mathbb{D}$, implies $f = \lambda[f]$.
(i) implies (iv): If $g \in H^q$, then $g = \lambda_1 BS[g]$ and we get $\frac{g}{f} = \frac{\lambda_1 BS[g]}{(\lambda[f])} = \left(\frac{\lambda_1}{\lambda}\right) BS\left[\frac{g}{f}\right] \in H^r$ in view of Riesz-Smirnov theorem and by the hypothesis that $g/f \in L^r$.
(iv) $\implies$ (i): Let $f = \lambda BS[f]$ and set $g = \min(|f|, 1)$. Then $[g] \in H^\infty$ and $|\frac{[g]}{f}| \leq 1$ a.e. on $\mathbb{T}$. By (iv) we get $\frac{[g]}{f} \in H^r$ ($r$ arbitrary). Again, we have $\frac{[g]}{f} = \lambda_1 B_1 S_1 \left[\frac{g}{f}\right] = \lambda_1 B_1 S_1 \frac{[g]}{[f]}$ (because $[[g]] = [g]$ and $\left[\frac{g}{f}\right] = \frac{[g]}{[f]}$), we get $1 \equiv \lambda\lambda_1 BB_1 SS_1 = \lambda_2 B_2 S_2$ with $|\lambda_2| = 1$, where $B_2$ is a Blaschke product and $S_2$ is a singular inner function. As $|B_2(z)| \leq 1$ and $|S_2(z)| \leq 1$ for all $z \in \mathbb{D}$, we get $|B_2| = |S_2| \equiv 1$ and hence $B_2 \equiv S_2 \equiv 1$. Thus, we conclude that $B = S = 1$, implies $f = \lambda[f]$.

It remains to show that (iv) and (v) are equivalent if $p = 2$. As (i)-(iii) are independent of choice of $q$ and $r$, we get equivalence between (iv) as well with $p = 2$, and arbitrary $q, r$ and with $p = q = r = 2$, (iv) is just earlier characterization of the outer function on $H^2$. □

*Remark* 5.7.8. In the family of Hardy spaces, dividing by an analytic function, even if it does not have any zero, is a delicate process and the result could be a function that does not belong to any Hardy space. For example, if $S$ is a singular inner function, then $1/S$ does not belong to any Hardy space (easily check!). However, at the same time, its boundary values are unimodular and one is (wrongly) tempted to say that $1/S$ is an inner function. The above result (Theorem 5.7.7 (iv), IMP) says that dividing by an outer function is legitimate as long as the boundary values remain in a Lebesgue space.

**Definition 5.7.9.** (Outer in $H^p$) Let $f \in H^p$, $p > 0$ and $f = \lambda BS[f]$. The function $[f]$ is called the outer part of $f$, and $\lambda BS$ is called the inner part of $f$. We write $[f] = f_{out}$ and $\lambda BS = f_{inn}$. If $f = \lambda[f]$, then $f$ is called **outer**.

It is clear from the above theorem that if $p = 2$, then definition of inner and outer functions coincide with previous ones.

**Corollary 5.7.10.** *Let $w \in L^1_+(\mathbb{T})$, and $p \geq 1$. The followings are equivalent.*

*(i) There exists $f \in H^p$, $f \not\equiv 0$ such that $|f|^p = w$ a.e. on $\mathbb{T}$.*

*(ii)* $\log w \in L^1$.

*Proof.* As $H^p \subset H^1$, and $p \geq 1$ (i) implies (ii) follows from the boundary uniqueness theorem Corollary 5.4.1.

Now (ii) implies (i) follows by taking $f = [w^{1/p}]$. Since if

$$f(z) := [w^{1/p}](z) = \exp\left(\int_{\mathbb{T}} P(z\bar{\xi}) \log |w(\xi)|^{1/p} dm(\xi)\right),$$

then by Theorem 5.7.1 (i), $f \in H^p(\mathbb{D})$.

Since

$$|f(z)|^p = \exp\left(\int_{\mathbb{T}} P(z\bar{\xi}) \log |w(\xi)| dm(\xi)\right)$$

by Fatou's theorem 5.6.2, we get $|f|^p = w$ a.e. on $\mathbb{T}$. □

## 5.8 Approximation by inner functions and Blaschke products

Using Fatou's theorem, we prove two important theorems on uniform approximation by inner functions.

**Theorem 5.8.1.** (*R. Douglas and W. Rudin, 1969*) *Let $\Sigma$ be the set of all inner functions. Then*

$$L^\infty(\mathbb{T}) = \operatorname*{clos}_{L^\infty}\left(\bar{\Theta}H^\infty : \Theta \in \Sigma\right) = \overline{\operatorname{span}}_{L^\infty}\left(\bar{\Theta}_1\Theta_2 : \Theta_1, \Theta_2 \in \Sigma\right). \tag{5.8.1}$$

*Moreover, any unimodular function in $L^\infty(\mathbb{T})$ belongs to*

$$\operatorname*{clos}_{L^\infty}(\Pi)\left(\bar{\Theta_1}\Theta_2 : \Theta_1, \Theta_2 \in \Sigma\right).$$

*Proof.* It is enough to show that $\chi_\sigma \in \overline{\operatorname{span}}_{L^\infty}\left(\bar{\Theta}_1\Theta_2 : \Theta_1, \Theta_2 \in \Sigma\right)$ for every Borel measurable set $\sigma$ in $\mathbb{T}$. Let

$$f_n = [n\chi_\sigma + \frac{1}{n}\chi_{\mathbb{T}\setminus\sigma}],\ n = 2, 3, \ldots$$

and $A_n = \{z \in \mathbb{C} : \frac{1}{n} < |z| < n\}$. It is clear that $f_n(\mathbb{D}) \subset A_n$ (by maximum principle) and $f_n(\mathbb{T}) \subset \partial A_n$. Now let $\phi_1(\zeta) = \zeta + \frac{1}{\zeta}$ for $\zeta \in \mathbb{C} \setminus \{0\}$, and $w : \phi_1(A_n) \to \mathbb{D}$ be a conformal

(Riemann) mapping of the ellipse $\phi_1(A_n)$ onto $\mathbb{D}$. Since the boundary of ellipse is smooth, $w$ can be continuously extended to clos $\phi_1(A_n)$, and hence

$$w \circ \phi_1 \circ f_n = \theta_1$$

is an inner function (because $\theta_1 \in H^\infty(\mathbb{D})$, and by Fatou's theorem $|\theta_1| = 1$ a.e. on $\mathbb{T}$). Since $w^{-1}$ is continuous on clos($\mathbb{D}$), it can be approximated by its Fejer polynomials. Therefore,

$$f_n + \frac{1}{f_n} = \phi_1 \circ f_n = w^{-1} \circ \theta_1 \in \operatorname*{span}_{L^\infty} (\theta_1^n : n \geq 0) .$$

Doing the same for the function $\phi_2(\zeta) = \zeta - \frac{1}{\zeta}$, we get an inner function $\theta_2$ such that $f_n - \frac{1}{f_n} \in \overline{\text{span}}_{L^\infty} (\theta_2^n : n \geq 0)$ . Hence $f_n \in \overline{\text{span}}_{L^\infty} \{\theta_1^k \theta_2^n : k, n \geq 0\}$, implies

$$|f_n|^2 \in \overline{\text{span}}_{L^\infty} \left(\theta_1^k \theta_2^n \theta_1^{-l} \theta_2^{-m} : k, n, l, m \geq 0\right).$$

Thus,

$$\chi_\sigma + \frac{1}{n^4} \chi_{\mathbb{T} \setminus \sigma} \in \overline{\text{span}}_{L^\infty} \left(\bar{\Theta}_1 \Theta_2 : \Theta_1, \Theta_2 \in \Sigma\right), \text{ for} n = 1, 2, \ldots.$$

Letting $n \to \infty$, we get $\chi_\sigma \in \overline{\text{span}}_{L^\infty} \left(\bar{\Theta}_1 \Theta_2 : \Theta_1, \Theta_2 \in \Sigma\right)$.

Let $u \in L^\infty(\mathbb{T})$, and $|u| = 1$ a.e. and $u_1 \in L^\infty(\mathbb{T})$ with $|u_1| = 1$ a.e. and $u = u_1^2$. Given $\epsilon > 0$, by (5.8.1) there exists $\varphi, \Theta_j \in \Sigma$ such that $|u_1 - \bar{\varphi} g| < \epsilon$, where $g = \sum_{j=1}^{n} a_j \Theta_j$, $a_j \in \mathbb{C}$. Set $\Theta = \prod_{j=1}^n \Theta_j$, and observe that $\bar{g}\Theta \in H^\infty$. Since $[\bar{g}\Theta] = [g]$ (because $|\bar{g}\Theta| = |\bar{g}|$), the inner-outer factorizations of $g$ and $\bar{g}\Theta$ are of the form $\bar{g}\Theta = v[g]$ and $g = w[g]$, where $v, w \in \Sigma$, and $1 - \epsilon < |[g]| < 1 + \epsilon$. Now, $|\bar{u_1} - \varphi \bar{g}| = |\bar{u_1} - \varphi \bar{\Theta} v[g]| < \epsilon$ gives

$$\Big| \frac{1}{\bar{u_1}} - \frac{1}{\phi \bar{\Theta} v[g]} \Big| < \frac{\epsilon}{1 - \epsilon}.$$

Since $|u_1 - a| < \epsilon$ and $|u_1 - b| < \epsilon$ implies that $|u_1^2 - ab| \leq |u_1 - a| + |a||u_1 - b|$, we obtain

$$\Big| u - \bar{\phi} w[g] \bar{\phi} \Theta \bar{v} \frac{1}{[g]} \Big| < \frac{2\epsilon}{1 - \epsilon},$$

which completes the proof. □

**Theorem 5.8.2.** (*O. Frostman, 1935*) *Let $\Theta$ be a (non-constant) inner function and $\zeta \in \mathbb{T}$. Then $b_{t\zeta} \circ \Theta$ are Blaschke products with simple zeros for a.e. $t \in (0, 1)$, where $b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}$, $\lambda \in \mathbb{D}$. In particular, $\Theta$ is a uniform limit of Blaschke products with simple zeros.*

*Proof.* Let $\zeta = 1$. Then we need to show that $H_t(z) := b_t \circ \Theta(z) = \frac{t - \Theta(z)}{1 - t\Theta(z)}, z \in \mathbb{D}$ is Blaschke product with simple zeros for all $t \in [0, 1)$. Let $\xi \in \mathbb{T}$, then the boundary function $|\tilde{H}_t(\xi)| = \left| \frac{t - \tilde{\theta}(\xi)}{1 - t\tilde{\theta}(\xi)} \right| = \left| \frac{t - \tilde{\theta}(\xi)}{\overline{\tilde{\theta}(\xi)} - t} \right| = \left| \frac{t - \tilde{\theta}(\xi)}{\overline{t - \tilde{\theta}(\xi)}} \right| = 1 \implies \tilde{H}_t \in H^\infty(\mathbb{T})$. Hence $H_t \in H^\infty(\mathbb{D})$.

By the unique canonical factorization of $H_t(z)$, $H_t(z) = \lambda BS[H_t](z)$ where

$$[H_t](z) = \exp \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \log |\tilde{H}_t(\xi)| dm(\xi) = \exp(0) = 1$$

since $|\tilde{H}_t(\xi)| = 1$. Hence $H_t(z) = \lambda BS$. Our claim is to show: $S = 1$, where

$$S(z) = \exp\left(-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu_t(\xi)\right), \mu_t \perp m, \mu_t \geq 0.$$

To show $S = 1$ we will show $\mu_t(\mathbb{T}) = 0$.

Then by Jensen's formula (5.3.1) (and expression of $S$ and $S \in H^\infty$ with $\|S\|_\infty \leq 1$), and the fact $|H_t(r\xi) \leq |S(r\xi)| \implies |S(r\xi)|^{-1} \leq |H_t(r\xi)|^{-1}$, we get the following:

$$\mu_t(\mathbb{T}) = \log |S(0)|^{-1} = \int_{\mathbb{T}} \log |S(r\xi)|^{-1} dm(\xi) \leq \int_{\mathbb{T}} \log |\tilde{H}_t(r\xi))|^{-1} dm(\xi) = g(r,t),$$

for all $t, r \in [0,1)$. Therefore, it is sufficient to check that $\lim\limits_{r\to 1} g(r,t) = 0$ a.e. $t \in (0,1)$. Now $\mu(\mathbb{T}) \leq g(r,t) \implies \int_0^1 \lim_{r\to 1} \mu_t(\mathbb{T}) \leq \int_0^1 \lim_{r\to 1} g(r,t)dt \implies \mu(\mathbb{T}) \leq \int_0^1 \lim_{r\to 1} g(r,t)dt$. We will show the right hand side is zero. For this we will show that

$$\int_0^1 \lim_{r\to 1} g(r,t)dt = \lim_{r\to 1} \int_0^1 g(r,t)dt$$

This happens due to DCT: $|g(r,t)| = |\int_{\mathbb{T}} \log |H_t(r\xi)|^{-1}|d\xi \leq \int_{\mathbb{T}} \log |H_t(0)|^{-1} d\xi = \log |H_t(0)|^{-1} \in L^1(0,1)$. So by DCT we can interchange the limit:

$$\begin{aligned}\int_0^1 \lim_{r\to 1} g(r,t)dt &= \lim_{r\to 1} \int_0^1 \int_{\mathbb{T}} \log |H_t(r\xi)| dm(\xi)dt \\ &= \lim_{r\to 1} \int_{\mathbb{T}} \int_0^1 \log |H_t(r\xi)| dt dm(\xi) = 0\end{aligned}$$

since $\int_0^1 \log |H_t(r\xi)|^{-1} dt = 0$. Let $u : \overline{\mathbb{D}} \to \mathbb{R}$, by $u(w) = \int_0^1 \log |b_t(w)|^{-1} dt = -\int_0^1 \log |b_t(w)| dt$. $u$ is continuous then

$$\begin{aligned}u(\mathbb{T}) &= -\int_0^1 \log |b_t(e^{i\varphi})| dt \\ &= -\int_0^1 \log |\frac{t - e^{i\varphi}}{1 - te^{i\varphi}}| dt \\ &= -\int_0^1 \{\log |t - e^{i\varphi}| - \log \overline{|t - e^{i\varphi}|}\} dt \\ &= 0.\end{aligned}$$

Therefore $\int_0^1 \log |\tilde{H}_t(\xi)|^{-1} dt = 0 \implies \mu(\mathbb{T}) = 0$.

■ The zeros of $b_\lambda \circ \Theta$ are simple if $\lambda - \Theta(z_j) \neq 0$, $\forall\, j$, where $(z_j)_{j\geq 1}$ are the zeroes of $\Theta'$.

Indeed, if $b_\lambda(\Theta(z)) = 0$, then $\lambda - \Theta(z_j) = 0$ and hence $(b_\lambda \circ \Theta)'(z) = b'_\lambda(\Theta(z))\Theta'(z) \neq 0$.

Finally, we show thar $u$ is continuous on $\bar{\mathbb{D}}$. Note that the integrals $\int_0^1 \log|1-tw|dt$ and $\int_0^1 \log|t-w|dt$ are similar and for $w = x+iy$, we have

$$\int_0^1 \log|t-w|^2 dt = \int_0^1 \log\{(t-x)^2 + y^2\}dt$$

is continuous in $x$ and $y$ ( for instance $\int_0^1 \log(t-x)^2 dt = \chi_{(0,1)} * \log(x^2)$ ). $\square$

## 5.9 Exercises

**Exercise 5.9.1.** Show that $H^2(\mathbb{D})H^2(\mathbb{D}) = H^1(\mathbb{D})$.

*Proof.* If $f, g \in H^2$, then $\|fg\|_1 \leq \|f_1\|_2\|g_1\|_2\| < \infty$ which implies $H^2H^2 \subseteq H^1$. For the converse, let $f \in H^1$ consider $G = \frac{f}{B}$ then $G \neq 0$ in $\mathbb{D}$. Hence $G = g^2$ for some function $g$.

Also we have $\|G\| = \|f\| \implies G \in H^1 \implies g \in H^2$. Take $h = Bg$. Since $B \in H^1(\mathbb{D})$ and $g \in H^2(\mathbb{D}) \subset H^1(\mathbb{D}) \implies h = Bg \in H^2(\mathbb{D})$ and $f = GB = g^2B = g(Bg) = gh$ $\square$

**Exercise 5.9.2.** $f \in H^1, f(\mathbb{T}) \subseteq \mathbb{R}$ then $f$ is a constant.

*Proof.* Since $f \in H^1$ for $z \in \mathbb{D}$,

$$f(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\tilde{f}(\zeta)dm(\zeta)(= \mathscr{P}f(z))$$

If $\tilde{f}(\mathbb{T}) \subseteq R$ and the Poison kernal $P_z(\zeta)$ is always real then $f(z)$ is real from the above integration. But the only analytic function which is real must be constant. $\square$

**Exercise 5.9.3.** Let $f \in H^{1/2}$. Assume that $f \geq 0$ a.e. on $\mathbb{T}$. Then $f$ is a constant.

*Proof.* Assume $f \not\equiv 0$. By the canonical factorization theorem we have: $f = Bg$ where $B$ is the Blaschke product associated with $f$ and $g$ belongs to $H^{1/2}$ and has no zeros on $\mathbb{D}$. That is why we can define $h = g^{1/2}$, and the function $h$ belongs to $H^1$ with $\|h\|_1 = \|g\|_{\frac{1}{2}}$. Clearly, $f = Bh^2$.

The condition $f \geq 0$ ensures that $f = |f|$ a.e. on $\mathbb{T}$. Hence, since $B$ is unimodular on $\mathbb{T}$, we have $Bh^2 = \bar{h}$ a.e. on $\mathbb{T}$.

Now on one hand we have, $Bh \in H^1$, and on the other hand $\bar{h} \in \bar{H}^1$. We know that $H^1 \cap \overline{H}^1$ contains only the constant functions. Therefore $Bh$ is a constant function. By the uniqueness of the canonical factorization this happens precisely when $B$ is a unimodular constant and $h$ is a constant. Thus eventually $h$ is a constant. $\square$

**Example 5.9.4.** If $f(z) = \exp(\frac{z+1}{z-1})$ then $f$ is a singular inner function.

*Proof.* Recall that $|e^w| = |e^{\operatorname{Re} w + i \operatorname{Im} w}| = |e^{\operatorname{Re} w}| = e^{\operatorname{Re} w}$. Hence $|f(z)| = \exp\left(\operatorname{Re}(\frac{z+1}{z-1})\right) = \frac{|z|^2-1}{|z-1|^2} < 0$ for $z \in \mathbb{D}$. It follows that $|f(z)| < 1 \forall z \in \mathbb{D}$. Thus $f \in H^\infty$. Moreover for $|z| = 1$ and

$z \neq 1$ implies $\operatorname{Re} \frac{z+1}{z-1} = 0$ and therefore $|\tilde{f}(e^{i\theta})| = 1$ for all $\theta \neq 0$. Since $e^w$ is never zero for any complex number $w$, it follows that $f$ is an inner function with no zeros on $\mathbb{D}$.

□

*Remark* 5.9.5. The function $f(z) = \exp(\frac{1+z}{1-z})$ is not an inner. This function is the reciprocal of the function in earlier example hence $|f(e^{i\theta})| = 1$ for $\theta \in (0, 2\pi)$. However for $0 < r < 1$

$$|f(r)| = \exp\left(\frac{1+r}{1-r}\right) \to \infty, uadr \to 1^-$$

Although $f$ has unimodular boundary value almost everywhere on $\mathbb{T}$, it is unbounded on $\mathbb{D}$ and hence is not an inner function. Thus when checking to see whether or not an analytic function is inner one must be careful to check at first that it is actually bounded on $\mathbb{D}$.

**Exercise 5.9.6.** Let $r > 0, s > 0, t > 0$ be such that $\frac{1}{r} = \frac{1}{s} + \frac{1}{t}$. Show that $H^r = H^s \cdot H^t$ and moreover $\|f_r\| = \min\{\|g\|_s \|h\|_t : g \in H^s, h \in H^t \text{ s.t. } f = gh\}$

*Proof.* By Holders inequality, if $g \in H^s(\mathbb{D}), h \in H^1(\mathbb{D})$ then $f = gh \in \operatorname{Hol}(\mathbb{D})$ and for every $\rho, 0 < \rho < 1$, we have $\|f_\rho\| \leq \|g_\rho\|_s \|h_\rho\|_t$, which implies $f \in H^r(\mathbb{D})$ and $\|f\|_r \leq \|g\|_s \|h\|_t$. Conversely, if $f \in H^r(\mathbb{D})$, with $f = \lambda BV[f]$ its Canonical factorization, then by $g = \lambda BV[f]^{r/s}, h = [f]^{r/t}$, we obtain $f = gh$ and $\|f\|_r = \|g\|_s \|h\|_t$. □

**Exercise 5.9.7.** Let $\lambda \in \mathbb{D}$ and $\varphi_\lambda$ be an evaluating functional on $H^p$, $1 \leq p \leq \infty$, i.e.

$$\varphi_\lambda(f) = f(\lambda), f \in H^p.$$

Show that $\|\varphi_\lambda\| = (1 - |\lambda|^2)^{-1/p}$.

*Proof.* When $p = 2, \varphi_\lambda(f) = f(\lambda) = \sum_{k \geq 0} \widehat{f}(k)\lambda^k = (f, k_\lambda)_{H^2}$, where

$$k_\lambda(z) = \sum_{k \geq 0} \overline{\lambda}^k z^k, z \in \mathbb{D},$$

is the Szego reproducing kernal of $H^2$, hence $\|\varphi_\lambda\| = \|k_\lambda\|_2 = (1 - |\lambda|^2)^{-1/2}$. When $p$-is arbitrary, recall that for every function $f$, $|f(\lambda)| \leq |[f]\lambda|$ and $\|f\|_p = \|[f]^{p/2}\|_2^{2/p}$ which leads to:

$$\begin{aligned}\|\varphi_\lambda\| = \sup\{|f(\lambda)| : f \in H^p, \|f\|_p \leq 1\} &= \sup\{|[f]^{p/2}(\lambda)|^{2/p} : \|[f]^{p/2}\| \leq 1\} \\ &= \left((1 - |\lambda|^2)^{-1/2}\right)^{2/p}.\end{aligned}$$

□

**Exercise 5.9.8.** ( Neuwirth and Newman, 1967 ) Let $f \in H^p(\mathbb{D}), p > 0$. Show that $f = \mathit{constant}$ if and only if the following hypothesis is verified:

(i) $p \geq 1$ and $f(\zeta)$ is real a.e. $\zeta \in \mathbb{T}$.

(ii) $p \geq 1/2$ and $f(\zeta) \geq 0$ a.e. $\zeta \in \mathbb{T}$.

Show that the conclusion no longer holds if $p < 1$.

*Proof.* Case (i) is evident, because in this case $f, \overline{f} \in H^1(\mathbb{T})$, which implies f=constant.

For (ii) see Exercise 5.9.3.

For the last assertion, consider the function $f_1 = i\frac{1+z}{1-z}$ respectively $f_2 = f_1^2$. It is easy to see that $f_1 \in H^p(\mathbb{D})$ for any $p < 1$ and $f_2 \in H^p(\mathbb{D})$ for any $p < 1/2$. □

**Exercise 5.9.9.** Let $f, g \in H^2$ and $h = fg$. Show that $|\widehat{h}(n)| \leq \sum_{k+j=n} |\widehat{f}(k)| \cdot |\widehat{g}(j)|$.

*Proof.* The Fourier series $g = \sum_{j\in\mathbb{Z}} \widehat{g}(z)z^j$ converges in $L^2(\mathbb{T})$ hence by Cauchy Schwartz's inequality the series $h = fg = \sum_{j\in\mathbb{Z}} \widehat{g}(z) f z^j$ converges in $L^1(\mathbb{T})$ and by continuity of $h \mapsto \widehat{h}(n)$, we obtain $\widehat{h}(n) = \sum_{j\in\mathbb{Z}} \widehat{f}(n-j)\widehat{g}(j)$; the result follows. □

**Exercise 5.9.10.** Let $\varphi(e^{it}) = i(t-\pi)$ for $0 < t < 2\pi$. Find the Fourier coefficients of $\varphi$.

*Proof.* $\widehat{\varphi}(0) = 0$ and for $k \neq 0$,

$$\begin{aligned}\widehat{\varphi}(k) &= \int_0^{2\pi} i(t-\pi)e^{-ikt}dt/2\pi \\ &= \Big[-(t-\pi)e^{-ikt}/2\pi k\Big]_{t=0}^{2\pi} + \int_0^{2\pi} e^{-ikt}dt/2\pi k \\ &= -1/k\end{aligned}$$

□

**Exercise 5.9.11.** (The Hilbert Inequality, 1908) Let $f, g \in H^2$. Show that

$$\left|\sum_{k,j\geq 0} \frac{\widehat{f}(k)\widehat{g}(k)}{k+j+1}\right| \leq \pi\|f\|_2\|g\|_2.$$

*Proof.* For $F, G \in L^2(\mathbb{T})$ and $\Phi \in L^\infty(\mathbb{T})$ just as in (a) above, we have $(\Phi F, \overline{G}) = \sum_{i+j+k=0} \widehat{\varphi}(i)\widehat{F}(k)\widehat{G}(j)$, which gives

$$(\varphi f, \overline{zg}) = -\sum_{k,j\geq 0} \frac{\widehat{f}(k)\widehat{g}(j)}{k+j+1}.$$

Then the result follows from

$$|(\varphi f, \overline{zg})| \leq \|\varphi f\|_2\|\overline{zg}\|_2 \leq \|\varphi\|_\infty\|f\|_2\|g\|_2 = \pi\|f\|_2\|g\|_2.$$

□

**Exercise 5.9.12.** (The Hardy Inequality, 1926): For every function $h \in H^1$, $\sum_{k\geq 0} \frac{|\widehat{h}(k)|}{k+1} \leq \pi \|h\|_1$.

*Proof.* By Exercise 5.9.6, $h = fg$ with $f, g \in H^2$ and $\|f\|_2^2 = \|g\|_2^2 = \|h\|_1$ and by Exercises 5.9.9 and 5.9.11

$$\sum_{k\geq 0} \frac{|\widehat{h}(k)|}{k+1} \leq \sum_{k\geq 0} \frac{\sum_{i+j=k} |\widehat{f}(i)||\widehat{g}(j)|}{k+1} \leq \pi \|f\|_2 \|g\|_2 = \pi \|h\|_1.$$

□

We have seen that every $H^p$ function $f(re^{i\theta})$ converges almost everywhere to an $L^p$ boundary function $f(e^{i\theta})$. It is important to know that whether $f(re^{i\theta})$ always tends to $f(e^{i\theta})$ in the sense of the $L^p$ mean or not.

**Exercise 5.9.13.** (Mean convergence theorem) If $f \in H^p(0 < p < \infty)$ then

$$\lim_{r\to 1} \int_0^{2\pi} |f(re^{i\theta})|^p = \int_0^{2\pi} |f(e^{i\theta})|^p \tag{5.9.1}$$

and

$$\lim_{r\to 1} \int_0^{2\pi} \left| f(re^{i\theta}) - f(e^{i\theta}) \right|^p d\theta = 0 \tag{5.9.2}$$

*Proof.* First let us prove 5.9.2 for $p = 2$. If $f(z) = \sum a_n z^n$ is in $H^2$, then $\sum |a_n|^2 < \infty$. But by Fatou's Lemma

$$\begin{aligned} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^2 d\theta \leq & \liminf_{\rho\to 1} \int_0^{2\pi} |f(re^{i\theta}) - f(\rho e^{i\theta})|^2 d\theta \\ =& 2\pi \sum_{n=1}^{\infty} |a_n|^2 (1 - r^n)^2, \end{aligned}$$

which tends to 0 ar $r \to 1$. This proves (5.9.2) and hence (5.9.1) for $p = 2$.

■ If $f \in H^p(0 < p < \infty)$, we use the factorization $f = Bg$. Since $[g(z)]^{p/2} \in H^2$, it follows from what we have just proved that

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^p d\theta \to \int_0^{2\pi} |g(e^{i\theta})|^p d\theta = \int_0^{2\pi} |f(e^{i\theta})|^p d\theta$$

This together with the Fatou's Lemma proves (5.9.1)

The following lemma can now be applied to deduce (5.9.2) from (5.9.1).

□

**Lemma 5.9.14.** *[12][p. 21] Let $\Omega$ be a measurable subset of $\mathbb{R}$ and let $\varphi_n \in L^p(\omega), 0 < p < \infty; n = 1, 2, \ldots$ As $n \to \infty$, suppose $\varphi_n(x) \to \varphi(x)$ a.e. on $\Omega$ and*

$$\int_\Omega |\varphi_n(x)|^p dx \to \int_\Omega |\varphi(x)|^p dx < \infty$$

*then*

$$\int_{\Omega} |\varphi_n(x) - \varphi(x)|^p dx \to 0.$$

**Corollary 5.9.15.** *If $f \in H^p$ for some $p > 0$, then*

$$\lim_{r\to 1} \int_0^{2\pi} \left|\log^+ |f(re^{i\theta})| - \log^+ |f(e^{i\theta})|\right| d\theta = 0$$

*Proof.* Immediately follows from Mean convergence theorem 5.9.13 and the following inequality:

$$|\log^+ a - \log^+ b| \le \frac{1}{p}|a-b|^p, a \ge 0, b \ge 0, 0 < p \le 1$$

For the proof the inequality see [12][p. 22] □

**Exercise 5.9.16.** [12][p. 34] A function $f$ analytic in $\mathbb{D}$ is representable in the form $f(z) = \mathscr{P}\varphi(z)$ i.e.

$$f(z) := \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) f(e^{it}) dt$$

as a Poisson-integral $\varphi \in L^1$ if and only if $f \in H^1$. In this case $\varphi(t) = f(e^{it})$ a.e.

*Proof.* If an analytic funcition $f(z)$ has the form $f(z) = \mathscr{P}\varphi(z)$ then

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \le \int_0^{2\pi} |\varphi(t)| dt$$

so that $f \in H^1$.

Conversely, suppose $f \in H^1$, and write

$$\Phi(z) := \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) f(e^{it}) dt$$

For any fixed $\rho, 0 < \rho < 1$

$$f(\rho z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) f(\rho e^{it}) dt$$

But by the Exercise 5.9.13 $\int_0^{2\pi} |f(\rho e^{it}) - f(e^{it})| dt \to 0$ as $\rho \to 1$, so $f(\rho z) \to \Phi(z)$. Hence $\Phi(z) = f(z)$. □

**Corollary 5.9.17.** *A function $f(z)$ is analytic in $|z| < 1$ is the Poisson integral of a function $\varphi \in L^p (1 \le p \le \infty)$ if and only if $f \in H^p$.*

# Chapter 6

# Szegö infimum and generalized Phragmén–Lindelöf principle

In this section, we consider two applications of the canonical Riesz-Smirnov factorization. Namely, the Szegö infimum $\mathrm{dist}(1, H_0^2(\mu))$ is expressed in terms of measure $\mu$, the cyclic functions of $L^2(\mathbb{T})$ are described. The classical logarithmic integral criterion for completeness of the polynomials, the case of incompleteness, and the closure of the polynomials $H^2(\mu)$ is described in terms of the outer function related to Radon-Nikodym derivative $w = \frac{d\mu}{dm}$. We consider outer functions, their extremal and extension properties, and distribution value properties. The important Smirnov subclass of Nevanlinna functions is considered. After transferring these results to an arbitrary simply connected domain of $\mathbb{C}$, we use these techniques to get a remarkably general Phragmén–Lindelöf type principle due to Smirnov (1920) and then by Helson (1960).

## 6.1 Szegö infimum and weighted polynomial approximation

**Theorem 6.1.1.** (*Szegö, Kolmogorov*) *Let $d\mu = wdm + d\mu_s$ be a Borel measure. Then*

$$\inf_{p \in \mathbb{P}_+^0} \int_{\mathbb{T}} |1-p|^2 d\mu = \exp\left(\int_{\mathbb{T}} \log w \, dm\right).$$

*Proof.* By the Theorem 4.7.1 two cases are possible

(i) If there exists $f \in H^2$ such that $|f|^2 = w$ a.e. $m$ then $dist^2 = 0$; otherwise

(ii) $\mathrm{dist}^2 = |\widehat{f}(0)|^2$

By the Corollary 5.7.10, Case (ii) $\Leftrightarrow \log w \in L^1$ holds if and only if $\log w \in L^1$ and in this case:

$$f(z) = \exp \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \log w^{\frac{1}{2}}(\xi) dm(\xi)$$

Since $f \in H^2$, $\widehat{f}(0) = f(0)$ and $|\widehat{f}(0)|^2 = |f(0)|^2 = \exp \int_{\mathbb{T}} \log w dm$.

□

Let $f \in L^2(\mathbb{T})$, and write $E_f = \overline{\text{span}}\{z^n f : n \geq 0\}$. If $E_f = L^2(\mathbb{T})$, we say $f$ is a cyclic vector. Note that the half of the trigonometric system $(z^n)_{n\geq 0}$ is far from being complete in $L^2(\mathbb{T})$, but multiplying by a suitable function $f$ one can get completeness property i.e. $\overline{\text{span}}\{z^n f : n \geq 0\} = L^2(\mathbb{T})$. It may happen that for different halfs of $(z^n)_{n\in\mathbb{Z}}$, nothing similar is true.

**Corollary 6.1.2.** *Let $f \in L^2$. Then $E_f = \overline{\text{span}}\{z^n f : n \geq 0\} = L^2$ if and only if $f(\xi) \neq 0$ a.e. on $\mathbb{T}$ and $\int_{\mathbb{T}} \log |f| dm = -\infty$.*

*Proof.* Two cases may possible: Either $zE_f = E_f$ or, $zE_f \subsetneq E_f$. In the first case by N-Weiner Theorem 3.0.4 there exists $\sigma \subset \mathbb{T}$ such that $E_\sigma = \chi_\sigma L^2(\mu)$. If the second case holds: $zE_f \subsetneq E_f$. $\Leftrightarrow$ there exists $\theta$ such that $|\theta| = 1$ and $E_f = \theta H^2$. Since $E_f = L^2 \implies zE_f = L^2$ again, hence only the first case is possible, so second case does not possible, i.e., $\forall \theta$ such that $|\theta| = 1$, $E_f \neq \theta H^2 \Leftrightarrow$ there does not exists $g \in H^2$ such that $z^n f = \theta g \ \forall n \Leftrightarrow 1.|f| = 1.|g| \Leftrightarrow |f| = |g| \Leftrightarrow \log |f| \in L^1$ by Corollary 5.7.10.

( $\Longleftarrow$ ) there exists $\sigma \in \mathbb{T}$ such that $E_f = \chi_\sigma L^2(\mathbb{T})$. As $f \in \chi_\sigma L^2(\mathbb{T})$ and $f \neq 0$ a.e. on $\mathbb{T}$ we get $\sigma = T$, and then $E_f = L^2(\mathbb{T})$. □

**Example 6.1.3.** (a) If $f(e^{i\theta}) = |1 - e^{i\theta}|^\alpha$, $\alpha > -\frac{1}{2}$, then $E_f \neq L^2(\mathbb{T})$.

(b) If $f(e^{i\theta}) = \exp\left(\frac{-1}{1-e^{i\theta}}\right)$, then $E_f = L^2(\mathbb{T})$.

The following two theorems are final statements on weighted polynomial approximation on the circle $\mathbb{T}$.

**Theorem 6.1.4.** *Let $\mu$ be a positive measure on $\mathbb{T}$ and let $w = \frac{d\mu}{dm}$ its Radon-Nikodym derivative. Then polynomials $\mathbb{P}_+$ are dense in $L^2(\mu)$ if and only if $\log w \notin L^1(\mathbb{T})$.*

*Proof.* Polynomials are dense in $L^2(\mu)$ if and only if the Szego distance is zero follows from Corollary 4.3.4. This holds if and only if there does not exists an outer such that $|f|^2 = w$ ( using Theorem 4.7.1), which is immediate from Corollary 5.7.10. □

**Theorem 6.1.5.** *Let $\mu$ be a positive measure on $\mathbb{T}$, let $d\mu = wdm + d\mu_s$ be its Lebesgue decomposition and suppose that $\log w \in L^1(\mathbb{T})$. Let $\phi \in H^2$ be the outer function defined by $\phi = [w^{\frac{1}{2}}]$. Then closure $H^2(\mu) = \text{clos}_{L^2(\mu)} \mathbb{P}_+$ is given by*

$$H^2(\mu) = L^2(\mu_s) \oplus (\phi^{-1} H^2) = L^2(\mu_s) \oplus \{f \in \text{Hol}(\mathbb{D}) : \ f\phi \in H^2\}.$$

*Proof.* Indeed, Corollary 4.3.1 gives $H^2(\mu) = H^2(wdm) \oplus L^2(\mu_s)$ and Lemma 4.3.3 and Theorem 6.1.1 show that $H^2(wdm)$ is 1-invariant (non-reducing) subspace of $L^2(wdm)$ ( see also Remark 4.3.2). $\Leftrightarrow$ $H^2(wdm) = \theta H^2$ for some $\theta$ such that $|\theta|^2 w = 1$ by the Helson Theorem 3.3.3 $\implies \theta = [w^{\frac{1}{2}}]^{-1} = \frac{1}{w^{\frac{1}{2}}}$. Hence $H^2(wdm) = \frac{1}{w^{1/2}} H^2 = \varphi^{-1} H^2$ since $\varphi = [w^{1/2}]$. □

## 6.2 Properties of Outer functions

Note that from Theorem 5.7.1 to define $[f]$ the condition $\log|f| \in L^1$ is sufficient, but the extra condition $f \in L^p$ ensures that $[f] \in H^p(\mathbb{D})$. In general, the definition of the outer function is defined as follows:

**Definition 6.2.1** ( Outer functions )**.** Let $h$ be a measurable function on $\mathbb{T}$ with $\log|h| \in L^1(\mathbb{T})$. An outer function ( of absolute value $|h|$ ) is a function $f = \lambda[h]$ with $|\lambda| = 1$ and, as in Theorem 5.7.1:

$$[h](z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|h(\zeta)| dm(\zeta)\right), \; |z| < 1.$$

Below we are discussing few properties of outer functions:

**Properties 6.2.2.** (i) An outer function $f$ admits non-tangential boundary limits $\tilde{f}$. Moreover, $f \in H^p(\mathbb{D}) \Leftrightarrow \tilde{f} \in L^p(\mathbb{T})$

*Proof.* By Fatou's theorem $\lim_{r\to 1} \log|[f]|(r\xi) = \lim_{r\to 1} \int_{\mathbb{T}} P_{r\xi}(\zeta) \log|f|(\zeta) dm(\zeta) = \log|\tilde{f}|(\xi)$ exists non-tangentially a.e. on $\mathbb{T}$. Hence $|[f](\xi)| = |\tilde{f}(\xi)| \implies |[f]| = |\tilde{f}|$.

If $\tilde{f} \in L^p(\mathbb{T})$ then $[f] \in H^p(\mathbb{D})$ follows from the Theorem 5.7.1 (i). If $[f] \in H^p(\mathbb{D})$ then $\tilde{f} \in L^p$ since $|[f]| = |\tilde{f}|$. □

(ii) Let $f \in H^p, p \geq 1$. Then $f$ is outer if and only if $E_f = \text{clos}_{H^p}(f P_a) = H^p (\Leftrightarrow f$ is cyclic in $H^p)$

(iii) If $f \in H^p$ and $\frac{1}{f} \in H^q$ $(p > 0, q > 0)$, then $f$ is outer.

*Proof.* $f = \lambda_1 B_1 S_1 [f]$ and $\frac{1}{f} = \lambda_2 B_2 S_2 [\frac{1}{f}] \implies \frac{1}{\lambda_1 B_1 S_1 [f]} = \lambda_2 B_2 S_2 [\frac{1}{f}] \implies 1 = \lambda B S [\frac{f}{f}] = \lambda B S \implies B = 1, S = 1($ since $|B| < 1, |S| < 1$ on $\mathbb{T})$ Similarly, $B_1 = B_2 = 1$ and $S_1 = S_2 = 1$. Hence $f$ is an outer ($\frac{1}{f}$ is also an outer.) □

(iv)

**Theorem 6.2.3.** *(Smirnov, 1928)*

*(a) If $f \in \text{Hol}(\mathbb{D})$ and $\text{Re}\, f(z) \geq 0$ for all $z \in \mathbb{D}$, then $f \in H^p$, $0 < p < 1$ (but perhaps $f \notin H^1(\mathbb{D})$) . Moreover, $f$ is an outer.*

*Proof.* Note that $\text{Re}\, f(z) \geq 0, \forall z \in \mathbb{D} \implies \text{Re}\, f(z) > 0, \forall z \in \mathbb{D}$. Indeed if there exists a point $z_0 \in \mathbb{D}$ such that $\text{Re}\, f(z_0) = 0$ then by maximal/minimum principle for harmonic functions $\text{Re}\, f = 0$ on $\mathbb{D}$, so $f$ is constant, identically equal to 0, a contradiction [see [11] p.150.]

As the values of $f$ are in the right-half plane:

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Re}(z) \geq 0\}$$

the function $z \longmapsto (f(z))^p$ is analytic and we can choose $\arg f(z)$ such that $|\arg f(z)^p| \leq p\pi/2$, $z \in \mathbb{D}$. Hence if $0 < p < 1$, then there exists $c_p > 0$ such that $|f(z)|^p \leq c_p$ $\mathrm{Re} f(z)^p$ [since $\mathrm{Re}\ f(z)^p = |f(z)|^p \cos(\arg(f(z))^p)$]. The MVT applied to the harmonic function $\mathrm{Re} f(z)^p$ gives

$$\int_0^{2\pi} |f(re^{i\theta})|^p \frac{dt}{2\pi} \leq \int_0^{2\pi} \mathrm{Re}(f(re^{i\theta})^p)/\cos(\pi p/2)\frac{dt}{2\pi} = \mathrm{Re}(f(0)^p)/\cos(\pi p/2)$$

for $0 \leq r < 1$. Hence $f \in H^p(\mathbb{D})$, $0 < p < 1$.

■Moreover, since $\mathrm{Re}\ (\frac{1}{f(z)}) \geq 0$ in $\mathbb{D}$, we have $f$ and $\frac{1}{f}$ in $H^p$, $0 < p < 1$. By Property (iii), $f$ is an outer function. □

(b) *More generally, if* $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) \neq 0$ *and* $\alpha := \sum_{z\in\mathbb{D}} |\arg(f(z))| < \infty$ *then* $f$ *is outer and* $f \in H^p(\mathbb{D})$ *for every* $0 < p < \pi/2\alpha$ *(but perhaps* $f \neq H^{\frac{\pi}{2\alpha}}(\mathbb{D})$*.)*

*Proof.* Apply the first case to $g = f^{\pi/2\alpha}$. □

(c) *For every* $h \in L^1(\mathbb{T})$, $\Gamma h \in \cap_{0<p<1} H^p(\mathbb{D})$ *for every* $0 < p < \pi/2\alpha$ *where*

$$\Gamma h(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} h(\zeta) dm(\zeta)$$

*Proof.* If $h \geq 0$ then $\mathrm{Re}\ \Gamma h(z) \geq 0$ in $\mathbb{D}$, hence $\Gamma h \in \cap_{0<p<1} H^p(\mathbb{D})$. The general case follows from $h = h_1 - h_2 + ih_3 - ih_4$ where $0 \leq h_j \leq |h|$. □

**Remark:** *By the Herglot'z Theorem 5.7.2, the general form of a function* $f \in \mathrm{Hol}(\mathbb{D})$ *with* $\mathrm{Re}(f) \geq 0$ *is*

$$\Gamma\mu(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) + ic$$

*where* $\mu$ *is a positive measure on* $\mathbb{T}$ *and* $c \in \mathbb{R}$.

**Example 6.2.4.** (Herglotz Integral) Let $\mu \in \mathcal{M}(\mathbb{T})$ such that

$$f_\mu = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu(\xi).$$

Then $f_\mu \in H^p, 0 < p < 1$ since $\mathrm{Re}\, f_\mu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2} d\mu = \int_{\mathbb{T}} P_z(\xi) d\mu \geq 0$ if $\mu \geq 0$ and $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ where $0 \leq \mu_j \leq |\mu|$.

■ If $\mu \geq 0$ then also $\mathrm{Re}(\frac{1}{f_\mu}) \geq 0 \implies \frac{1}{f_\mu} \in H^p$, hence $f_\mu$ is an outer.

**Example 6.2.5.** (Cauchy Integral) If $f$ is integrable then $F(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{it} f(e^{it})}{e^{it} - z} dt \implies F(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{it}}{e^{it} - z} d\mu(t)$. If $\mu \geq 0$ then $\mathrm{Re}\{\frac{e^{it}}{e^{it} - z}\} = \frac{1 - r\cos(t-\theta)}{1 - 2r\cos(t-\theta) + r^2} > 0$. Hence $f \in H^p, 0 < p < 1$.

(v) If $f \in H^\infty$ and $\|f\|_\infty \leq 1$, then $1 + f$ is outer.

*Proof.* $\mathrm{Re}(1+f) \geq 0$ and apply Theorem 6.2.3 (a) □

(vi) The set of outer functions is a commutative group for standard point-wise-point multiplication.

(vii) Let $f,\, g \in H^p (p > 0)$

(a) Then $fg$ is outer if and only if $f, g$ are outer.

*Proof.* Let $f = \lambda_1 B_1 S_1 [f]$ and $g = \lambda_2 B_2 S_2 [g]$, hence $fg = (\lambda_1\lambda_2) B_1 B_2 S_1 S_2 [fg]$, then use the uniqueness part of the Smirnov Canonical Factorization Theorem 5.7.6. □

(b) Let $f$ be an outer function and let $|f| \leq |g|$, then $g$ is an outer.

*Proof.* Obviously, $\frac{f}{g} \in H^\infty$ and $\frac{f}{g}$ has no zeros in $\mathbb{D}$. By Theorem 5.7.6 we get the representation $\frac{f}{g} = \lambda S F$, where $F$ is outer. Suppose that $g$ is not outer. Then $g = \lambda_1 S_1 F_1$ with $S_1$ is a non-trivial singular inner function and $f = (\lambda\lambda_1)(SS_1)(FF_1)$ with $SS_1 \not\equiv$ constant, which contradicts the hypothesis. □

(c) If $f \in H^p(\mathbb{D})$, $p \geq 1$ and $\inf\limits_{z\in\mathbb{D}} |f(z)| > 0$, then $f$ is outer.

*Proof.* It is clear that for $g \in H^q$ $(q \geq 1)$ we have $\frac{g}{f} \in H^q$ and hence by Theorem 5.7.7 (iv) $f$ is outer. □

**Theorem 6.2.6.** *Let $p > 0$.*

*(i) Let $f_n \in H^p$ be a sequence of outer functions with $f_n(0) > 0$. If $|f_n| \searrow$ on $\mathbb{T}$, then $f(z) = \lim\limits_{n\to\infty} f_n(z)$, $z \in \mathbb{D}$ exists uniformly on compact sets. Moreover, if $\lim\limits_{n\to\infty} f_n(0) = 0$, then $f \equiv 0$, otherwise $f$ is an outer $H^p$ function.*

*(ii) Let $f \in H^p$ be an outer function. Then there exists a sequence of outer functions $f_n \in H^p$ and $\inf\limits_{z\in\mathbb{D}} |f_n(z)| > 0$, $n \geq 1$, $|f_n| \searrow |f|$ on $\mathbb{T}$ (and hence on $\mathbb{D}$) and $f(z) = \lim\limits_{n\to\infty} f_n(z)$, $z \in \mathbb{D}$.*

*Proof.* (i) As the functions $f_n$ are outer, we have

$$\log |f_n(z)| = \int_{\mathbb{T}} P(z\bar{\xi}) \log |f_n(\xi)| dm(\xi).$$

To show the uniform convergence of $f_n$, it is enough to show that $f_n$ is uniformly Cauchy sequence. For this, we will show $\log|f_n(z)|$ is a uniformly Cauchy.

$$\begin{aligned} |\log|f_n(z)| - \log|f_{n+p}(z)|| &= \Big|\int_{\mathbb{T}} P(z\bar{\xi}) \log \frac{|f_n(\xi)|}{|f_{n+p}(\xi)|} dm(\xi)\Big| \\ &\leq \sup_{|z|\leq R} |P(z\bar{\xi})| \int_{\mathbb{T}} \Big|\log \frac{|f_n(\xi)|}{|f_{n+p}(\xi)|}\Big| dm(\xi) \\ &= \text{const} \int_{\mathbb{T}} \log \frac{|f_n(\xi)|}{|f_{n+p}(\xi)|} dm(\xi) \\ &= \text{const} \Big( \int_{\mathbb{T}} \log|f_n(\xi)| dm(\xi) - \int_{\mathbb{T}} \log|f_{n+p}(\xi)| dm(\xi) \Big). \end{aligned}$$

The conclusion is followed by monotone convergence theorem.

Suppose that $\inf\limits_{n\geq 1} f_n(0) = 0$, then

$$\lim_{n\to\infty} \int_{\mathbb{T}} \log|f_n| dm = \lim_{n\to\infty} \log f_n = -\infty.$$

For a point $z_0 \in \mathbb{D}$, we have $P(z_0\bar{\xi}) \leq \frac{1+|z_0|}{1-|z_0|} = C_0$. Hence,

$$\log|f_n(z_0)| \leq C_0 \int_{\mathbb{T}} \log|f_n| dm.$$

We conclude that $\lim\limits_{n\to\infty} \log|f_n(z_0)| = -\infty$ and similarly for all $z \in \mathbb{D}$ and we get $f \equiv 0$. If $\inf\limits_{n\geq 1} f_n(0) > 0$ and $|f_n| \searrow h$ on $\mathbb{T}$, then

$$\int_{\mathbb{T}} \log h dm = \lim_{n\to\infty} \int_{\mathbb{T}} \log|f_n| dm > -\infty,$$

and hence $\log h \in L^1$. Now, it is obvious that $\lim\limits_{n\to\infty} f_n(z) = f(z)$ with $f = [h]$.

(ii) Without loss of generality, we may assume that $f(0) > 0$. Set $f_n = [|f| + \delta_n]$, where $\delta_n > 0$ an appropriate sequence with $\lim\limits_{n\to\infty} \delta_n = 0$ and $\int_{\mathbb{T}} \log(|f| + \delta_n) dm < \infty$. Then $f_n$ satisfies the desired properties.

□

## 6.3 The Nevanlinna ($N$) and Smirnov ($N_+$) classes

We know that Nevanlinna class can be represented as

$$N = \Big\{ f \in \text{Hol}(\mathbb{D}) : \text{ there exist } f_1, f_2 \in \bigcup_{p>0} H^p \text{ such that } f = f_1/f_2 \Big\}$$

and let

$$\mathcal{D} = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \text{ there exist } f_1, f_2 \in \bigcup_{p>0} H^p \text{ such that } f = f_1/f_2 \text{ and } f_2 \text{ is outer} \Big\}$$

be the **Smirnov class** (sometimes denoted by $N_+$).

**Lemma 6.3.1.** *We have*

$$N = \{ f \in \mathrm{Hol}(\mathbb{D}) : \textit{ there exist } f_1, f_2 \in H^\infty \textit{ such that } f = f_1/f_2 \} \textit{ and}$$

$$\mathcal{D} = \{ f \in \mathrm{Hol}(\mathbb{D}) : \textit{ there exist } f_1, f_2 \in H^\infty \textit{ such that } f = f_1/f_2 \textit{ and } f_2 \textit{ is outer} \}.$$

*Proof.* Let $f \in N, f \not\equiv 0$ and $f = \frac{f_1}{f_2}$, where $f_1, f_2 \in H^p$'s have canonical factorizations $f_1 = \lambda[f_1]B_1S_1$ and $f_2 = \lambda[f_2]S_2$. Set $F_1 = \lambda[\min(1, |f|)]B_1S_1$ and $F_2 = [\min(|f|^{-1}, 1)]S_2$. Clearly $F_1, F_2 \in H^\infty$ and since $|f|. \min(|f|^{-1}, 1) = \min(1, |f|)$, we also get $f = \frac{F_1}{F_2}$.

$$\begin{aligned} [|f|].[\min(|f|^{-1}, 1)] &= [\min(1, |f|)] \\ \implies [f] = [|f|] &= \frac{[\min(1, |f|)]}{[\min(|f|^{-1}, 1)]} \end{aligned}$$

Now, $\frac{F_1}{F_2} = \frac{\lambda[\min(1,|f|)]B_1S_1}{[\min(|f|^{-1},1)]S_2} = \lambda[f]\frac{B_1S_1}{S_2} = \lambda[f]B_1S_3$.

Hence $f = \frac{f_1}{f_2} = \frac{\lambda[f_1]B_1S_1}{[f_2]S_2} = \lambda[\frac{f_1}{f_2}]B_1S_3 = \lambda[f]B_1S_3 = \frac{F_1}{F_2}$. □

**Definition 6.3.2.** (Outer in Nevanlinna class) A function $f \in N$ is called outer if there exist two outer functions $f_1, f_2$ such that $f = \frac{f_1}{f_2}$.

**Properties 6.3.3.** (of the class $\mathcal{D}$ and Nevanlinna outer functions)

(a) If $f$ is outer, then $f \in \mathcal{D}$.

(b) If $f_1$ and $f_2$ is outer, then so is $f_1 f_2$.

(c) If $f_1 f_2$ are outer, and $f_1, f_2 \in \mathcal{D}$, then $f_1, f_2$ are outer.

(d) If $f_1, f_2 \in \mathcal{D}$, then $f_1 f_2 \in \mathcal{D}$.

(e) If $F \in \mathrm{Hol}(\mathbb{D})$, $G \in \mathcal{D}$ and $|F| \le |G|$ in $\mathbb{D}$, then $F \in \mathcal{D}$.

To verify (c), just let $G = \frac{G_1}{G_2}$ with $G_1, G_2 \in H^\infty$, and $G_2$ outer. By hypothesis $|G_2 F| \le |G_1|$ in $\mathbb{D}$, and hence $G_2 F \in H^\infty$. We conclude that $F = \frac{G_2 F}{G_2} \in \mathcal{D}$

**Theorem 6.3.4.** (*Generalized Maximum Principle*) *Let $f \in \mathcal{D}$ and $g$ be an outer function in $N$. If $|f| \le |g|$ on $\mathbb{T}$, then $|f| \le |g|$ on $\mathbb{D}$.*

*Proof.* Let $f = \frac{f_1}{f_2}$ and $g = \frac{g_1}{g_2}$ where $f_2, g_1$ and $g_2$ are outer functions in $H^\infty$ and $f_1 \in H^\infty$. By assumption $|f_1 g_2| \le |f_2 g_1|$ on $\mathbb{T}$ and hence $|f_1 g_2| \le |[f_1 g_2]| \le |[f_2 g_1]| = |f_2 g_1|$ in $\mathbb{D}$. □

*Remark* 6.3.5. This result is not true in general if $f \in N \setminus \mathcal{D}$ and/or if $g$ is not outer.

Let us recall that by Fatou's theorem every $f \in H^\infty$ has a non-tangential limit a.e. on $\mathbb{T}$ and the boundary function satisfies:

$$\int_{\mathbb{T}} \log|f| dm > -\infty,$$

that means the non-tangential limits of $f$ are non-zero a.e. From here we see that:

**Proposition 6.3.6.** *Every function in $N$ class has a non-tangential limit a.e. on $\mathbb{T}$.*

**Proposition 6.3.7.** $H^p \subset N_+$

*Proof.* Hint: If $f \in H^p \setminus \{0\}$ then $f = \lambda BS[f]$ where

$$[f](z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|f(\zeta)| dm(\zeta)\right).$$

Now $\log = \log^+ - \log^-$ and consider $f_1, f_2$ corresponding to functions $\log^+$ and $\log^-$. □

So we have the relation: $H^p \subset N_+ \subset N$.

**Theorem 6.3.8.** *(Smirnov Theorem) $f \in N_+$ and its boundary limit function belongs to $L^p$ then $f \in H^p$ i.e. $N_+ \cap\ L^p = H^p$.*

*Proof.* The proof depends on the Arithmetic-Geometric Mean Inequality:

$$\exp\left(\int_{\mathbb{T}} \log h d\sigma\right) \le \int_{\mathbb{T}} h d\sigma,$$

where $h$ is a non-negative function on $\mathbb{T}$ which is integrable.
If $f \in N^+$ then $f = g_1/g_2$ where $g_1, g_2 \in H^\infty$ and $g_2$ is outer. Since the presence of an inner factor in $g_1$ will not affect whether or not $f \in H^2$, we can also assume that $g_1$ is also an outer. Using the definition of an outer function applied to functions $g_1$ and $g_2$ we see that

$$\frac{g_1}{g_2}(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \frac{|g_1(\zeta)|}{|g_2(\zeta)|} dm(\zeta).\right)$$

Furthermore, for each $r \in (0, 1)$ and $w \in \mathbb{T}$

$$\left|\frac{g_1}{g_2}(rw)\right|^2 = \exp\left(\int_{\mathbb{T}} P_{rw}(\zeta) \log \frac{|g_1(\zeta)|^2}{|g_2(\zeta)|^2} dm(\zeta\right)$$

Now apply the Arithmetic-Geometric Mean inequality to the function $|g_1/g_2|$ and the measure $P_{rw} dm$:

$$\left|\frac{g_1}{g_2}(rw)\right|^2 \le \int_{\mathbb{T}} \left|\frac{g_1}{g_2}(\zeta)\right|^2 P_{rw} dm(\zeta). \tag{6.3.1}$$

Integrate both sides:

$$\begin{aligned}
\int_{\mathbb{T}} |f(rw)|^2 dm(w) &= \int_{\mathbb{T}} \frac{g_1}{g_2}(rw)|^2 dm(w) \\
&\leq \int_{\mathbb{T}} \left( \int_{\mathbb{T}} \left| \frac{g_1}{g_2}(\zeta) \right|^2 P_{rw}(\zeta) dm(\zeta) \right) dm(w) \\
&= \int_{\mathbb{T}} \left( \int_{\mathbb{T}} |f(\zeta)|^2 P_{rw}(\zeta) dm(\zeta) \right) dm(w) \\
&= \int_{\mathbb{T}} |f(\zeta)|^2 \left( \int_{\mathbb{T}} P_{rw}(\zeta) dm(w) \right) dm(\zeta) \\
&= \int_{\mathbb{T}} |f|^2 dm
\end{aligned}$$

Thus $\sup_{0<r<1} \int_{\mathbb{T}} |f(rw)|^2 dm(w) \leq \int_{\mathbb{T}} |f|^2 dm$, which implies $f \in H^2$.
To prove the second statement of the theorem, observe that if $f \in N^+$ and $f|_{\mathbb{T}} \in L^\infty$ then as before we can assume $f = g_1/g_2$ and $g_1, g_2$ are bounded outer functions. By (6.3.1) we see that

$$\begin{aligned}
|f(rw)|^2 = \left| \frac{g_1}{g_2}(rw) \right|^2 &\leq \int_{\mathbb{T}} \left| \frac{g_1}{g_2}(\zeta) \right|^2 P_{rw}(\zeta) dm(\zeta) \\
&= \int_{\mathbb{T}} |f(\zeta)|^2 P_{rw}(\zeta) dm(\zeta) \\
&\leq \|f|_{\mathbb{T}}\|_\infty^2 \int_{\mathbb{T}} P_{rw} dm(zeta) \\
&= \|f|_{\mathbb{T}}\|_\infty^2,
\end{aligned}$$

which implies $f \in H^\infty$. □

*Remark* 6.3.9. Smirnov's theorem is no longer true for $f \in N$ even when $f$ is analytic on $\mathbb{D}$. For instance the function

$$f(z) = \exp\left(\frac{1+z}{1-z}\right)$$

which is the reciprocal of the atomic inner function described in Example 5.9.4 belongs to $N$ class, analytic on $\mathbb{D}$ and has boundary values of unit modulus a.e. on $\mathbb{T}$. However it does not belongs to $H^2$ since as in Remark 5.9.5

$$|f(r)| = \exp\left(\frac{1+r}{1-r}\right), r \in (0,1)$$

which does not satisfy the necessary growth condition for an $H^2$ function as described in see[8](p. 59):

$$|f(\lambda)| \leq \frac{\|f\|}{\sqrt{1-|\lambda|^2}}, f \in H^2.$$

The original definition of the Nevanlinna class is different from definition in 6.3. $f \in N$ if and only if

$$\sup_{0\le r<1}\int_{\mathbb{T}} \log^+ |f(r\xi)| d\xi < \infty.$$

The equivalence of the two definitions is not at all obvious; the proof can be found in Nevanlinna and Nevanlinna (1922), Privalov (1941), Duren (1970) [12][p.16], and Koosis (1998) [4] . We will state the theorem as follows:

**Theorem 6.3.10.** *[12][p.16] A function analytic in the unit disk belong to the class $N$ if and only if it is the quotient of two bounded analytic function.*

*Proof.* ($\Leftarrow$) Suppose first that $f(z) = \varphi(z)/\psi(z)$ where $\varphi, \psi$ are analytic and bounded in $\mathbb{D}$. There is no loss of generality in assuming $|\varphi(z)| \le 1, |\psi(z)| \le 1$ and $\psi(0) \ne 0$. Then

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \le -\int_0^{2\pi} \log |\psi(re^{i\theta})| d\theta.$$

But by Jensen's formula (see Ahlforse, p. 206)

$$\frac{1}{2\pi}\int_0^{2\pi} \log |\psi(re^{i\theta})| d\theta = \log |\psi(0)| + \sum_{|x_n|<r} \log \frac{r}{|z_n|},$$

where $z_n$ are zeroes of $\psi$. This shows that $\int \log |\psi|$ increases with $r$, so $f \in N$.
($\Rightarrow$) Let $f(z) \not\equiv 0$ be of class $N$. Let $f$ has a zero of multiplicity $m \ge 0$ at the origin, so that $z^{-m} f(z) \to \alpha \ne 0$ as $z \to 0$. Let $z_n$ be the other zeroes of $f$, repeated according to multiplicity and arranged so that $0 < |z_1| \le |z_2| \le \cdots < 1$. If $f(z) \ne 0$ on the circle $|z| = \rho < 1$, the function

$$F(z) = \log \left\{ f(z) \frac{\rho^m}{z^m} \Pi_{|z_n|<\rho} \left( \frac{\rho^2 - \overline{z}_n z}{\rho(z - z_n)} \right) \right\}$$

is analytic in $|z| \le \rho$, and $\operatorname{Re} F(z) = \log |f(z)|$ on $|z| = \rho$. Hence by analytic completion of the Poisson formula:

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \log |f(\rho e^{it})| \frac{\rho e^{it} + z}{\rho e^{it} - z} dt + iC.$$

This is sometimes called the Poisson-Jensen formula. After exponentiation, it takes of the form $f(z) = \varphi_\rho(z)/\psi_\rho(z)$ where

$$\varphi_\rho(z) = \frac{z^m}{\rho^m} \Pi_{|z_n|<\rho} \frac{\rho(z - z_n)}{\rho^2 - \overline{z}_n z} . \exp \left\{ -\frac{1}{2\pi}\int_0^{2\pi} \log^- |f(\rho e^{it})| \frac{\rho e^{it} + z}{\rho e^{it} - z} dt + iC \right\}$$

$$\psi_\rho(z) = \exp \left\{ -\frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(\rho e^{it})| \frac{\rho e^{it} + z}{\rho e^{it} - z} dt \right\}$$

Now choose a sequence $\{\rho_k\}$ increasing to 1 such that $f(z) \ne 0$ on the circles $|z| = \rho_k$. Let

$\Phi_k(z) = \varphi_{\rho_k}(\rho_k z;); \Psi_k(z) = \psi_{\rho_k}(\rho_k z)$. Then $f(\rho_k z) = \Phi_k(z)/\Psi_k(z)$ in $\mathbb{D}$. But the functions are analytic in the unit disk, and $|\Phi_k(z)| \le 1$, $|\Psi_k(z)| \le 1$. Hence $\{\Phi_k\}$ and $\{\Psi_k\}$ are normal families, and there exists a sequence $\{k_i\}$ such that $\Phi_{k_i}(z) \to \varphi(z)$ and $\Psi_{k_i}(z) \to \psi(z)$ uniformly in each disk $|z| \le R < 1$. The function $\varphi, \psi$ are analytic in unit disk and $|\varphi(z)| \le 1, |\psi(z)| \le 1$. According to the definition of $\psi_\rho$ the fact that $\int \log^+ |f|$ is bounded gives a uniform estimate $|\Psi_k(0)| \ge \delta > 0$, so $\psi(z) \not\equiv 0$. Thus $f = \varphi/\psi$ and the proof is completed. □

The importance of this theorem is that it allows properties of functions in $N$ to be deduced from the corresponding properties of bounded analytic functions. The boundary behavior, for example, can now be discussed.

**Theorem 6.3.11.** *For each $f \in N$, the non-tangential limit $f(e^{i\theta})$ exists almost everywhere and $\log|f(e^{i\theta})|$ is integrable unless $f(z) \equiv 0$. If $f \in H^p$ for some $p > 0$, then $f(e^{i\theta}) \in L^p$.*

*Proof.* Assuming $f(z) \not\equiv 0$, represent in the form $\varphi(z)/\psi(z)$, where $|\varphi(z)| \le 1$ and $|\psi(z)| \le 1$. Since $\varphi$ and $\psi$ are bounded analytic functions, they have non-tangential limits $\varphi(e^{i\theta})$ and $\psi(e^{i\theta})$ almost everywhere. Appleaing to Fatou's Lemma we have

$$\int_0^{2\pi} |\log|\varphi(e^{i\theta})||d\theta \le \liminf_{r\to 1} \left\{ -\int_0^{2\pi} \log|\varphi(re^{i\theta})d\theta \right\}$$

But $\int \log \varphi(re^{i\theta})d\theta$ increases with $r$, by Jensen's theorem. Hence $\log|\varphi(e^{i\theta})| \in L^1$ and similarly for $\psi$. In particular $\psi(e^{i\theta})$ cannot vanish on a set of positive measure. The radial limit $f(e^{i\theta})$ therefore exists almost everywhere, and $\log|f(e^{i\theta})| \in L^1$. Another application of Fatou's lemma shows that $f(e^{i\theta}) \in L^p$ if $f \in H^p$. □

The theorem says that if $f \in N$ and if $f(e^{i\theta}) = 0$ on a set of positive measure, then $f(z) \equiv 0$. In other words, a function of class $N$ is uniquely determined by its boundary values on any set of positive measure.

It is evident from the representation $f = \varphi/\psi$ that $\int \log^- |f(re^{i\theta})|d\theta$ is bounded if $f \in N$. Hence $f \in N$ if and only if $\int \left|\log|f(re^{i\theta})|\right| d\theta$ is bounded.

## 6.4 A conformally invariant framework

Here we consider the classes $\mathrm{Nev}(\Omega)$ and $\mathcal{D}(\Omega)$, where $\Omega$ is a simply connected domain ($\neq \mathbb{C}$), that is, domains that are conformally equivalent to the open unit $\mathbb{D}$.

**Definition 6.4.1.** Define

$$H^\infty(\Omega) = \{f \in \mathrm{Hol}(\Omega) : \|f\|_{H^\infty} = \sup_{z\in\Omega} |f(z)| < \infty\}$$

and

$$N(\Omega) = \{f \in \mathrm{Hol}(\Omega) : \text{ there exist } f_1, f_2 \in H^\infty(\Omega) \text{ such that } f = f_1/f_2\}.$$

For $\omega : \mathbb{D} \to \Omega$ be an onto conformal map. A function $f \in \text{Nev}(\Omega)$ is called outer if $f \circ w$ is an outer in $\text{Nev}(\mathbb{D})$. With this definition, we get

$$\mathcal{D}(\Omega) = \{f \in \text{Hol}(\Omega) : \text{ there exist } f_1, f_2 \in H^\infty(\Omega) \text{ such that } f = f_1/f_2 \text{ and } f_2 \text{ is outer}\}.$$

The following two results are simple factorization to $\Omega$ of the corresponding well known facts in $\mathbb{D}$. Note if $\omega : \Omega \to \mathbb{D}$ extends to a homeomorphism of clos $(\Omega)$ onto clos $(\mathbb{D})$, then we say $\Omega$ is **Jordan domain**.

**Lemma 6.4.2.** (*Generalized Maximum Principle*) *Let $\Omega$ be a Jordan domain. Let $\lambda \in \partial\Omega$, $f \in \mathcal{D}(\Omega) \cap C(\text{clos}(\Omega) \setminus \{\lambda\})$ and let $g$ be an outer function such that $g \in C(\text{clos}(\Omega) \setminus \{\lambda\})$ and $|f| \le |g|$ on $\partial\Omega \setminus \{\lambda\}$. Then $|f| \le |g|$ on $\Omega$.*

**Lemma 6.4.3.** *Let $f \in H^\infty(\Omega)$. Then $f$ is outer if and only if there exists a sequence of outer functions $(f_n)_{n\ge1} \in H^\infty(\Omega)$ such that*

$$\inf_{z\in\Omega} |f_n(z)| > 0,\ n \ge 1, \quad \lim_{n\to\infty} f_n(z) = f(z), \quad |f_n(z)| \searrow |f(z)|, z \in \Omega.$$

**Corollary 6.4.4.** *Let $\Omega_1 \subset \Omega_2$ be two simply connected domains and $f \in N(\Omega_2)$.*

*(i) If $f$ is outer on $\Omega_2$, then $f|_{\Omega_1}$ is outer on $\Omega_1$.*

*(ii) If $f \in \mathcal{D}(\Omega_2)$, then $f|_{\Omega_1} \in \mathcal{D}(\Omega_1)$.*

## 6.5 The generalized Phragmén–Lindelöf principle

The result of Theorem 6.3.4 and Lemma 6.4.2 are, in fact, the versions of the Phragmén–Lindelöf principle. The difference is that, in general, the mejorants are not given by analytic functions.

Let $\Omega$ be a Jardon Domain, let $M$ and $M_*$ be two non-negative functions on $\Omega$, and let $\omega \in C(\partial\Omega \setminus \{\lambda\})$, where $\lambda \in \partial\omega$, $\Omega > 0$. Then $M_*$ is called Phragmén–Lindelöf majorant for $M$ and $\omega$ if for every $f \in \text{Hol}(\Omega) \cup C(\text{clos}(\Omega) \setminus \{\lambda\})$ with $|f| \le M$ on $\partial\Omega \setminus \{\lambda\}$ we have $|f| \le M_*$.

**Theorem 6.5.1.** (*Generalized Phragmén–Lindelöf principle*) *Let $f \in \mathcal{D}(\Omega)$ and $G \in N(\Omega) \cap C(\text{clos}(\Omega) \setminus \{\lambda\})$ be such that $M \le |F|$ on $\Omega$, $\omega \le |G|$ on $\partial\Omega \setminus \{\lambda\}$. Then either there exists an outer function $[\omega \circ \omega]$ (and then $M_* = |[\omega \circ \omega] \circ \omega^{-1}|$ is a Phragmén–Lindelöf majorant for $M$ and $\omega$) or $f \equiv 0$ for all $f \in \text{Hol}(\Omega) \cup C(\text{clos}(\Omega) \setminus \{\lambda\})$ such that $|f| \le M$ on $\Omega$ and $|f| \le \omega$ on $\partial\Omega\{\lambda\}$ (and then $M_* = 0$).*

*Proof.* In view of ($e$) of Properties 6.3.3, the inequalities $|F| \le M \le |F|$ show that $f \in (\Omega)$. If there exists $f \not\equiv 0$, $f \in N(\Omega)$ such that

$$|f \circ \omega| \le \omega \circ \omega \le |G \circ \omega|$$

on $\mathbb{T} \setminus \omega^{-1}(\{\lambda\})$, then we can define the outer function $[\omega \circ \omega]$. Applying Lemma 6.4.2 we get $|f \circ \omega| \leq |[\omega \circ \omega]|$ on $\mathbb{T} \setminus \omega^{-1}(\{\lambda\})$ and hence the desired result. □

## 6.6 Exercises

**Exercise 6.6.1.** Let $b$ be a non-constant function in the closed unit ball of $H^\infty$. Put

$$f = \frac{1}{1-b}.$$

Then $f$ is an outer in $H^p$, for $0 < p < 1$.

*Proof.* Since $b \in \{\|f\|_\infty \leq 1 : f \in H^\infty\}$ by the maximal principle, $|b(z)| < \|b\|_\infty \leq 1$ for each $z \in \mathbb{D}$. Hence $f$ is analytic on $\mathbb{D}$. Moreover we have

$$\operatorname{Re} \frac{1}{1-b(z)} = \frac{1-\operatorname{Re} b(z)}{|1-b(z)|^2} \geq \frac{1-|b(z)|^2}{|1-b(z)|^2} > 0uad(z \in \mathbb{D}).$$

Hence by Smirnov Theorem 6.2.3, $f$ is an outer in $H^p$. □

**Exercise 6.6.2.** If a polynomial $p$ has no zero in the open disc $\mathbb{D}$, then $p$ is outer.

*Proof.* Consider $p(z) = \text{const} \prod_{i=1}^{n} \Big(1 - \dfrac{z}{\xi_i}\Big)$, $|\xi_i| \geq 1$. As $|z| < 1$ and $|\xi_i| \geq 1$, we have $\operatorname{Re}\big(1 - \frac{z}{\xi_i}\big) \geq 0$. By applying Theorem 6.2.3 and the Property ( (a).) □

# Chapter 7

# Harmonic analysis in $L^2(\mathbb{T}, \mu)$

The main result of this section is the Helson- Szegö theorem characterizing those $L^2(\mathbb{T}, \mu)$ in which the Fourier series of every function $f \in L^2(\mathbb{T}, \mu)$ converges in the norm topology. This is one of the main results of harmonic analysis on the circle group $\mathbb{T}$. It is closely related to generalized Fourier series with respect to a minimal sequence; harmonic conjugates, the Riesz projections, and weighted estimates for Hilbert singular integrals.

**Definition 7.0.1.** A sequence $(x_n)_{n\geq 1}$ in Banach Space $X$ is called **minimal** if $x_n \notin M_n = \overline{\text{span}}\{x_k : k \neq n\}$, and is called **uniformly minimal** if $\inf\limits_{n\geq 1} \text{dist}\left(\frac{x_n}{\|x_n\|}, M_n\right) > 0$.

To proceed we need a corollary of the Hahn Banach Theorem.

**Proposition 7.0.2.** *Let $M$ be a linear subspace of a normed linear space $X$, and let $x_0 \in X$. Then $x_0 \in \overline{M}$ if and only if there does not exists a bounded linear functional $f$ on $X$ such that $f(x) = 0$ $\forall x \in M$ but $f(x_0) \neq 0$ (in fact it is $1$).*

*Proof.* ($\Longleftarrow$) If $x_0 \in \overline{M}$, $f$ is a bounded linear functional on $X$ and $f(x) = 0$ $\forall x \in M$. The continuity of $f$ shows that $f(x_0) = 0$ (since $x_0 \in \overline{M}$). So there does not exists a bounded linear functional $f$ on $X$ such that $f(x) = 0$ $\forall x \in M$.
($\Longrightarrow$) $x_0 \notin \overline{M}$. Then $\exists$ a $\delta > 0$ such that $\|x - x_0\| > \delta, \forall x \in M$. Let $M'$ be the subspace generated by $M$ and $x_0$ and define $f : M' \to \mathbb{C}$ by $f(x + \lambda x_0) = \lambda$ if $x \in M$ and $\lambda$ is a scaler. Since $\delta|\lambda| \leq |\lambda|\|x_0 + \overline{\lambda}x\| = \|\lambda x_0 + x\| \implies |f(x + \lambda x_0)| = |\lambda| \leq \frac{1}{\delta}\|\lambda x_0 + x\|$. Also $f(x) = 0$ on $M$ and $f(x_0) = 1$. By the Hahn Banach Theorem there exists unique $\tilde{f}$ which extends $f$ from $M'$ to $X$. $\square$

**Lemma 7.0.3.** *(i) A sequence $(x_n)_{n\geq 1} \subset X$ is minimal if and only if there exists $f_n \in X^*$ such that $(x_k, f_n) = \delta_{kn}$. Such a pair $((x_n)_{n\geq 1}, (f_k)_{k\geq 1})$ will be called biorthonormal and $f_n$, $n \geq 1$ coordinate functionals.*

*(ii) $(x_n)_{n\geq 1} \subset X$ is uniformly minimal if and only if there exists a sequence $(f_n)_{n\geq 1}$ of coordinate functionals such that $\sup\limits_{n\geq 1} \|x_n\| \|f_n\| < \infty$.*

*Proof.* (i) By Hahn-Banach theorem, if $x_n \notin M_n$, then there exists a sequence $f_n \in X^*$ with $\|f_n\| = 1$, $f_n(x_n) = \|x_n\|$, $\tilde{f}_n(x_n) = 1$, $\tilde{f}_n = \frac{f_n}{\|x_n\|}$.

(ii) Moreover for any subspace $E \subset X$,

$$\operatorname{dist}(x, E) = \sup\{|f(x)| : f \in X^*, f|_{\mathbb{E}} \equiv 0, \|f\| \leq 1\}.$$

For this, if $x \in E$ then both sides are equal. So firstly we will show "$\leq$". When $x \notin E$, by Hahn- Banach theorem there exists $\tilde{f} \in X^*$ such that $\tilde{f}(x) = \operatorname{dist}(x, E)$, and $\tilde{f}(E) = 0$ with $\|\tilde{f}\| \leq 1$. Implies

$$\operatorname{dist}(x, E) = |\tilde{f}(x)| \leq \sup\{|f(x)| \ : \ f \in X^*, f|_{\mathbb{E}} \equiv 0, \|f\| \leq 1\}.$$

For the other inequality, let $y \in E$, then we have

$$|f(x)| = |f(x-y)| \leq \|f\|\|x-y\| \leq \|x-y\|,$$

and hence $|f(x)| \leq \inf\limits_{y \in E} \|x-y\| = \operatorname{dist}(x, E)$. This implies

$$\sup\{|f(x)| : f \in X^*, f|_{\mathbb{E}} \equiv 0, \|f\| \leq 1\} \leq \operatorname{dist}(x, E).$$

Thus,

$$\sup\{|f(x)| : f \in X^*, f|_{\mathbb{E}} \equiv 0, \|f\| \leq 1\} = \operatorname{dist}(x, E).$$

Now, replacing $f$ by $f/f(x)$, it follows that

$$\inf \Big\{\|f\| : f \in X^*, f|_{\mathbb{E}} \equiv 0, f(x) = 1\} = \frac{1}{\operatorname{dist}(x, E)}.$$

(If $\phi \neq S \subset (0, \infty)$ then $\frac{1}{\sup(S)} = \inf \frac{1}{S} = \inf_{s \in S} \frac{1}{s}$)

**Main Proof:** Apply this to $x = x_n$, $E = M_n$, and let $f_n \in X^*$ be the corresponding coordinate functionals with minimal norm. Then,

$$\operatorname{dist}\Big(\frac{x_n}{\|x_n\|}, M_n\Big) = \frac{1}{\|x_n\|} \operatorname{dist}(x_n, M_n) = \frac{1}{\|x_n\|} \frac{1}{\|f_n\|}.$$

Thus,

$$\inf_{n \geq 1} \operatorname{dist}\Big(\frac{x_n}{\|x_n\|}, M_n\Big) > 0 \text{ if and only if} \sup_{n \geq 1} \|x_n\| \, \|f_n\| < \infty.$$

□

**Definition 7.0.4.** To a minimal sequence $(x_n)$ we associate the (formal) Fourier series

$$x \sim \sum_{n\geq 1}(x, f_n)x_n,\ x \in X.$$

The operator $x \longmapsto P_n x = (x, f_n)x_n$ is called the projection on the $n^{th}$ Fourier component (or the co-ordinate projection with respect to the biorthogonal pair $((x_n)_{n\geq 1}, (f_k)_{k\geq 1})$.

*Remark* 7.0.5. We have $\|P_n\| = \|f_n\|\|x_n\|$ (because $f_n(x_n) = 1$).

*Proof.* $\|P_n(x_n)\| = |f_n(x_n)|\|x_n\| = 1.\|x_n\| = \|f_n\|\|x_n\|$ (since $f_n(x_n) = 1,$ and $1 = \|f_n\|$). Also, since $P_n x = (x, f_n)x_n$ we have

$$\sup_{x\neq 0}\frac{\|P_n(x)\|}{\|x\|} \leq \|f_n\|\|x_n\|$$
$$\implies \|P_n\| = \|f_n\|\|x_n\|,$$

because at the point $x_n$ the function value attends its maximum. □

**Definition 7.0.6.** A sequence $(x_n)$ in Banach space $X$ is called a basis of $X$ if for all $x \in X$ there exists a unique sequence $(a_n) \subset \mathbb{C}$ such that $x = \sum_{k\geq 1} a_k x_k$. Note that $a_n = a_n(x)$ A sequence $x_n$ is called a basis sequence if it is basis in $\overline{\text{span}}_X\{x_n : n \geq 1\}$.

**Theorem 7.0.7.** (*S. Banach, 1932*) *Let $(x_k)$ be a basis of the Banach space X. Then $(x_k)$ is uniformly minimal and $f_k(x) = a_k(x)$, $x \in X$ are the coordinate functionals.*

**Definition 7.0.8.** Let $X$ be a Banach space and let $(x_n)_{n\in\mathbb{Z}}$ be a family in $X$. Then it is called **symmetric basis** if for all $x \in X$, there exists a unique $(a_k(x))_{k\in\mathbb{Z}} \subset \mathbb{C}$ such that $x = \lim_{n\to\infty}\sum_{k=-n}^{n} a_k(x)x_k$. It is called **non-symmetric** if $x = \lim_{n,m\to\infty}\sum_{k=-m}^{n} a_k(x)x_k$.

**Lemma 7.0.9.** *Let $\chi = (x_k)_{k\in\mathbb{Z}}$ and $(f_k)_{k\in\mathbb{Z}}$ be a biorthogonal pair in a Banach space X. Set $P_{m,n} = \sum_{k=-m}^{n} (.\,, f_k)x_k,\ \ m, n \in \mathbb{Z}$. Then*

*(i) $\chi$ is a symmetric (respectively non-symmetric) basis if and only if $\sup_{n\geq 1}\|P_{-n,n}\| < \infty$ (respectively $\sup_{m,n}\|P_{m,n}\| < \infty$) and $\chi$ is complete.*

*(ii) If $\chi$ is a (at least symmetric) basis, then $(f_k)_{k\in\mathbb{Z}}$ is total, i.e. $f_k(x) = 0$ for all $k \in \mathbb{Z}$ implies $x = 0$.*

*(iii) For $\sigma \subset \mathbb{Z}$, define $\chi_\sigma = \overline{\text{span}}\{x_k : k \in \sigma\}$ and $\chi^\sigma = \overline{\text{span}}\{x \in X : f_k(x) = 0$ for all $k \notin \sigma\}$. If $\chi$ is a basis, then for all $\sigma \subset \mathbb{Z}$, we have $\chi_\sigma = \chi^\sigma$.*

*Proof.* (i) Since $\chi$ is a basis, $\lim_{m,n} P_{m,n}x = x$ for all $x \in \mathcal{L}in\{x_k : k \in \mathbb{Z}\}$. By the UBP (uniform bounded principle: pt-wise bounded implies uniform bounded) $\sup_{m,n} \|P_{m,n}\| < \infty$.

(ii) If $f_k(x) = 0$ for all $k \in \mathbb{Z}$, then $P_{-n,n}x = 0$ for all $n \geq 1$. Hence $x = 0$.

(iii) The inclusion $\chi_\sigma \subset \chi^\sigma$ is clear (even for minimal families). On the other hand, if $x \in X^\sigma$, then $x = \lim_{n\to\infty} P_{-n,n}x$ with $P_{-n,n}x \in X_\sigma$. Hence $x \in X_\sigma$.

□

## 7.1 Skew projections

Let $L, M$ be two subspaces of a vector space $X$ such that $L \cap M = \{0\}$. Define $P : L + M \to X$ by $P(x + y) = x$, then $P^2 = P$, $P|_L = id$ and $P|_M = 0$. Then $P$ is called **skew projection** onto $L$ parallel to $M$ and denoted as $P := P_{L||M}$.

**Lemma 7.1.1.** *Let $L, M$ be two subspaces of a Banach space $X$ verifying $L \cap M = \{0\}$. Then*

*(i) $P_{L||M}$ is continuous if and only if $P_{\bar{L}||\bar{M}}$ is well defined and continuous (here $\bar{L} =$ clos $L$ and $\bar{M} =$ clos $M$).*

*Proof.* Let $x + y \in L + M, x \in L, y \in M$. Then $P_{L||M}$ is continuous $\iff \|P_{L||M}(x + y)\| = \|x\| \leq c\|x + y\|$ for every $x \in L, y \in M \iff \|\overline{x}\| \leq C\|\overline{x} + \overline{y}\|$, $\overline{x} \in L$, and $\overline{y} \in M \iff P_{\overline{L}||\overline{M}}$ is continuous. □

*(ii) If $L, M$ are closed, then $P_{L||M}$ is continuous if and only if $L + M =$ clos $(L + M)$.*

*Proof.* Apply closed graph theorem for the operator $T = P_{L||M}$. □

**Definition 7.1.2.** Let $L, M$ be two subspaces of a Hilbert space H. Define angle $\alpha \in [0, \frac{\pi}{2}]$ (or minimal angle) between $L$ and $M$ by

$$\cos\langle L, M\rangle = \cos\alpha = \sup_{x\in L,\, y\in M} \frac{|\langle x, y\rangle|}{\|x\|\|y\|}.$$

NOTATION: We write $\alpha = \langle L, M\rangle$.

*Remark* 7.1.3. $L \perp M$ if and only if $\alpha = \frac{\pi}{2}$.

**Lemma 7.1.4.** *With the above notations we have*

$$\cos\langle L, M\rangle = \cos\langle\bar{L}, \bar{M}\rangle = \|P_{\bar{M}}P_{\bar{L}}\|$$

*and*

$$\sin\langle L, M\rangle = \sin\langle\bar{L}, \bar{M}\rangle = \|P_{L||M}\|^{-1},$$

*where the symbols have obvious meaning.*

*Proof.* Clearly, $\sup_{y\in M\setminus\{0\}} \frac{|(P_{\bar{M}}x,y)|}{\|y\|} = \|P_{\bar{M}}x\|$. Moreover, $\langle x,y\rangle = \langle P_{\bar{M}}x,y\rangle$ for $y\in M$ and hence

$$\begin{aligned}\cos\langle L,M\rangle = \sup_{0\neq x\in L,\,0\neq y\in M} \frac{|\langle x,y\rangle|}{\|x\|\|y\|} &= \sup_{0\neq x\in L,\,0\neq y\in M} \frac{|\langle P_{\overline{M}}x,y\rangle|}{\|x\|\|y\|}\\ &= \sup_{0\neq x\in L}\frac{1}{\|x\|}\sup_{0\neq y\in M}\frac{|\langle P_{\overline{M}}x,y\rangle|}{\|y\|}\\ &= \sup_{0\neq x\in L}\frac{\|P_{\bar{M}}x\|}{\|x\|}.\end{aligned}$$

But

$$\sup_{0\neq x\in L}\frac{\|P_{\bar{M}}x\|}{\|x\|} = \sup_{0\neq x\in L}\frac{\|P_{\bar{M}}P_{\bar{L}}x\|}{\|x\|} = \sup_{0\neq x\in H}\frac{\|P_{\bar{M}}P_{\bar{L}}x\|}{\|x\|} = \|P_{\bar{M}}P_{\bar{L}}\|.$$

Hence $\cos\langle L,M\rangle = \|P_{\bar{M}}P_{\bar{L}}\|$

Now,

$$\begin{aligned}\|P_{L||M}\|^2 &= \sup_{0\neq x\in L,\,0\neq y\in M}\frac{\|P_{L||M}(x+y)\|^2}{\|x+y\|^2}\\ &= \sup_{0\neq x\in L,\,0\neq y\in M}\frac{\|x\|^2}{\|x+y\|^2}\\ &= \sup_{0\neq x\in L}\frac{\|x\|^2}{\inf_{0\neq y\in M}\|x+y\|^2}\\ &= \sup_{0\neq x\in L}\frac{\|x\|^2}{\|(1-P_{\bar{M}})x\|^2}.\end{aligned}$$

This now gives

$$\sin^2\langle L,M\rangle = 1-\cos^2\langle L,M\rangle = 1-\sup_{0\neq x\in L}\frac{\|P_{\bar{M}}x\|^2}{\|x\|^2} = \inf_{0\neq x\in L}\frac{\|(1-P_{\bar{M}})x\|^2}{\|x\|^2} = \frac{1}{\|P_{L||M}\|^2.}$$

So $\sin\langle L,M\rangle = \frac{1}{\|P_{L||M}\|}$. □

**Corollary 7.1.5.** *The projection $P_{L||M}$ is continuous if and only if $\|P_{\bar{L}}P_{\bar{M}}\|<1$ (and hence if and only if $\langle L,M\rangle>0$). Moreover, $\|P_{L||M}\| = \|P_{M||L}\|$.*

*Proof.* $P_{L||M}$ is continuous $\Leftrightarrow$ $\|P_{L||M}\|$ exists and $>0$ $\Leftrightarrow$ $\frac{1}{P_{L||M}}$ exists and $>0$ $\Leftrightarrow$ $\sin\langle L,M\rangle > 0 \Leftrightarrow \langle L,M\rangle>0$. Since $\sin\langle L,M\rangle>0 \Leftrightarrow \cos\langle L,M\rangle<1 \Leftrightarrow \|P_{\overline{M}}P_{\overline{L}}\|<1$ by Lemma 7.1.4 □

## 7.2 Bases of exponentials in $L^2(\mathbb{T},\mu)$

Now, let $X = L^2(\mathbb{T},\mu)$, where $\mu$ is a finite Borel measure, and $x_k = e^{ikt}$, $k\in\mathbb{Z}$ (or, $x_k = z^k$, $k\in\mathbb{Z}$).

**Lemma 7.2.1.** *If $(e^{ikt})_{k\in\mathbb{Z}}$ is a basis of $L^2(\mu)$ then $\mu_s \equiv 0$.*

*Proof.* Let $\sigma_n = \{k : k > n\}$, let $L^2_{\sigma_n} = \overline{\text{span}}_{L^2(\mu)}\{z^k : k > n\}$, and let $f_k$ be coordinate functionals associated to $(e^{ikt})_{k\in\mathbb{Z}}$, then

$$\bigcap_{n\geq 1} L^2_{\sigma_n} = \{x \in L^2(\mu) : f_k(x) = 0 \text{ for all}\, k \in \mathbb{Z}\} = \{0\}$$

($\because x \in L^2(\mu) \implies x = \sum_{k\in\mathbb{Z}}\langle x, f_k\rangle z^k = \sum_{k\in\mathbb{Z}} f_k(x) z^k$ and $f_k(x) = 0$ since $f_k \perp L^2(\sigma_k)$ for all $k \geq 1$( by Proposition 7.0.2 ) $\implies x = 0$ (by Banach theorem 7.0.7). Clearly, $L^2_{\sigma_n}$ is an invariant subspace, and $z^n \in L^2_{\sigma_n}$ and $z^n \neq 0$ on $\mathbb{T}$. So it can be deduced (as in Corollary 4.3.1) that $L^2_{\sigma_n} = L^2_{\sigma_n}(\mu_a) + L^2(\mu_s)$ for all $n \in \mathbb{Z}$. But then also $\bigcap_{n\geq 1} L^2_{\sigma_n} \supset L^2(\mu_s)$, implies $L^2(\mu_s) = 0$. □

*Remark* 7.2.2. For studying exponential basis in $L^2(\mathbb{T},\mu)$ one can restrict to measure which is absolutely continuous with respect to the Lebesgue measure $m$, $d\mu = wdm$, $w \in L^1_+(\mathbb{T}, m)$.

**Lemma 7.2.3.** (*Kolmogorov, 1941*) *Let $w \geq 0$, $w \in L^1_+$. Then $(z^n)_{n\in\mathbb{Z}}$ is a minimal sequence in $L^2(wdm)$ if and only if $\frac{1}{w} \in L^1(\mathbb{T})$.*

*Proof.* Due to biorthogonality, we have

$$\delta_{n,k} = (z^n, f_k)_{L^2(wdm)} = \int_{\mathbb{T}} z^n \bar{f}_k wdm,\; n, k \in \mathbb{Z}.$$

So we deduce that $\bar{f}_k w = \bar{z}^k$, $k \in \mathbb{Z}$, that is $f_k = \frac{z^k}{w}$, $k \in \mathbb{Z}$. Hence

$$f_k \in L^2(wdm) \text{ if and only if } \int_{\mathbb{T}} \frac{1}{w^2} wdm < \infty.$$

□

## 7.3 Riesz Projection

Let $\mathbb{P}, \mathbb{P}_+$ be as earlier and $\mathbb{P}_- = \text{span}\{e^{ikt} : k < 0\}$. Define the Riesz projection $P_+$ by

$$P_+ f = \sum_{k\geq 0} \hat{f}(k) e^{ikt},\; f \in \mathbb{P}.$$

Then

$$P_+ = P_{\mathbb{P}_+||\mathbb{P}_-}.$$

Let also

$$P_{m,n} f = \sum_{k=m}^{n} \hat{f}(k) e^{ikt},\; f \in \mathbb{P},\; m, n \in \mathbb{Z},\; m \leq n.$$

The following result gives the principle link between the problem of bases and the norm estimation of the Riesz projection.

**Lemma 7.3.1.** *Let $w \in L^1_+$. Then the followings are equivalent.*

*(i)* $(z^k)_{k\in\mathbb{Z}}$ *is a nonsymmetric basis of* $L^2(wdm)$.

*(ii)* $\sup_{n,m\in\mathbb{Z}} \|P_{m,n}\| < \infty$.

*(iii)* $(z^k)_{k\in\mathbb{Z}}$ *is a symmetric basis of* $L^2(wdm)$.

*(iv)* $\sup_{n\in\mathbb{Z}} \|P_{-n,n}\| < \infty$.

*(v) The Riesz projection* $P_+$ *is continuous on* $L^2(wdm)$.

*(vi)* $\langle P_+, P_-\rangle > 0$ *(or* $\langle H^2_+, H^2_-\rangle > 0$, *where* $H^2_\pm = \operatorname{clos}_{L^2(wdm)} \mathbb{P}_\pm$.

*Proof.* In view of Lemma 7.0.9 we get (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv). It is also clear that (ii) implies (iv). Using Lemma 7.1.4 and Corollary 7.1.5 we obtain (v) $\Leftrightarrow$ (vi). Next, we verify that (iv) implies (v). Pick $f \in \mathbb{P}$, then for $n = n(f)$ sufficiently large, we get (using the relation: $\widehat{z^{-n}f}(k) = \widehat{f}(n+k)$), $P_+f = z^nP_{-n,n}z^{-n}f$, so $\|P_+f\| = \|P_{-n,n}z^{-n}f\| \le \|P_{-n,n}\|\|f\|$ implies $\|P_+\| \le \sup_{n\ge1} \|P_{-n,n}\|$. It remains to show that (v) implies (ii). Note that

$$P_{m,n}f = z^{n+1}(1-P_+)z^{-(n+m+1)}P_+z^mf, \quad f \in \mathbb{P}.$$

But then

$$\|P_{m,n}f\| = \|(1-P_+)z^{-(n+m+1)}P_+z^mf\| \le \|P_+\|\|P_+z^mf\| \le \|P_+\|^2\|f\|$$

for all $f \in \mathbb{P}$, since $\|1-P_+\| = \|P_+\|$, (by Corollary 7.1.5). Hence the result follows. □

## 7.4 Harmonic conjugates

In order to get the desired characterization of exponential type bases in $L^2(\mu)$, we need a result of analytic type, namely, the so-called harmonic conjugation on $\mathbb{T}$ (or $\mathbb{D}$).

**Theorem 7.4.1.** *Let $u \in L^2(\mathbb{T})$ be a real valued function. Then there exist a unique real valued function $v \in L^2(\mathbb{T})$ such that $\hat{v}(0) = 0$ and $u + iv \in H^2$. The mapping $u \mapsto v$ is linear and continuous with $\|v\| \le \|u\|$.*

*Proof.* Let $u = \sum_{n\in\mathbb{Z}} \hat{u}(n)e^{int} \in L^2$. Then $\bar{u} = \sum_{n\in\mathbb{Z}} \bar{\hat{u}}(n)e^{-int}$. Since $u$ is real valued, $\bar{u} = u \Leftrightarrow \bar{\hat{u}}(n) = \hat{u}(-n)$, $n \in \mathbb{Z}$. Define

$$f = \hat{u}(0) + 2\sum_{n\ge1} \hat{u}(n)z^n.$$

Then $f \in H^2$ and

$$\operatorname{Re} f = \frac{1}{2}(f+\bar{f}) = \hat{u}(0) + \sum_{n\geq 1}\hat{u}(n)e^{int} + \sum_{n\geq 1}\bar{\hat{u}}(n)e^{-int} = u.$$

This means that $v = \operatorname{Im} f$ will satisfy the conclusion of the theorem. Next, we show that $v$ is unique. If $u+iv = u+iv_1 \in H^2$, then $v - v_1 \in H^2$. As $v - v_1$ is real valued $\overline{v-v_1} \in H^2$. But this is possible only if $v - v_1 = c$. Also $c = v(0) - v_1(0) = \hat{v}(0) - \hat{v}_1(0) = 0$[since for $v, v_1 \in H^2 \implies v(0) = \widehat{v}(0), v_1(0) = \widehat{v}_1(0)$; and $\widehat{v}(0) = \widehat{v}_1(0) = 0$ from assumption.] Finally, we have

$$v = \operatorname{Im} f = \frac{f-\bar{f}}{2i} = \frac{1}{i}\Big(\sum_{n\geq 1}\hat{u}(n)e^{int} - \sum_{n\geq 1}\bar{\hat{u}}(n)e^{-int}\Big) = \frac{1}{i}\Big(\sum_{n>0}\hat{u}(n)e^{int} - \sum_{n<0}\hat{u}(n)e^{int}\Big).$$

The process $u \longmapsto v$ is linear and

$$\|v\|^2 = \sum_{k\neq 0}|\hat{u}(k)|^2 \leq \|u\|^2,$$

and if $\hat{u}(0) = 0$, then $\|u\| = \|v\|$. □

**Definition 7.4.2.** The function $v$ is called Harmonic conjugate of $u$. Let $v = \tilde{u}$. The mapping $H : L^2(\mathbb{T}) \to L^2(\mathbb{T}),\ u \longmapsto \tilde{u}$ is called the **Hilbert transform**.

## 7.5 Different formula for $\tilde{u}$

(a) We can translate the above formula for $\tilde{u}$ in terms of Riesz projections

$$\tilde{u} = \frac{1}{i}(P_+u - P_-u) - \frac{1}{i}\hat{u}(0).$$

In particular, if $\hat{u}(0) = 0$, then $\tilde{u} = \frac{1}{i}(P_+u - P_-u)$. Also, we have $f = u + i\tilde{u} = 2P_+u - \hat{u}(0)$.

(b) If $u$ verify the conditions of the theorem, then $f = u+iv \in H^2$ and $u = \operatorname{Re} f$. As $f$ extends to $\mathbb{D}$ so Re $f$ does as well. For $z \in \mathbb{D}$, $u(z) = \operatorname{Re} f * P_z = u * P_z$. Since the Poisson kernel verify $P_z(\zeta) = \operatorname{Re}\Big(\dfrac{\zeta+z}{\zeta-z}\Big)$, we get $u(z) = \operatorname{Re} f_1(z)$, where

$$f_1(z) = \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}u(\zeta)dm(\zeta).$$

Note that $f_1 \in \operatorname{Hol}(\mathbb{D})^{*1}$ and Re $f_1 = u$, $f_1(0) = \int_{\mathbb{T}} udm \in \mathbb{R}$. By uniqueness, we have $f = f_1$ and

$$\tilde{u} = \operatorname{Im} f = \operatorname{Im} f_1 = \int_{\mathbb{T}} \operatorname{Im}\Big(\frac{\zeta+z}{\zeta-z}\Big)u(\zeta)dm(\zeta) = \int_0^{2\pi} Q_r(\tau-t)u(e^{it})\frac{dt}{2\pi}$$

where $z = re^{it}$ and

$$Q_r(t) = \operatorname{Im}\Big(\frac{\zeta + z}{\zeta - z}\Big) = \frac{2r\sin t}{1 - 2r\cos t + r^2}.$$

*1 [

$$uad\frac{e^{it}+z}{e^{it}-z} = 1 + 2\sum_{n=1}^{\infty} z^n e^{-int} \implies \frac{1}{2\pi}\int_{-\pi}^{\pi}(\frac{e^{it}+z}{e^{it}-z})f(e^{it}))dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})dt + \frac{2}{2\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} z^n e^{-int} f(e^{it})dt$$

$$= \widehat{f}(0) + 2\sum_{n=1}^{\infty}\widehat{f}(n)z^{-n}.$$

Since it has a power series it is analytic. (See [9] p.12 )]

*Remark* 7.5.1. For $r \to 1$, $Q_r \sim \dfrac{\sin t}{1-\cos t} = \cot(t/2)$. In fact, one can show that

$$\tilde{u}(\tau) = (u * \cot(./2))(\tau) = \int_0^{2\pi} u(\tau - t)\cot(t/2)\frac{dt}{2\pi}$$

in the sense of Cauchy principle valued integral.

## 7.6 The Helson-Szegö theorem

**Theorem 7.6.1.** *Let $\mu$ be a finite Borel measure on $\mathbb{T}$. Then the followings are equivalent.*

*(i) The family $(z^n)_{n\in\mathbb{Z}}$ is a (symmetric or nonsymmetric) basis of $L^2(\mu)$.*

*(ii) The Riesz projection $P_+$ is bounded on $L^2(\mu)$.*

*(iii) The angle satisfies $\sin\langle P_+, P_-\rangle > 0$.*

*(iv) $d\mu = |h|^2 dm$, where $h \in H^2$ is an outer function such that $\operatorname{dist}\Big(\dfrac{\bar{h}}{h}, H^\infty\Big) < 1$.*

*(v) $d\mu = wdm$, where $w = e^{u+\tilde{v}}$ and $u, v$ are real valued bounded functions and $\|v\|_\infty < \frac{\pi}{2}$ (condition (HS)).*

The proof of the theorem will be given in several steps based on the following lemmas.

**Lemma 7.6.2.** *The mapping $j : H^2 \times H^2 \to H^1$, $(\phi, \psi) \longmapsto \phi\psi$ is continuous and symmetric. Moreover, $j(B^2 \times B^2) = B^1$, where $B^p$ is the unit ball in $H^p$.*

*Proof.* The continuity follows from the Cauchy Schwarz inequality $\|\phi\psi\|_1 \le \|\phi\|_2\|\psi\|_2$. For surjectively, let $f \in H^1$, then $f = \lambda BS[f]$. Write $\phi = \lambda BS[f]^{\frac{1}{2}}$ and $\psi = [f]^{\frac{1}{2}}$ then $\phi\psi \in H^2$. □

**Lemma 7.6.3.** *Let $E$ be a subspace of the Banach space $X$, and $\Phi \in X^*$. Then*

$$\|\Phi|_E\| = \inf\{\|\Psi\|_{X^*} : \Psi = \Phi \text{ on } E\} = \inf\{\|\Phi + \alpha\|_{X^*} : \alpha \in X^* \text{ and } \alpha|_E = 0\}$$

*Proof.* The inequality "$\leq$" is clear. For "$\geq$" apply Hahn-Banach theorem. Let $\Psi' = \Phi|_E$. Then

$$\|\Psi\|_{X^*} = \sup_{x\in X} |\Psi(x)| \geq \|\Psi'\|_{X^*} = \sup_{x\in X} |\Psi'(x)| = \|\Phi|_E\|.$$

By Hahn-Banach theorem, there exists $\Psi' \in X^*$ such that $\|\Phi|_E\| = \|\Psi'\|_{X^*}$, and hence the result follows. □

**Lemma 7.6.4.** *Let $f \in H^1$ and suppose that $f(\mathbb{T}) \subset A \subset \mathbb{C}$. Then $f(\mathbb{D}) \subset \operatorname{conv} A)$ (the closed convex hull of $A$).*

*Proof.* Observe that for $z = rw \in \mathbb{D}, |w| = 1$ we have $f(z) = P_z * f = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} f(\zeta)d\zeta \in \operatorname{conv}(A)$. However, $\operatorname{conv}(A) = \cap H$ where the intersection is taken over all the half-planes: $H = \{z \in \mathbb{C} : \operatorname{Re}(az + b) \geq 0\}$ containing $A$, $a, b \in \mathbb{C}$. Since $P_r > 0$ and $\int_{\mathbb{T}} P_r dm(\xi) = 1$, we see that the condition $\operatorname{Re}(af(\zeta + b) \geq 0)$ for a.e. $\zeta \in \mathbb{T}$ as $f(\zeta) \in A \subset H \implies \operatorname{Re}(af(z) + b) \geq 0 \implies f(z) \in \operatorname{conv}(A)$ □

**Lemma 7.6.5.** ($V$. *Smirnov, A. Kolmogorov*) *Let $v \in L^\infty(\mathbb{T})$ be a real valued function then $e^{\lambda\tilde{v}} \in L^1(\mathbb{T})$ if $\lambda\|v\|_\infty < \frac{\pi}{2}$.*

*Proof.* It is sufficient to show that $\|u\|_\infty < \frac{\pi}{2}$ implies $e^{\tilde{u}} \in L^1$. Set $f = e^{-i(u+i\tilde{u})}$, which is well defined in $\mathbb{D}$, since $u + i\tilde{u} \in H^2$. Clearly $|f| = e^{\tilde{u}}$ and $|\arg f| = |u| < \dfrac{\pi(1-\epsilon)}{2}$ for some $\epsilon > 0$ (on $\mathbb{T}$ and hence on $\mathbb{D}$ in view of Lemma 7.6.4). The same reasoning as in (Theorem 6.2.3) now gives $f \in H^1$ and hence $|f| = e^{\tilde{u}} \in L^1(\mathbb{T})$. □

*Proof.* **Implication (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) of Helson-Szegö theorem.**
Recall that we may restrict to $d\mu = wdm$, $w \in L^1_+(\mathbb{T})$. By Lemma 7.3.1 we get the equivalence of (i),(ii) and (iii).
Next we show (i) and (ii) are equivalent to (iv) (see Figure 7.1): Note that if the sequence $(z^n)_{n\in\mathbb{Z}}$ is a basis, then we can see from Banach's (Theorem 7.0.7) and Kolmogorov's (Lemma 7.2.3) that $\frac{1}{w} \in L^1$ and hence $\log w \in L^1$ (this can be justified without using Banach theorem as $\bar{z} \notin H^2(\mu)$ we get $\log w \in L^1$). In view of the later observation, we suppose that there exists an outer function $h \in H^2$ such that $|h|^2 = w$. Thus,

$$(f,g)_{L^2(\mu)} = \int_{\mathbb{T}} f\bar{g}wdm = \int_{\mathbb{T}} fh\bar{g}h\frac{\bar{h}h}{h^2}dm = \int_{\mathbb{T}} (fh)(\bar{g}h)\frac{\bar{h}}{h}dm = \int_{\mathbb{T}} FG\frac{\bar{h}}{h}dm$$

for all $f \in \mathbb{P}_+$ and $g \in \mathbb{P}_-$ and therefore,

$$\|f\|^2_{L^2(\mu)} = \int |fh|^2 dm = \|F\|^2_{L^2(\mathbb{T})}, \quad \|g\|^2_{L^2(\mu)} = \|G\|^2_{L^2(\mathbb{T})}.$$

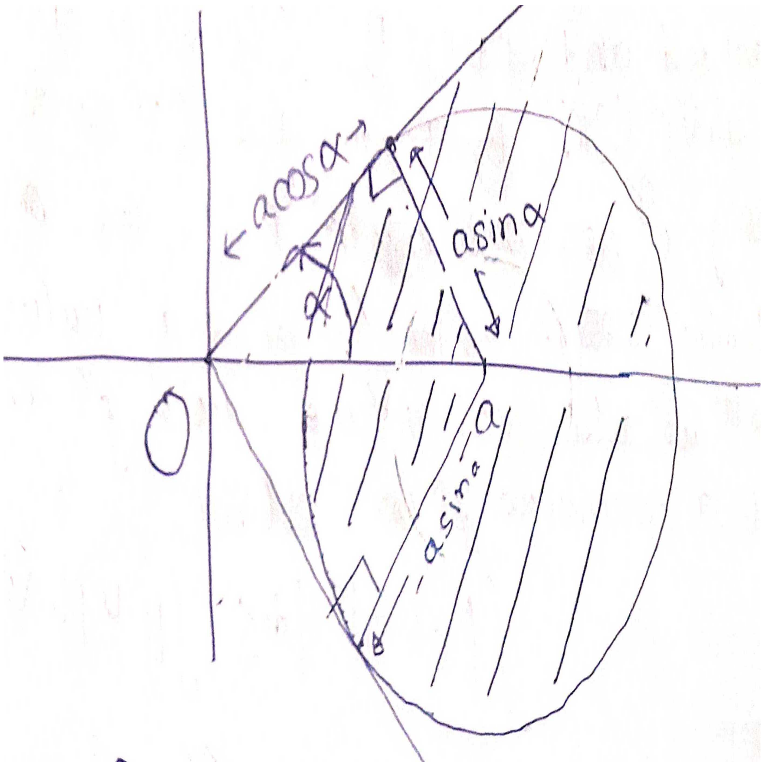


Figure 7.1: Geometry used in the proof of the Helson–Szegő theorem (schematic).

Clearly $F = fh \in H^2$, since $\bar{g} \in \mathbb{P}_+^0$, we get $G \in H_0^2$. By definition of outer function, it follows that $\overline{\text{span}}\{F = fh : f \in \mathbb{P}_+\} = H^2$, and also $A := \{F = fh : f \in \mathbb{P}_+, \|F\| \le 1\}$ is dense in the unit ball $B^2$ of $H^2$. For the same reason, we see that $B := \{G = \bar{g}h : g \in \mathbb{P}_-, \|G\| \le 1\}$ is dense in $B^2 \cap H_0^2$. We deduce that

$$\cos\langle \mathbb{P}_+, \mathbb{P}_-\rangle_{L^2(\mu)} = \sup\{|(f,g)| : f \in \mathbb{P}_+, g \in \mathbb{P}_- \ \|f\|_L^2(\mu) \le 1,\ \|g\|_L^2(\mu) \le 1\}$$

$$= \sup\Big\{\Big|\int_{\mathbb{T}} FG\frac{\bar{h}}{h}dm\Big| : F \in A,\ G \in B\Big\}.$$

Set $\Phi(k) = \int\limits_{\mathbb{T}} k(\frac{\bar{h}}{h})dm$, $k \in L^1(\mathbb{T})$. As $\bar{h}/h \in L^\infty(\mathbb{T})$, we get $\Phi \in (L^1(\mathbb{T}))^*$. By (Lemma 7.6.2), we see that the angle $\langle \mathbb{P}_+, \mathbb{P}_-\rangle = \|\Phi|_{H_0^1}\|$, and by means of (Lemma 7.6.3), we can express it in terms of $h$:

$$\cos\langle \mathbb{P}_+, \mathbb{P}_-\rangle_{L^2(\mu)} = \|\Phi|_{H_0^1}\| = \operatorname*{dist}_{L^\infty(\mathbb{T})}\Big(\frac{\bar{h}}{h}, (H_0^1)^\perp\Big) = \operatorname*{dist}_{L^\infty(\mathbb{T})}\Big(\frac{\bar{h}}{h}, H^\infty\Big).$$

The last equality is the consequence of the relation

$$(H_0^1)^\perp = \{g \in L^\infty : \int_{\mathbb{T}} gf\,dm = 0 \text{ for all } f \in H_0^1\} = H^\infty.$$

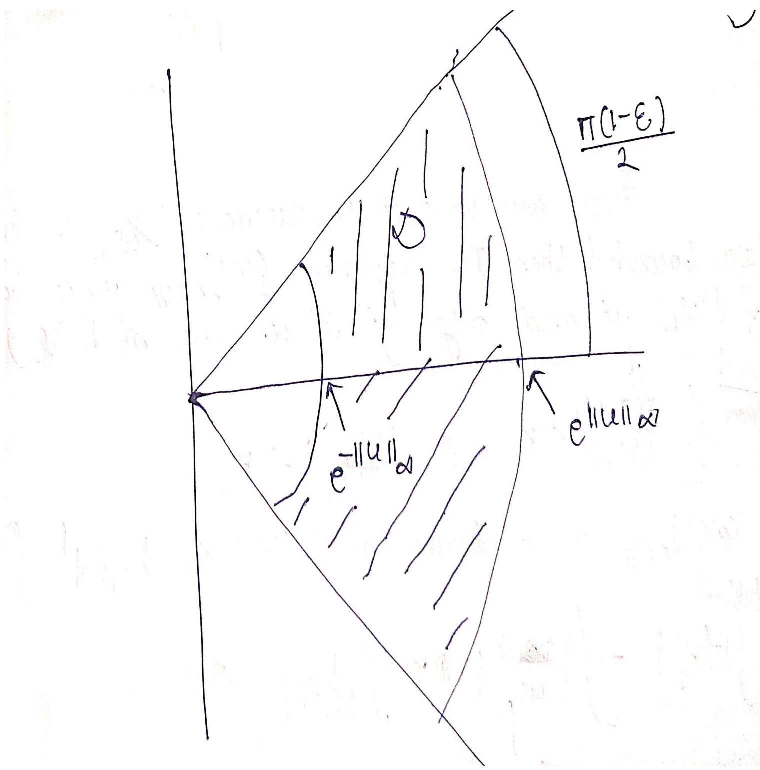


Figure 7.2: Fig2

Now, we conclude that $\cos\langle\mathbb{P}_+,\mathbb{P}_-\rangle<1$ if and only if $\log w\in L^1$, $w=|h|^2$ for an outer function $h\in H^2$ satisfying $\mathrm{dist}_{L^\infty(\mathbb{T})}(\frac{\bar h}{h},H^\infty)<1$, that is (i) and (ii) are equivalent to (iv).

**Proof of implication (iv) $\implies$ (v):**(See **Fig 2**) Suppose $\mathrm{dist}_{L^\infty(\mathbb{T})}(\frac{\bar h}{h},H^\infty)<1$, where $h$ is a outer and $|h|^2=w$. Then there exists $g\in H^\infty$ such that $\|\frac{\bar h}{h}-g\|_\infty<1$. That is for $\epsilon>0$, we have $|\frac{\bar h}{h}-g|<1-\epsilon$ a.e. on $\mathbb{T}$, and hence $||h|^2-gh^2|<(1-\epsilon)|h|^2$ a.e. on $\mathbb{T}$. Setting $a=|h(\xi)|^2>0$ for $\xi\in\mathbb{T}$, we see that $|a-gh^2|<(1-\epsilon)a$.

Geometrically, it means that if $\alpha\in(0,\frac{\pi}{2})$ is such that $\sin\alpha=1-\epsilon$, and $A=\{z:|\arg z|<\alpha\}$, then we get $gh^2(\mathbb{T})\subset A$ (cf. Figure 1).

From (Lemma 7.6.4) we get $gh^2(\mathbb{D})\subset A$, so $\log gh^2$ is analytic in $\mathbb{D}$. We set $v=-\operatorname{Im}\log gh^2=-\arg gh^2$ and get $|\tilde v|=\operatorname{Re}\log gh^2+c=\log|gh|^2+c$, where $c$ has to be chosen such that $\tilde v(0)=0$. We obtain $\log gh^2=\tilde v-iv-c$ and $gh^2=e^{\tilde v-iv-c}$ on $\mathbb{T}$, we have $|\frac{\bar h}{h}-g|<1-\epsilon$, which implies that $|1-|g||<1-\epsilon$, hence $\epsilon\le|g|\le2-\epsilon$. Finally, $|h|^2=\dfrac{e^{\tilde v-c}}{|g|}=e^{u+\tilde v}$, where $u=-\log|g|-c\in L^\infty(\mathbb{T})$ and $\|v\|_\infty<\frac{\pi}{2}$.

**Proof of implication (v) implies (iv):**

Let $wdm=e^{u+\tilde v}dm$, where $u,v\in L^\infty(\mathbb{T})$ are real valued and $\|v\|_\infty<\frac{\pi}{2}$. Clearly $\log w=u+\tilde v\in L^1$ and by (Lemma 7.6.5) we have $w\in L^1(\mathbb{T})$. Hence there exists an outer function $h\in H^2$ such that $|h|^2=w$. Thus $\log|h|^2=u+\tilde v$ and $\log h^2=u+\tilde v+i(u+\tilde v)^\sim=u+\tilde v+i(\tilde u-v+c)$ for some constant $c\in\mathbb{R}$. Setting $g=e^{-(u+i\tilde u)-ic}$ we obtain, in view of $|g|=e^{-u}$, a bounded holomorphic function $g \in H^\infty$. Moreover,

$$\frac{h}{\bar{h}}g = \frac{h^2}{|h|^2}g = \exp(i(\tilde{u} - v + c) - u - i\tilde{u} - ic) = \exp(-u - iv),$$

where $\|v\|_\infty < \frac{\pi}{2}$. This gives the following estimates on $\mathbb{T}$.

$$e^{-\|u\|_\infty} \leq \Big|\frac{h}{\bar{h}}g\Big| \leq e^{\|u\|_\infty}, \quad \Big|\arg(\frac{h}{\bar{h}})g\Big| = |v| < \pi\frac{(1-\epsilon)}{2}.$$

(cf. Figure 2). The value of $(\frac{h}{\bar{h}})g$ thus belongs to

$$\mathcal{D} := \Big\{z \in \mathbb{C} :\ e^{-\|u\|_\infty} \leq |z| \leq e^{\|u\|_\infty}, \quad |\arg z| < \pi\frac{(1-\epsilon)}{2}\Big\}.$$

For $\lambda$ sufficiently big and some $\delta > 0$ we have $B(\lambda, (1-\delta)\lambda) \supset \operatorname{clos}\mathcal{D}$ or $\lambda^{-1}B(\lambda,(1-\delta)\lambda) = B(1, 1-\delta) \supset \lambda^{-1}\operatorname{clos}\mathcal{D}$. Then $\lambda^{-1}\frac{h}{\bar{h}}g \in B(1,1-\delta)$ a.e. on $\mathbb{T}$. In other words, $|\lambda^{-1}(\frac{h}{\bar{h}})g - 1| < 1-\delta$ a.e. on $\mathbb{T}$, and $|\lambda^{-1}g - (\frac{\bar{h}}{h})| < 1-\delta$ a.e. $\mathbb{T}$. As $g \in H^\infty$, this gives $\operatorname{dist}_{L^\infty(\mathbb{T})}(\frac{\bar{h}}{h}, H^\infty) < 1$. □

## 7.7 An example

Let $\omega(e^{it}) = |t|^\alpha$, $t \in (-\pi,\pi)$, $\alpha \in \mathbb{R}$. Then for $\alpha \geq 1$ we have $1/\omega \notin L^1(\mathbb{T})$ and $(e^{int})_{n\in\mathbb{Z}}$ cannot be uniformly minimal in view of Lemma 7.2.3. For $\alpha \leq -1$, $\omega \notin L^1$. Thus, the only interesting case is $|\alpha| < 1$.

First note that if the quotient $\omega_1/\omega_2$ and $\omega_2/\omega_1$ are bounded, then the sequence $(e^{int})_{n\in\mathbb{Z}}$ is a basis of $L^2(\omega_1)$ if and only if it is one of $L^2(\omega_2)$. [ $|\frac{w_1}{w_2}| < K$ and $|\frac{w_2}{w_1}| < K_1$. By the Lemma 7.2.3, $(e^{int})_{n\in\mathbb{Z}}$ is a basis of $L^2(w_1) \Leftrightarrow \frac{1}{w_1} \in L^1$. Now

$$\int |\frac{1}{w_2}| \leq \int |\frac{K}{w_1}| = K\int \frac{1}{|w_1|} < \infty \Rightarrow \frac{1}{w_2} \in L^1 \Leftrightarrow$$

$(e^{int})_{n\in\mathbb{Z}}$ is a basis of $L^2(w_2)$ by Lemma 7.2.3. Similarly the other case follows.]

The identity map $f \longmapsto f$ is an isomorphism from $L^2(\omega_1)$ to $L^2(\omega_2)$.

Next, let $\omega_1 = \omega$ and $\omega_2 = (1-e^{it})^\alpha$. Then

$$\log\omega_2 = \log|1-e^{it}|^\alpha = \alpha\operatorname{Re}\arg(1-e^{it}) := u.$$

Necessarily, we get

$$\begin{aligned}
\tilde{u}(t) &= \alpha \arg(1 - e^{it}) = \alpha \arg(e^{it/2}(e^{-it/2} - e^{it/2}) \\
&= \alpha \arg(e^{it/2}(-2i \sin t/2). \\
&= \begin{cases} \alpha(t/2 - \pi/2) & \text{if } t > 0 \\ \alpha(\pi/2 - t/2) & \text{if } t < 0. \end{cases}
\end{aligned}$$

We deduce that $\|\tilde{u}\|_\infty = |\alpha| \frac{\pi}{2} < \frac{\pi}{2}$ if $|\alpha| < 1$. Hence $(e^{int})_{n \in \mathbb{Z}}$ is a basis in $L^2(|t|^\alpha dt) \Leftrightarrow |\alpha| < 1$.

# Chapter 8

# Transfer to the upper half-plane

In this section, we give an outline of the Hardy-space theory in the half-plane and on the line. We restrict ourselves to the key results only: an isometric correspondence between Hardy-space in the disc and in the half-plane, the canonical factorization, the Fourier transform representation (Paley-Wiener theorem), and invariant subspaces.

## 8.1 A unitary mapping from $L^p(\mathbb{T})$ to $L^p(\mathbb{R})$

Let $\omega : \mathbb{D} \to \mathbb{C}$, $\omega(z) = i\frac{1+z}{1-z}$, be the usual conformal mapping of the disc $\mathbb{D}$ to the upper half-plane $\mathbb{C}_+ = \{\xi \in \mathbb{C} : \mathrm{Im}\ \xi > 0\}$.

The restriction to the boundary $\omega|_{\mathbb{T}}$ is a one to one correspondence between $\mathbb{T} \setminus \{1\}$ and $\mathbb{R}$. The inverse $\omega^{-1}$, $\omega^{-1}(x) = \dfrac{x-i}{x+i}$ has Jacobian $|J(x)| = \dfrac{2}{1+x^2}$, $x \in \mathbb{R}$. Hence the mapping

$$U = U_p : L^p(\mathbb{T}) \to L^p(\mathbb{R})$$
$$U_p f(x) = \left(\frac{1}{\pi(x+i)^2}\right)^{1/p} f(\omega^{-1}(x)),\ x \in \mathbb{R},$$

is an isomorphic isomorphism (unitary for $p = 2$) of the space $L^p(\mathbb{T})$ onto $L^p(\mathbb{R})$.

First, we give three descriptions of the image under $U$ of the Hardy-space $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$, then pass to arbitrary $p$, $1 \le p \le \infty$. Clearly, $U_p H^p(\mathbb{T})$ is a closed subspace of $L^p(\mathbb{R})$.

## 8.2 Cauchy kernel and Fourier transform

The first description of $U_2 H^2(\mathbb{T})$ is straightforward.

**Lemma 8.2.1.**

$$U_2 H^2(\mathbb{T}) = \overline{\mathrm{span}}_{L^2(\mathbb{R})}\Big\{\frac{1}{x-\bar{\mu}} : \mathrm{Im}\ \mu > 0\Big\}.$$

To prove this we first need the following proposition:

**Proposition 8.2.2.** $H^2(\mathbb{D}) = \overline{\text{span}}\{c_\lambda = \frac{1}{\overline{\lambda}-z} : \lambda \in \mathbb{D}\}$

*Proof.* From Corollary 5.2.2 for $f \in H^2$, and for each $\lambda \in \mathbb{D}$ the evaluation map $\varphi_\lambda$ is bounded and by Riesz-representation theorem it takes of the form: $\varphi_\lambda(f) = f(\lambda) = \langle f, c_\lambda \rangle$ where $c_\lambda \in H^2$ is unique. We now calculate $c_\lambda$ and see that it is $\frac{1}{\overline{\lambda}-z}, z \in \mathbb{D}$. For each $\lambda \in \mathbb{D}$, the function $\lambda \to \frac{1}{\overline{\lambda}-z} \in H^2$, since

$$\frac{1}{\overline{\lambda}-z} = \sum_{n\geq 0} \overline{\lambda}^n z^n \text{ and } (\overline{\lambda}^n) \in \ell^2(\mathbb{N}_0)$$

and so

$$\|\frac{1}{1-\overline{\lambda}z}\| = \|\sum_{n\geq 0} \overline{\lambda}^n z^n\| = \left\langle \sum_{n\geq 0} \overline{\lambda}^n z^n, \sum_{n\geq 0} \overline{\lambda}^n z^n \right\rangle^{\frac{1}{2}} = \frac{1}{1-\|\lambda\|^2} < \infty.$$

Furthure, if $f = \sum_{n\geq 0} a_n z^n \in H^2$ then

$$\left\langle f, \frac{1}{1-\overline{\lambda}z} \right\rangle = \sum_{n\geq 0} a_n \lambda^n = f(\lambda).$$

By the uniqueness of the Riesz-representation theory: $c_\lambda = \frac{1}{\overline{\lambda}-z}$. Moreover, $\|c_\lambda\|^2 = \langle c_\lambda, c_\lambda \rangle = c_\lambda(\lambda) = \frac{1}{1-|\lambda|^2}$. $c_\lambda$ is called the **Cauchy Kernal or Szego Kernal**. The space $H^2$ is called the **Reproducing Kernal Hilbert space.**

It is easy to check that the set $D = \{c_\lambda : \lambda \in \mathbb{D}\}$ is linearly independent. Also if $f \in H^2$ is orthogonal to $c_\lambda, \forall \lambda \in \mathbb{D}$ then $f = 0$ ( since $f(\lambda) = \langle f, c_\lambda \rangle$). Hence $D$ is dense in $H^2$. (A set $D$ in $X$ is dense if and only if $D^\perp = \{0\}$.)

□

*Proof of Lemma 8.2.1.* Since $H^2(\mathbb{T}) = \overline{\text{span}}_{L^2(\mathbb{T})}\left\{\frac{1}{1-\overline{\lambda}z} : |\lambda| < 1\right\}$, and $U_2$ is an isometry, we have

$$H^2(\mathbb{T}) = \overline{\text{span}}_{L^2(\mathbb{T})}\left\{U_2(1-\bar{\lambda}z)^{-1} = \frac{C_\lambda}{z-\overline{\omega(\lambda)}} : \lambda \in \mathbb{D}\right\} = \overline{\text{span}}\{\frac{1}{z-\overline{\mu}} : \text{Im}\,\mu > 0\}.$$

Clearly, $\mu = \omega(\lambda)$ runs over the entire upper half-plane $\mathbb{C}_+$. □

Now, we recall that Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$,

$$\mathcal{F}(f)(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ixz} dx,$$

$$\mathcal{F}^{-1}(f)(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{ixz} dx$$

are unitary mapping of $L^2(\mathbb{R})$ onto itself.

**Lemma 8.2.3.** *$U_2 H^2 = \mathcal{F}^{-1} L^2(\mathbb{R}_+)$, where $L^2(\mathbb{R}_+) = \{f \in L^2(\mathbb{R}) : f = 0$ on $(-\infty, 0)\}$.*

*Proof.* Compute the inverse Fourier transform of the function $\chi_{\mathbb{R}_+}e^{i\lambda x} \in L^2(\mathbb{R}_+)$, where $\operatorname{Im}\lambda > 0$ :

$$\mathcal{F}^{-1}(\chi_{\mathbb{R}_+}e^{i\lambda x}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\chi_{\mathbb{R}_+}e^{i\lambda x}e^{ixz}dx = \frac{1}{\sqrt{2\pi}}\frac{1}{i(z+\lambda)}\left[ix(z+\lambda)\right]_{x=0}^{\infty} = \frac{i}{\sqrt{2\pi}}\frac{1}{z-(-\lambda)},$$

where $-\lambda = \mu$ runs, again, over the entire half-plane $\mathbb{C}_+$. Since $\mathcal{F}^{-1}$ is an isometry, Lemma 8.2.3 reduces to the proof of the following identity:

$$L^2(\mathbb{R}_+) = \overline{\operatorname{span}}\{\chi_{\mathbb{R}_+}e^{i\lambda x}\ \operatorname{Im}\lambda > 0\}.$$

The equality follows from the injectivity (classical Fourier uniqueness theorem) of the Fourier transform $\mathcal{F}$. Let $f \in L^2(\mathbb{R}_+)$ and suppose that $f \perp \chi_{\mathbb{R}_+}e^{i\lambda x}$ for all $\lambda$ with $\operatorname{Im}\lambda > 0$.

$$\begin{aligned}
&\int_{\mathbb{R}} f(x)\chi_{\mathbb{R}_+}e^{-\lambda x}dx = 0\\
&\implies \int_{\mathbb{R}} f(x)\chi_{\mathbb{R}_+}e^{-x}e^{-iyx}dx = 0 \text{ (putting } \lambda = y+i)\\
&\implies \mathcal{F}(f\chi_{\mathbb{R}_+}e^{-x})(y) = 0\ (\forall y \in \mathbb{R})\\
&\implies f\,\chi_{\mathbb{R}_+}e^{-x} = 0 \ \text{ a.e. on } \mathbb{R}\ [\text{ since } \widehat{f} = 0 \implies f = 0]\\
&\implies f = 0
\end{aligned}$$

□

## 8.3 The Hardy space $H^p_+ = H^p(\mathbb{C}+)$

Here we see from real line $\mathbb{R}$ to the half-plane $\mathbb{C}_+$. We identify the subspace $U_pH^p \subset L^2(\mathbb{R})$ with the space of boundary values of a certain holomorphic space in the half-plane $C_+$. Note that $\omega^{-1}(z) = \frac{z-i}{z+i}$ is a conformal mapping from $\mathbb{C}_+$ to $\mathbb{D}$.
Hence the same formula as above, $U_p : H^p(\mathbb{C}^+) \to H^p(\mathbb{D})$

$$U_pf(z) = \left(\frac{1}{\pi(z+i)}\right)^{1/p} f(\omega^{-1}(z)),\ \operatorname{Im} z > 0$$

defines a holomorphic function in $\mathbb{C}_+$ for all $f \in H^p(\mathbb{C}_+)$.
Moreover, $\omega^{-1}$ is still conformal at the boundary points $r \in \mathbb{R}$ and transfers a Stolz angle in $\mathbb{C}_+$, $\{x+iy :\ |x-r| < cy\}$, into a Stolz angle in $\mathbb{D}$. Now, Fatou's theorem implies that the functions $U_pf,\ f \in H^p(\mathbb{D})$, have non-tangential boundary limits $(U_p(f))_{\mathbb{R}}$ a.e. on $\mathbb{R}$, $U_p(f_{\mathbb{T}}) = (U_pf)_{\mathbb{R}}$. Hence in order to get another characterization of $U_pH^p(\mathbb{T})$, it remains to describe $U_pH^p(\mathbb{D})$ in intrinsic terms as a subset of $\operatorname{Hol}(\mathbb{C}_+)$. This is done in the next theorem. But, first we define Hardy classes on $\mathbb{C}_+$.

**Definition 8.3.1.** Hardy space $H^p_+ = H^p(\mathbb{C}_+)$, $0 < p \le \infty$, is the class of functions $g \in \mathrm{Hol}(\mathbb{C}_+)$ such that

$$\|g\|_{H^p_+} = \sup_{y>0} \Big( \int_{\mathbb{R}} |g(x+iy)|^p dx \Big)^{\frac{1}{p}} < \infty,$$

with the usual modification for $p = \infty$. In order to compare $H^p(\mathbb{C}_+)$ with $U_p H^p(\mathbb{D})$, we need the following simple result.

**Lemma 8.3.2.** *(i) Let $\gamma$ be an arbitrary circle in $\overline{\mathbb{D}}$. Then*

$$\int_\gamma |f(z)|^p |dz| \le 2 \int_{\mathbb{T}} |f(z)|^p |dz|$$

*for all $f \in H^p(\mathbb{D})$, $1 \le p < \infty$, here $|dz|$ stands for the arc length measure.*

*(ii) Let $g \in H^p(\mathbb{C}_+)$, $1 \le p < \infty$ and $z \in \mathbb{C}_+$, then*

$$|g(z)| \le \Big( \frac{2}{\pi \,\mathrm{Im}\, z} \Big)^{\frac{1}{p}} \|g\|_{H^p_+}.$$

*Proof.* (i) First let $p = 1$. For $u \in L^1(\mu)$, denote by $u_*$ be the harmonic extension of $u$ in the unit disc,

$$u_*(z) = \int_{\mathbb{T}} u(\zeta) \frac{1-|z|^2}{|\zeta - z|^2} dm(\zeta),\ z \in \mathbb{D}.$$

We show that $u \longmapsto u_*|_\gamma$ is a bounded operator from $L^1(\pi)$ to $L^1(\gamma)$ of norm at most $4\pi$. Indeed,

$$\begin{aligned} \int_\gamma |u_*(z)||dz| &\le \int_\gamma |u(\zeta)| \frac{1-|z|^2}{|\zeta - z|^2} dm(\zeta)|dz| \\ &= \int_{\mathbb{T}} |u(\zeta)| \Big( \int_\gamma \frac{1-|z|^2}{|\zeta - z|^2} |dz| \Big) dm(\zeta) \\ &= 2\pi r \int_{\mathbb{T}} |u(\zeta)| \frac{1-|c|^2}{|\zeta - c|^2} dm(\zeta), \end{aligned}$$

where $\gamma = \gamma(c, r)$. In the last inequality, we have used the MVT for harmonic functions applied to the Poisson kernel $P_z(\zeta) = \mathrm{Re}\,\big(\frac{z+\zeta}{z-\zeta}\big)$. Since $2\pi dm(z) = |dz|$ on $\mathbb{T}$, $r \le 1 - |c|$ and $\frac{1-|c|^2}{|\zeta - c|^2} \le \frac{1+|c|}{1-|c|} \le \frac{2}{1-|c|}$, we get the desired inequality. For an arbitrary $p$, $1 \le p < \infty$, we have $|u_*|^p \le (|u|^p)_*$, from Holder's inequality, and the result follows.
(ii) Using the MVT in the disc, $D = \{x + iy \,:\, |\lambda - (x+iy)| < \mathrm{Im}\,\lambda\}$, Holder's inequality, and

what is sometimes called the “rolling a disk" trick:

$$\begin{aligned}|g(\lambda)| &= \frac{1}{\pi(\operatorname{Im}\lambda)^2}\int_D |g\,dxdy| \\ &\leq \frac{1}{\pi(\operatorname{Im}\lambda)^2}(\int_{\mathbb{D}}|g|^p dxdy)^{\frac{1}{p}}(\int_{\mathbb{D}} 1dxdy)^{\frac{1}{q}} \\ &\leq \Big(\frac{1}{\pi(\operatorname{Im}\lambda)^2}\Big)\Big(\int_D |g|^p dxdy\Big)^{\frac{1}{p}}(\pi(\operatorname{Im}\lambda)^2)^{\frac{1}{q}} \\ &\leq \Big(\frac{1}{\pi(\operatorname{Im}\lambda)}\Big)^{2(1-\frac{1}{q})}\Big(\int_0^{2\operatorname{Im}\lambda} dy\int_{\mathbb{R}}|g(x+iy)|^p dy\Big)^{\frac{1}{p}} \\ &\leq \Big(\frac{2}{(\pi\operatorname{Im}\lambda)}\Big)^{\frac{1}{p}}\|g\|_{H^p_+}.\end{aligned}$$

□

The following theorem is one of the main result of this section.

**Theorem 8.3.3.** *Let $1 \leq p \leq \infty$. Then $U_pH^p(\mathbb{D}) = H^p(\mathbb{C}_+)$.*

*Proof.* If $g \in \operatorname{Hol}(\mathbb{C}_+)$, $y > 0$, and $Uf = g$, then

$$\int_{\mathbb{R}} |g(x+iy)|^p dx = \frac{1}{2\pi}\int_{C_y} |f(z)|^2|dz|,$$

where $C_y$ is the circle in $\mathbb{D}$ having the interval $[\frac{y-1}{y+1}, 1]$ as diameter and being tangent to the unit circle $\mathbb{T}$ at the point 1. A line on the upper half plane at a distance $y$ parallel to $x$-axis maps to the circle $C_y := \{z : |z - \frac{y}{y+1}| = \frac{1}{y+1}\}$, i.e., to check! for a point $(x + iy_0)$ in the line parallel to $x$-axis in the upper half plane maps to $C_{y_0}$ under $w^{-1}$. i.e., $w^{-1}(x + iy_0) = \frac{x+iy_0-i}{x+iy_0+i}$ satisfies $|z - \frac{y_0}{y_0+1}| = \frac{1}{y_0+1}$. There are two points to be noted from the above discussions (**Fig 3**):
(i) Infinite straight-line parallel to $x$-axis on the upper-half plane wraps around the circle $C_y$
(ii) The region $\operatorname{Im} z \geq y > 0$ maps into the inside of the circle $C_y$, easily check that $(0, 2y)$ maps to center of the circle $(\frac{y}{y+1}, 0)$. Now

$$\begin{aligned}\int_{\mathbb{R}} |g(x+iy_0)|^p dx &= \int_{-\infty}^{+\infty}\left|\frac{1}{\pi(x+iy_0+i)}\right| |f(\omega^{-1}(x+iy_0))|^p dx \\ = \int_{-\infty}^{+\infty}&\frac{1}{\pi(x^2+(y_0+1)^2)}\left|f\left(\frac{x+iy_0-i}{x+iy_0+i}\right)\right|^p dx \\ = \int_{C_{y_0}}&|f(z)|^p\frac{|dz|}{2\pi}\end{aligned}$$

So it remains to verify that

$$\sup_{0<r<1}\int_{\mathbb{T}}|f(r\xi)|^p|d\xi| < \infty \Leftrightarrow \sup_{y>0}\int_{C_y}|f|^p|dz| < \infty, \forall f \in \operatorname{Hol}(\mathbb{D}).$$

( $\implies$ ) Using Lemma 8.3.2 (i) $\implies$ : for any closed curve $\gamma \in \mathbb{D}$,

$$\begin{aligned} &\int_\gamma |f(z)|^p dz \le 2\int_{\mathbb{T}} |f(z)|^p dz \\ &\implies \sup \int_\gamma |f(z)|^p dz \le 2 \sup \int_{\mathbb{T}} |f(z)|^p dz \\ &\implies \sup \int_\gamma |f(z)|^p dz < \infty \\ &\implies \sup_{y>0} \int_{C_y} |f(z)|^p dz < \infty \text{ [for } \gamma = C_y] \end{aligned}$$

( $\impliedby$ )To prove the converse, let $g \in H^p_+$. By Lemma 8.3.2 (ii), $g$ is bounded on every half-plane $\operatorname{Im} z \ge y > 0$. Hence $g \circ w$ is bounded on the disc $\operatorname{int}(C_y)$. Since the function $(1-z)$ is outer on the $\operatorname{int}(C_y)$ (no-zero inside the interior) and $f = \pi\Big(\big(\frac{2i}{1-z}\big)^2\Big)^{\frac{1}{p}}(g\, o\, w) \in L^p(C_y)$,[*1] we get $f \in H^p(C_y)$ by the integral maximum principle 5.7.7 (iv). (We use the previous theory for the following classes $H^p(D)$ over disc $\mathbb{D} = \operatorname{int}\,(C_y)$, instead of the unit disc $\mathbb{D}$; the corresponding modifications, including the very definition of $H^p(\mathbb{D})$, do not cause any difficulties and can be obtained by a linear change of variable). Now, applying Lemma 8.3.2(i) to the circle $\gamma(r) = \{z \in \mathbb{C} : |z| = r\} \subset \operatorname{int}(C_y)$, we get

$$\int_{\gamma(r)} |f(z)|^p |dz| \le 2 \sup_{y>0} \int_{C_y} |f(z)|^p |dz|.$$

In fact, the Poisson representation (Corollary 8.4.1) implies that for $g \in H^p_+$, the norms

$$\Big(\int_{\mathbb{R}} |g(x+iy)|^p dx\Big)^{\frac{1}{p}}$$

are monotonically increasing in $y > 0$ and tend to $\|g|_{\mathbb{R}}\|_{L^p}$ as $y \to 0$ (to see this, use approximate identity properties of the Poisson kernel). This shows that $\|g|_{\mathbb{R}}\|_{L^p} = \|g\|_{H^p_+}$. ☐

[*1] $[\int_{C_y} (\pi(\frac{2i}{1-z})^2)|go\,w|^p(z)dz = \int_{\{\text{line passing through y}\}} |go\,w|^p dw = \int_{AB} |g(w)|^p dw < \infty$ since $g \in H^p(\mathbb{C}_+) \implies$

$$\sup_{y>0} \Big(\int_{\mathbb{R}} |g(x+iy)|^p dx\Big)^{\frac{1}{p}} < \infty \implies \forall y > 0, \int_{\mathbb{R}} |g(x+iy)|^p dx < \infty$$

]

**Theorem 8.3.4.** (*R. Paley and N. Wiener, 1934*)

$$H^p(\mathbb{C}_+) = \mathcal{F}^{-1} L^2(\mathbb{R}_+)$$

*Proof.* This is immediate from Lemma 8.2.3 and Theorem 8.3.3. ☐

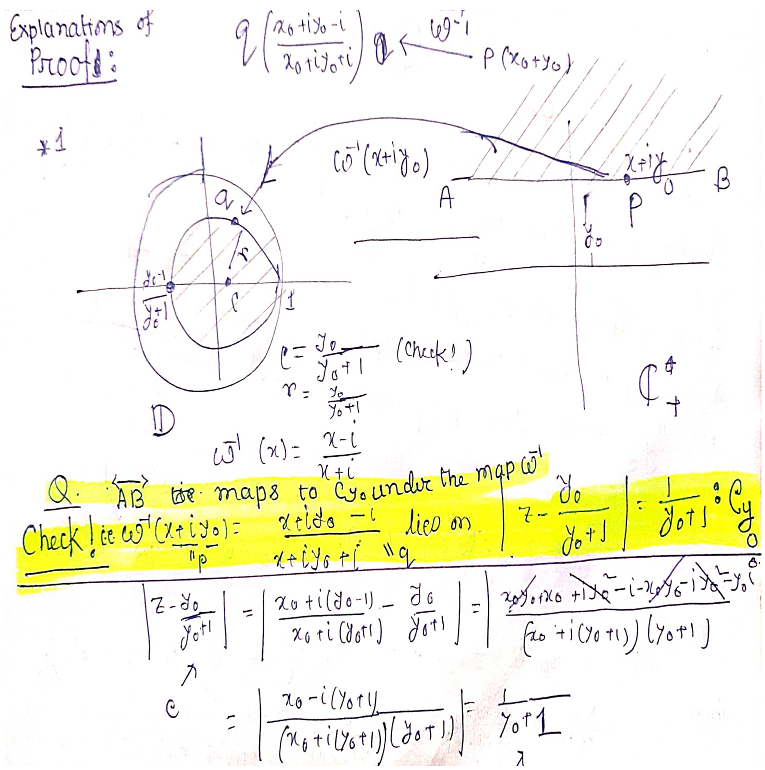


Figure 8.1: Fig3

## 8.4 Canonical factorization and other properties

The following properties are straightforward consequences of the change of variables from Section 8.1, Theorem 8.3.3, and the corresponding facts from $H^p$ theory in the disc $\mathbb{D}$.

**Corollary 8.4.1.** (*Poisson formula*) *If* $f \in H^p(\mathbb{C}_+)$, $1 \leq p \leq \infty$, *then*

$$f(x+iy) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2} f(t)dt, \; y > 0.$$

*Proof.* $f \in H^P(\mathbb{C}_+) \implies$ there exists $g \in H^p(\mathbb{D})$ such that $U_p g(z) = f(z), z \in \mathbb{C}_+ \implies f(z) = (\frac{1}{\pi(z+i)})^{1/p} g(\frac{z-i}{z+i}), z \in \mathbb{C}_+$. Now put $w = \frac{z-i}{z+i} \in \mathbb{D}$ for $z \in \mathbb{C}_+$; then $(\frac{1}{z+i})^2 = (\frac{1-w}{2i})^2$ hence $f(z)$ can be re written as

$$f(z) = \left(\frac{1}{\pi}(\frac{1-w}{2i})^2\right)^{1/p} g(w) \text{ for } z \in \mathbb{C}_+ \text{ and } w \in \mathbb{D}.$$
$$= h(w) \in H^p(\mathbb{D}) \; \left[\text{since} \left(\frac{1}{\pi}(\frac{1-w}{2i})^2\right)^{1/p} \text{ is bounded on } \mathbb{D} \text{ and } g \in H^p(\mathbb{D})\right]$$

Now using Poisson formula for $h$ on $\mathbb{D}$ :

$$\begin{aligned}
f(z) = f(x+iy) = h(w) &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{h}(\xi)\frac{1-|w|^2}{|\xi - w|^2}|d\xi| \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \tilde{h}(\frac{t-i}{t+i})\frac{1-|\frac{z-i}{z+i}|^2}{|\frac{t-i}{t+i} - \frac{z-i}{z+i}|^2}\frac{2dt}{1+t^2} \ [\text{since } \xi = \frac{t-i}{t+i}, w = \frac{z-i}{z+i}] \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \tilde{f}(t)\frac{2y}{|t-z|^2}dt \\
&= \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\tilde{f}(t)dt, y>0
\end{aligned}$$

□

**Corollary 8.4.2.** (*Boundary uniqueness theorem*) *If* $f \in H^p(\mathbb{C}_+)$, $1 \le p \le \infty$ *and* $f \neq 0$, *then*

$$\int_{\mathbb{R}} \frac{|\log|f(x)||}{1+x^2}dx < \infty.$$

*Proof.* Let $f \in H^p(\mathbb{C}_+) \implies f(z) = h(w)$ for $z \in \mathbb{C}_+, w \in \mathbb{D}$ and $h \in H^p(\mathbb{D})$ By the boundary uniqueness theorem for the disk:

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} |\log|h(\xi)|||d\xi| < \infty \\
\implies &\frac{1}{2\pi}\int_{\mathbb{R}} \left|\log|\tilde{f}(t)|\right| \frac{2dt}{1+t^2} < \infty \\
\implies &\int_{\mathbb{R}} \frac{|\log|\tilde{f}(t)||}{1+t^2}dt < \infty.
\end{aligned}$$

□

**Corollary 8.4.3.** (*Blaschke condition and Blaschke product*) *If* $f \in H^p(\mathbb{C}_+)$, $1 \le p \le \infty$, *and* $f \neq 0$, *then*

$$\sum \frac{\operatorname{Im}\lambda_n}{1+|\lambda_n|^2} < \infty,$$

*where* $\lambda_n$ *are the zero of* $f$ *in* $\mathbb{C}_+$ *(counting multiplicities). The corresponding Blaschke product (having similar properties as in* $\mathbb{D}$*) is*

$$B(z) = \prod_n \epsilon_n \frac{z-\lambda_n}{z-\bar{\lambda}_n},\ z \in \mathbb{C}_+,$$

*where* $\epsilon_n = \frac{|\lambda_n^2+1|}{\lambda_n^2+1}$ *(by definition,* $\epsilon_n = 1$ *for* $\lambda_n = i$*).*

*Proof.* Let $f \in H^P(\mathbb{C}_+), 1 \le p \le \infty$ and $f \neq 0$. Then there exists $g \in H^p(\mathbb{D})$ such that $U_p g = f$. Now, $f(\lambda_n) = 0 \implies U_p g(\lambda_n) = 0 \implies \left(\frac{1}{\pi(\lambda_n+i)^2}\right)^{1/p} g(\frac{\lambda_n - i}{\lambda_n + i}) = 0 \implies g(\gamma_n) = 0$ where

$\gamma_n = \frac{\lambda_n - i}{\lambda_n + i} \in \mathbb{D}$. So $\lambda_n' s$ are roots of $f$ if and only if $\gamma_n' s$ are roots of $g$.

$$|\gamma_n| = \frac{|\lambda_n - i|}{|\lambda_n + i|} \implies |\gamma_n|^2 = \gamma_n\overline{\gamma_n} = \frac{\lambda_n - i}{\lambda_n + i}.\frac{\overline{\lambda}_n + i}{\overline{\lambda}_n - i} = \frac{|\lambda_n|^2 + \lambda_n i - i\overline{\lambda}_n + 1}{|\lambda_n|^2 - \lambda_n i + i\overline{\lambda_n} + 1} = \frac{1 + |\lambda_n|^2 - 2y_n}{1 + |\lambda_n|^2 + 2y_n}$$

where $y_n = \operatorname{Im}(\lambda_n)$.
Calculate $1 - |\gamma_n|^2 = \frac{4y_n}{1+|y_n|^2+2y_n}$.
We have $g \in H^p(\mathbb{D})$. So $|\gamma_n| \to 1$ when $n \to \infty$ as $\sum_{n\geq 1}(1 - |\gamma_n|) < \infty$ since $\sum a_n < \infty \implies \lim a_n = 0$. So $\lim_{n\to\infty}(|\lambda_n| - 1) = 0 \implies \lim_{n\to\infty} |\lambda_n| = 1)$ since $\lim_{n\to\infty} \frac{1-|\lambda_n|^2}{1-|\lambda_n|} = \lim_{n\to\infty}(1 + |\lambda_n|) = 2(\neq 0)$. So $\sum(1 - |\gamma_n|) < \infty \Leftrightarrow \sum(1 - |\gamma_n|^2) < \infty$ (Limit comparison Test of the series). Now consider the series: $\sum \frac{y_n}{1+|\lambda_n|^2}$

$$\frac{1 - |\gamma_n|^2}{\frac{y_n}{1+|\lambda_n|^2}} = \frac{\frac{4y_n}{1+|y_n|^2+2y_n}}{\frac{y_n}{1+|\lambda_n|^2}} = \frac{4(1 + |\lambda_n|^2)}{1 + |\lambda_n|^2 + 2y_n} \to 4$$

( If $|\lambda_n| \to 1$ in $\mathbb{C}_+ \implies |\lambda_n| \to x$ axis $\implies \operatorname{Im}\lambda_n = y_n \to 0$ )
Hence by Comparison Test $\sum(1 - |\gamma_n|^2) < \infty \Leftrightarrow \sum \frac{y_n}{1+|\lambda_n|^2} < \infty$. Hence the desired Blaschke condition is: $\sum \frac{\operatorname{Im}(\lambda_n)}{1+|\lambda_n|^2} < \infty$.
■ The Blaschke factor for $g \in H^p(\mathbb{D})$ is $\Pi b_{\gamma_n} \frac{\gamma_n - w}{1-\overline{\gamma}_n w}$ for $w \in \mathbb{D}$ and $g(\gamma_n) = 0$. Here $b_{\gamma_n} = \frac{|\gamma_n|}{\gamma_n} = \frac{|\frac{\lambda_n - i}{\lambda_n + i}|}{\frac{\lambda_n - i}{\lambda_n + i}} = \frac{(\lambda_n + i)|\lambda_n - i|}{(\lambda_n - i)|\lambda_n + i|} = \frac{|\lambda_n^2 + 1|(\lambda_n + i)}{(\lambda_n - i)|\lambda_n + i|^2} = \frac{|\lambda_n^2 + 1|(\lambda_n + i)(\lambda_n + i)}{(\lambda_n^2 + 1)(\lambda_n + i)(\overline{\lambda_n + i})} = \frac{|\lambda_n^2 + 1|\lambda_n + i}{\lambda_n^2 + 1\lambda_n + 1}$
Now

$$\frac{\gamma_n - w}{1 - \overline{\gamma}_n w} = \frac{\frac{\lambda_n - i}{\lambda_n + i} - \frac{z-i}{z+i}}{1 - \frac{(\overline{\lambda_n} + i)(z-i)}{(\overline{\lambda}_n - i)(z+i)}} = \frac{2i(\lambda_n - z)(\overline{\lambda}_n - i)}{2i(\overline{\lambda}_n - z)(\lambda_n + i)}. = \frac{(z - \lambda_n)\overline{(\lambda_n + i)}}{(z - \overline{\lambda}_n)(\lambda_n + i)}$$

$B(z) = \Pi_n b_{\gamma_n} \frac{\gamma_n - w}{1-\overline{\gamma}_n w} = \Pi_n \epsilon_n \frac{z - \lambda_n}{z - \overline{\lambda}_n}$ where $\epsilon_n = \frac{|\lambda_n^2 + 1|}{\lambda_n^2 + 1}$.
Now $\lambda_n = i \implies \gamma_n = 0 \implies w$ is a factor of $B(w), w \in \mathbb{D} \implies \frac{z-i}{z+i}$ is a factor of $B(z)$ and obviously $\epsilon_n = 1$. □

**Theorem 8.4.4.** *Each function $f \in H^p(\mathbb{C}_+)$; $1 \leq p \leq \infty$, has a unique factorization of the form $f = \lambda BV[f]$, where $\lambda \in \mathbb{T}$, $B$ is the Blaschke product constructed from the zeroes of $f$, $V$ is a singular inner function (an $H^\infty$ function having no zeroes in $\mathbb{C}_+$ and with unimodular boundary values on $\mathbb{R}$) of the form*

$$V(z) = e^{iaz}V_v(z) = e^{iaz} \exp\left(i \int_{\mathbb{R}} \frac{1 + tz}{t - z} dv(t)\right),$$

*where $a \geq 0$, and $v$ is a finite positive singular measure on $\mathbb{R}$, $[f]$ is the Schwarz-Herglotz outer factor of the form*

$$[f](z) = \exp\Big(\frac{1}{\pi i} \int_{\mathbb{R}} \frac{1 + tz}{t - z} \log|f(t)| \frac{dt}{1 + t^2}\Big),\ z \in \mathbb{C}_+$$

*Proof.* Let $f \in H^p(\mathbb{C}_+)$. Then there exists $g \in H^p(\mathbb{D})$ such that $f(z) = g(w)$ for $z \in \mathbb{C}_+$ and $w \in \mathbb{D}$. Now

$$[g](w) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{\xi + w}{\xi - w} \log|\tilde{g}(\xi)||d\xi|\right]$$

Putting $\xi = \frac{t-i}{t+i}$ and $w = \frac{z-i}{z+i}$ we have:

$$\frac{t-i}{t+i} \pm \frac{z-i}{z+i} = \frac{\{tz + 1 + it - iz\} \pm \{tz + 1 - i(t-z)\}}{(t+i)(z+i)}$$

$$\implies \frac{\xi + w}{\xi - w} = \frac{1 + tz}{i(t-z)}$$

Hence $[f](z) = [g](\frac{z-i}{z+i}) = \exp\left(\frac{1}{\pi i}\int_{\mathbb{R}} \frac{1+tz}{t-z} \log|\tilde{f}(t)| \frac{dt}{1+t^2}\right), \ z \in \mathbb{C}_+$

■ As $g \in H^p(\mathbb{D})$, $g$ can be written as $g = \lambda BS[g]$. Here $S(w) = \exp\left[-\int_{\mathbb{T}} \frac{\xi+w}{\xi-w} d\mu(\xi)\right]$ for $w \in \mathbb{D}$ and $\xi \in \mathbb{T}$ and $\mu \perp m$. $\therefore S(\frac{z-i}{z+i}) = \exp\left[-\int_{\mathbb{R}} \frac{1+tz}{i(t-z)} d\mu\left(\frac{t-i}{t+i}\right)\right]$

$$S(w) = \exp\left[-\frac{1+w}{1-w}\mu(\{1\}) - \int_{\mathbb{T}\setminus\{1\}} \frac{\xi + w}{\xi - w} d\mu(\xi)\right]$$

$$\implies S\left(\frac{z-i}{z+i}\right) = \exp\left[i\mu(\{1\})z - \int_{\mathbb{R}} \frac{1+tz}{i(t-z)} d\mu\left(\frac{t-i}{t+i}\right)\right] [\because \frac{i(1+w)}{1-w} = -z]$$

$$\therefore V(z) = e^{i\alpha z} \exp\left[\int_{\mathbb{R}} \frac{1+tz}{i(t-z)} d\nu(t)\right]$$

when $\alpha = \mu\{1\}, d\nu(t) = d\mu\left(\frac{t-i}{t+i}\right) = \frac{2}{1+t^2} d\mu(t).$ □

*Remark* 8.4.5. It is clear from the previous computations that other facts of the Hardy Nevanlinna theory of Sections 3 and 4 in the disc can be transferred to the half-plane. In particular, the properties of the inner outer factorization from subsections 6.2-6.3 still hold with corresponding modifications caused by the change of variables. For instance, a function $f \in H^p(\mathbb{C}_+)$ having an analytic continuation across a point $x \in \mathbb{R}$ has singular representing measure zero in a neighborhood of this point. To find the point mass of the singular measure, the logarithmic residues of Section 4 (to be added) can be redefined and computed and so on and so on. In particular, the point mass at $\infty$ is $a = -\lim_{y\to\infty} \frac{1}{y} \log|f(iy)|$.

## 8.5 Invariant subspaces

Here we consider translation invariant subspaces of $L^2(\mathbb{R})$ and their Fourier dual objects - character invariant subspaces.

## 8.6 Duality between translation and multiplication by characters

Define the translation operator $\tau_s$ by

$$(\tau_s f)(x) = f(x-s),\ x \in \mathbb{R},\ \text{for } s \in \mathbb{R}.$$

This is a group of unitary operators on $L^2(\mathbb{R})$. A subspace $E \subset L^2(\mathbb{R})$ (closed, as always) is said to be (translation) 2-invariant and if $\tau_s E \subset E$ for all $s \in \mathbb{R}$, and (translation) 1-invariant if $\tau_s E \subset E$ for all $s \geq 0$ but not for (all) $s < 0$. The Fourier image of the translation operator $\tau_s$ is the multiplication operator by the corresponding character $e^{isx}$ of the group $\mathbb{R}$:

$$\tau_s(\mathcal{F}f) = \mathcal{F}(e^{is}f),\ \text{for all } f \in L^2(\mathbb{R}).$$

Without any risk of confusion, we write $e^{isx}$ both for the function $x \longmapsto e^{isx}$ and for the multiplication operator by this function, $f \longmapsto e^{isx}f$. Hence, we have

$$\tau_s = \mathcal{F}e^{isx}\mathcal{F}^{-1},$$

that is, the groups $(\tau_s)_{s\in\mathbb{R}}$ and $(e^{isx})_{s\in\mathbb{R}}$ are unitarily equivalent (conjugate) via the Fourier transform.

We use the same terminology as above for $e^{isx}$ -invariant subspaces. A subspace $E \subset L^2(\mathbb{R})$ is (character) 2-invariant if $e^{isx}E \subset E$ for all $s \in \mathbb{R}$, and (character) 1-invariant if $e^{isx}E \subset E$ for $s \geq 0$ but for (all) $s < 0$. Hence, $E$ is an 1- or 2- character invariant if and only if its Fourier image $\mathcal{F}E$ is a 1- or 2- translation invariant subspace.

Clearly, the Hardy space $H^2(\mathbb{C}_+)$ is a character 1-invariant subspace, and $\mathcal{F}H^2(\mathbb{C}_+) = L^2(\mathbb{R}_+)$ is translation 1-invariant.

Below, we will derive analogue of the Wiener theorem 3.0.4 and Beurling Helson theorem 3.1.1 for character invariant subspaces. First, we prepare the transfer of these results to $L^2(\mathbb{R})$ by means of the operator $U_2$.

**Lemma 8.6.1.** *Let $u_s = \exp\left(s\frac{z+1}{z-1}\right)$ $s \in \mathbb{R}$, and let $E$ be a (closed) subspace of $L^2(\mathbb{R})$. The $E$ is a 2-invariant subspace (with respect to the shift operator $f \longmapsto zf$) if and only if $u_s E \subset E$, for all $s \in \mathbb{R}$, and $E$ is 1-invariant subspace if and only if $u_s E \subset E$, for all $s \geq 0$, but not for (all) $s < 0$.*

*Proof.* If $b \in H^\infty$, and $E$ is a $z$-invariant subspace of $L^2(\mathbb{T})$, then $bE \subset E$. Indeed, by DCT, we have

$$\lim_{r\to 1} \|bf - b_r f\|_2 = 0,\ \text{for all } f \in E,$$

where $b_r(z) = b(rz)$.

On the other hand, $z^n f \in E$, for $n \geq 0$ and therefore, $b_r f \in E$, since Taylor series of $b_r$ is absolutely convergent on $\mathbb{T}$. Hence $bf \in E$. The same holds true for $\bar{b} \in H^\infty$ and $\bar{z}-$invariant subspace $E$. These prove the "only if" part of the lemma.
By analogous reasoning, to prove the converse, it suffices to show that the function $z$ is the bounded pointwise limit of functions $\phi_s = \dfrac{u_s - (1-s)}{u_s - (1+s)}$ as $s \to 0_+$. We have $\mathrm{Re}(1 - u_s(\zeta)) \geq 0$, and hence $|\phi_s(\zeta)| \leq 1$, for $\zeta \in \mathbb{T}$. On the other hand, using the standard formula
$e^{sw} = 1 + sw + o(s)$ as $s \to 0_+$, we easily get $\lim_{s\to 0} \phi_s(\zeta) = \zeta$ for $\zeta \in \mathbb{T} \setminus \{1\}$. □

**Theorem 8.6.2.** (*P. Lax, 1959*) *Let $E$ be a subspace of $L^2(\mathbb{R})$.*

*(i) $E$ is a (character) 2-invariant subspace if and only if $E = \chi_\Sigma L^2(\mathbb{R})$ for a measurable subset $\Sigma \subset \mathbb{R}$.*

*(ii) $E$ is a (character) 1-invariant subspace if and only if $E = \mathcal{F}_q H^2(\mathbb{C}_+)$ for a measurable function $q$ on $\mathbb{R}$ with $|q| = 1$ a.e.*

*Proof.* Lemma 8.6.1 shows that $E$ is 2 or 1-invariant if and only if its preimage $U_2^{-1}E \subset L^2(\mathbb{T})$ has the same property with respect to the shift operator on $L^2(\mathbb{R})$. The results thus follow by applying theorems 3.0.4, 3.1.1 and Theorem 8.3.3. □

**Corollary 8.6.3.** *Let $E$ be a subspace of $L^2(\mathbb{R})$.*

1. *$E$ is translation 2-invariant if and only if $E = \mathcal{F}\chi_\Sigma L^2(\mathbb{R})$ for a measurable subset $\Sigma \subset \mathbb{R}$.*
2. *$E$ is translation 1-invariant if and only if $E = \mathcal{F}qH^2(\mathbb{C}_+)$ for a measurable function $q$ on $\mathbb{R}$ with $|q| = 1$ a.e.*

Indeed, it suffices to use Theorem 8.6.2 and duality of Subsection 8.6.

**Corollary 8.6.4.** *(i) If $F \subset H^2(\mathbb{C}_+)$, then $\overline{\mathrm{span}}_{H^2_+}\{e^{isx}F : s \geq 0\} = \Theta H^2(\mathbb{C}_+)$, where $\Theta$ is the g.c.d of the inner factors of $f \in F$.*

*(ii) If $F \subset L^2(\mathbb{R}_+)$, then $\overline{\mathrm{span}}_{L^2(\mathbb{R}_+)}\{\tau_s F : s \geq 0\} = \mathcal{F}(\Theta H^2(\mathbb{C}_+))$, where $\Theta$ is the g.c.d of the inner factors of $\mathcal{F}^{-1}f$, $f \in F$.*

*(iii) If $f \in L^2(\mathbb{R})$, then $\overline{\mathrm{span}}_{L^2(\mathbb{R})}\{e^{isx}f : s \in \mathbb{R}\} = L^2(\mathbb{R})$ if and only if $f \neq 0$ a.e. on $\mathbb{R}$.*

*(iv) If $f \in L^2(\mathbb{R})$, then $\overline{\mathrm{span}}_{L^2(\mathbb{R})}\{e^{isx}f : s \geq 0\} = L^2(\mathbb{R})$ if and only if $f \neq 0$ a.e. and*

$$\int_{\mathbb{R}} (1 + x^2) \log |f| dx = -\infty$$

*(v) If $f \in L^2(\mathbb{R})$, then $\overline{\mathrm{span}}_{L^2(\mathbb{R})}\{\tau_s f : s \geq 0\} = L^2(\mathbb{R})$ if and only if $\mathcal{F}f \neq 0$ a.e. on $\mathbb{R}$*

*(vi) If $f \in L^2(\mathbb{R})$, then $\overline{\text{span}}_{L^2(\mathbb{R})}\{\tau_s f : s \geq 0\} = L^2(\mathbb{R})$ if and only if $\mathcal{F}f \neq 0$ a.e. and*

$$\int_{\mathbb{R}} (1 + x^2) \log |\mathcal{F}f| dx = -\infty.$$

Indeed, it suffices to use Theorem 8.6.2 and Corollary 8.6.3 and the corresponding properties of $z$-invariant subspaces of $L^2(\mathbb{R})$.

**Theorem 8.6.5.** *(Cauchy Representation) Assume that $1 \leq p < \infty$.*

*(i) Let $F(z)$ belongs to $H^p(\mathbb{C}_+)$ and let $F(x)$ be its boundary function. Then $F(x) \in L^p(-\infty, \infty)$. $F(z) =$*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(t)}{t - z} dt, y > 0 \text{ and} \tag{8.6.1}$$

$$0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)}{t - z}, y < 0. \tag{8.6.2}$$

*(ii) Let $F(x)$ be any function in $L^p(-\infty, \infty)$ satisfying (6.2). Then (6.1) and the Poisson representation (Corollary 8.4.1 ) define one and the same function $F(z)$ on $\mathbb{C}_+$. $F(z)$ belongs to $H^p(\mathbb{C}_+)$ and the non-tangential boundary function is equal to $F(x)$ a.e.*

*Proof.* (i) By Fatou's lemma and the definition of $H^p(\mathbb{C}_+)$, we have:

$$\int_{\infty}^{\infty} |F(x)|^p dx \leq \liminf_{y \to 0} \int_{-\infty}^{\infty} |F(x + iy)|^p dx < \infty \implies F \in L^p(-\infty, \infty).$$

Let $G(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)}{t-z} dt, y \neq 0$ Then $G(z)$ is homomorphic separately for $y > 0$ and $y < 0$. For $y > 0$

$$\begin{aligned} G(z) - G(\overline{z}) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[ \frac{1}{t - z} - \frac{1}{t - \overline{z}} \right] F(t) dt \\ &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{F(t)}{(t - x)^2 + y^2} dt \\ &= F(z). \end{aligned}$$

Since $F(z)$ and $G(z)$ are homomorphic on $\mathbb{C}_+$ so is $G(\overline{z}), z \in \mathbb{C}_+$. But

$$\overline{G(\overline{z})} = -\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\overline{F}(t)}{t - z} dt, z \in \mathbb{C}_+$$

is also homomorphic. Since $\overline{G}(\overline{z})$ and $G(\overline{z})$ are both holomorphic, hence $G(\overline{z})$ is constant on $\mathbb{C}_+$. Since $G(-iy) \to 0$ as $y \to \infty$, $G(\overline{z} \neq 0)$ on $\mathbb{C}_+$. Thus (8.6.1) and (6.2) holds.

(ii) Assuming

$$0 = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(t)}{t-z}dt, \forall y<0 \implies 0 = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(t)}{t-\overline{z}}dt, \forall y>0 \implies 0 = G(\overline{z}), \forall y>0$$

In (i) we have proved: $G(z) - G(\overline{z}) = F(z)$ for $y > 0 \implies G(z) = F(z)$. Applying Holders inequality:

$$\begin{aligned}
\int_{-\infty}^{\infty}|F(x+iy)|^p dx &= \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{y/\pi}{(t-x)^2+y^2}F(t)dt\right|^p dx \\
&= \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\left[\frac{y/\pi}{(t-x)^2+y^2}F(t)\right]^{1/p}\left[\frac{y/\pi}{(t-x)^2+y^2}\right]^{1/q}dt\right|^p dx \\
&\leq \int_{-\infty}^{\infty}\left|\left(\int_{-\infty}^{\infty}\frac{y/\pi}{(t-x)^2+y^2}|F(t)|dt\right)^{1/p}\left(\int_{-\infty}^{\infty}\frac{y/\pi}{(t-x)^2+y^2}dt\right)^{1/q}\right|^p dx \\
&\leq \int_{-\infty}^{\infty}\left|\left(\int_{-\infty}^{\infty}\frac{y/\pi}{(t-x)^2+y^2}|F(t)|dt\right)\left(\int_{-\infty}^{\infty}\frac{y/\pi}{(t-x)^2+y^2}dt\right)^{p/q}\right| dx \\
&\leq \int_{-\infty}^{\infty}\left|\left(\int_{-\infty}^{\infty}\frac{y/\pi}{(t-x)^2+y^2}|F(t)|^p dt\right)\right| dx \\
&\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y/\pi}{(t-x)^2+y^2}|F(t)|^p dx dt \\
&\leq \int_{-\infty}^{\infty}|F(t)|^p dt
\end{aligned}$$

This shows that $F \in H^p(\mathbb{C}_+)$ □

## 8.7 Cauchy kernels and $L^p$- decomposition

**Theorem 8.7.1.** *(i) Show that* $H^p(\mathbb{C}_+) = \overline{\operatorname{span}}_{L^2(\mathbb{R})}\left\{\frac{1}{x-\bar{\mu}} : \operatorname{Im}\mu > 0\right\}$ *for* $1 \leq p \leq \infty$. *(Hint: Use* $H^p(\mathbb{C}_+) = U_p H^p$ *and solve* $U_p f = \frac{1}{x-\bar{\mu}}$*).*

*(ii) Let* $1 < p < \infty$. *Show that* $L^p(\mathbb{R}) = H^p(\mathbb{C}_+) \oplus H^p(\mathbb{C}_-)$, *where* $\oplus$ *stands for the orthogonal sum for* $p = 2$ *and direct sum for* $p \neq 2$.

*(iii) Let*

$$Cf(z) = \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(t)}{t-z}dt, \quad z \in \mathbb{C}\setminus\mathbb{R}$$

*be the Cauchy integral of* $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, *then the followings are equivalent.*

*(a)* $f \in H^p(\mathbb{C}_+)$.

*(b)* $Cf = f_*$, *where* $f_*$ *stands for the Poisson integral extension.*

*(c)* $Cf(z) = 0$ *for* $\operatorname{Im} z < 0$.

*Proof.* Previously solved. □

**Theorem 8.7.2.** (*The Paley Wiener theorem*) *An entire function $E$ is called of exponential type if*

$$\overline{\lim}_{|z|\to\infty}\frac{\log|E(z)|}{|z|}<\infty;$$

*the limit itself is the* type *of $E$. Let $\mathcal{E}_a$ = set of all entire functions of exponential type $\le a$. For $a>0$, show that the followings are equivalent.*

*(i) $E\in\mathcal{E}_a$ and $E|_{\mathbb{R}}\in L^2(\mathbb{R})$.*

*(ii) There exists $f\in L^2(\mathbb{R})$ such that $\mathcal{F}f=E$ and $\operatorname{supp} f\in[-a,a]$.*

Hint: For (ii) $\implies$ (i), estimate the exponential type of $E$ applying the Cauchy inequality to the Fourier transform of $f$ :

$$|E(z)|=\Big|\int_{-a}^{a}e^{-ixz}f(x)dx\Big|\le\|f\|_2\Big(\frac{e^{2a|\operatorname{Im}z|}-1}{\operatorname{Im}z}\Big)^{\frac12}\le(2a)^{\frac12}e^{a|\operatorname{Im}z|}.$$

Moreover, $\|E\|_2=\|f\|_2$ by Plancherel's theorem:
(i) $\implies$ (ii): First suppose that $E|_{\mathbb{R}}\in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$. Then by Phragmèn-Lindelöf theorem $|E(z)|\le\|E\|_\infty e^{a|\operatorname{Im}z|}$, for $z\in\mathbb{C}$, implies

$$|E_\lambda(z)|=\frac{i\lambda}{z+i\lambda}e^{aiz}E(z)\in H^2(\mathbb{C}_+),\ \ \lambda>0.$$

The Paley Wiener theorem 8.3.4 entails that $\mathcal{F}(E_\lambda)=0$ a.e. on $(-\infty,0)$ and hence $\mathcal{F}(e^{aiz}E)=0$ on $(-\infty,a)$ (because $\lim\limits_{\lambda\to\infty}\|E_\lambda-e^{aiz}E\|_{L^2(\mathbb{R})}=0$). Therefore, $\mathcal{F}(E)=\tau_a\mathcal{F}(e^{iaz}E)=0$ a.e on $(-\infty,-a)$. Similarly $\mathcal{F}(E)=0$ a.e. on $(a,\infty.)$ and we get (ii). In general case, replace $E$ by $E^\epsilon(z)=\int_{\mathbb{R}}E(z-t)\phi_\epsilon(t)dt$, where $\phi_\epsilon(t)=\epsilon^{-1}\phi(\frac{t}{\epsilon})$, $\phi\ge0$ is compactly supported in $\mathbb{R}$. It is easy to see that $E^\epsilon\in\mathcal{E}_{a+\epsilon}$ and $\operatorname{supp}(E^\epsilon)\subset[-a-\epsilon,a+\epsilon]$, and we have $\lim\limits_{\epsilon\to0}\|E^\epsilon-E\|_{L^2(\mathbb{R})}=0$.

**Question 8.7.3.** (a) Show that $f\in H^2(\mathbb{C}_+)$ if and only if $f\in L^2(\mathbb{R})$ and $\mathcal{F}(f)=0$ a.e. on $\mathbb{R}$.

(b) Find $f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ such that $L^2(\mathbb{R})=\overline{\operatorname{span}}_{L^2(\mathbb{R})}(\tau_s f : s\in\mathbb{R})$ and $L^1(\mathbb{R})\neq\overline{\operatorname{span}}_{L^1(\mathbb{R})}\{\tau_s f : s\in\mathbb{R}\}$ (Hint: Consider $f=\chi_{(a,b)}$.)

(c) **Riesz Brother's theorem for** $\mathbb{R}$: Let $\mu$ be a complex Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}e^{ist}d\mu(t)=0$ for all $s>0$. Show that $\mu<<m$.

## 8.8 Exercises

**Exercise 8.8.1.** $H^1(\mathbb{C}_+)=H^2(\mathbb{C}_+)H^2(\mathbb{C}_+)$

*Proof.* We know that $w : \mathbb{D} \to \mathbb{C}_+$ is a conformal map.

$$\begin{aligned} F \in H^1(\mathbb{C}_+) &\implies F \cdot w' \in H^1(\mathbb{D}) \\ &\implies F \cdot w' = G_1 \cdot G_2 \text{ where } G_1, G_2 \in H^2(\mathbb{D}) \\ &\implies F = [G_1 \cdot (w')^{-1/2}][G_2 \cdot (w')^{-1/2}] \end{aligned}$$

Now define two functions $g_1, g_2$ by the following forms:

$$\begin{aligned} &g_1 \bullet w = G_1 \cdot (w')^{-1/2} \\ &g_2 \bullet w = G_2 \cdot (w')^{-1/2} \\ &\implies (g_1 \bullet w)(w')^{1/2} = G_1 \in H^2(\mathbb{D}) \\ &\implies (g_2 \bullet w)(w')^{1/2} = G_2 \in H^2(\mathbb{D}) \\ &\implies g_1, g_2 \in H^2(\mathbb{C}_+) \end{aligned}$$

and

$$\begin{aligned} &F = (g_1 \bullet w)(g_2 \bullet w) \\ &\implies F \bullet w^{-1} = g_1 \cdot g_2 \\ &\implies f = g_1 \cdot g_2. \end{aligned}$$

□

# Chapter 9

# Problem Sets

## 9.1 Problem Set I

1. Determine the validity (TRUE/FALSE) of each of the following statements, providing rigorous justification in every case.

   (a) Every subspace of $L^2(\mathbb{T}, m)$ of dimension greater than one is simply invariant.

   (b) Let $H^2 = \overline{\text{span}}\{z^n : n \geq 0\}$. Is it true that $H^2 \perp zH^2$?

   (c) If $0 \neq f \in H^2$, then $E_f = \overline{\text{span}}\{z^n f : n \geq 0\}$ is a reducing subspace of $H^2$.

   (d) Let $\mu$ be a finite measure on $\mathbb{T}$. Is $E_f$ necessarily a reducing subspace of $L^2(\mu)$?

   (e) If $\Theta \in H^2$ is an inner function, does it follow that

   $$\overline{\text{span}}\{z^n \Theta : n \geq 0\} = \Theta H^2?$$

   (f) Is $H^2(\mathbb{T}, m) \cap L^\infty(\mathbb{T}, m)$ dense in $L^2(\mathbb{T}, m)$?

   (g) Let $\mu$ be a finite Borel measure on $\mathbb{T}$. If $\bar{z}^2 \in H^2(\mu)$, does it follow that $H^2(\mu) = z^2 H^2(\mu)$?

   (h) Let $f = \chi_{[0, \frac{\pi}{2}]}$. Does it follow that

   $$\overline{\text{span}}\{z^n f : n \geq 0\}$$

   is a non-reducing subspace of $H^2(\mathbb{T}, m)$?

(i) Suppose $0 \leq \mu \ll m$. Can it happen that $H^2(\mu)$ is a proper reducing subspace of $L^2(\mu)$?

2. Let $\mu$ be a finite Borel measure on $\mathbb{T}$. Prove or disprove that

$$L^2(\mu) = L^2(\mu) \cdot L^2(\mu).$$

3. Let $\mu$ be a finite Borel measure on $\mathbb{C}$. Prove or disprove that for every $f \in L^2(\mathbb{C}, \mu)$ there exist $g, h \in L^2(\mathbb{C}, \mu)$ such that $f = gh$.

4. Let $w \in L^1_+(\mathbb{T}, m) = \{g \in L^1(\mathbb{T}, m) : g \geq 0\}$. Suppose there exists $f \in H^2$ such that $|f|^2 = w$ a.e. on $\mathbb{T}$. Show that there exists a unique outer function $f_o$ satisfying $|f_o|^2 = w$ a.e. on $\mathbb{T}$.

5. Let $\mu$ be a finite Borel measure on $\mathbb{T}$. Define $H^2_0(\mu) = zH^2(\mu)$. Show that

$$H^2_0(\mu) = H^2_0(\mu_a) \oplus L^2(\mu_s),$$

where $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of $\mu$.

6. Let $\mu$ be a finite Borel measure on $\mathbb{T}$. Prove that the following are equivalent:

   (i) There exists a non-reducing subspace $E \subset L^2(\mu)$ with $zE \subset E$.

   (ii) There exists a nonzero complex measure $\nu$ absolutely continuous with respect to $\mu$ and orthogonal to $\mathbb{P}_+$, i.e.
$$\int_{\mathbb{T}} z^n \, d\nu = 0 \quad \forall \, n \geq 1.$$

7. Let $\mu$ be a finite measure on $\mathbb{T}$. Show that

$$zE \subseteq E \subset L^2(\mu) \quad \Rightarrow \quad zE = E$$

if and only if $m$ is not absolutely continuous with respect to $\mu$.

8. Let $\mu$ be a compactly supported finite measure on $\mathbb{C}$. Show that every reducing subspace $E$ of $L^2(\mu)$ is of the form
$$E = \chi_\sigma L^2(\mu),$$
for some Borel set $\sigma \subset \mathbb{C}$.

9. Let $L^\infty(\mathbb{T}, m)$ denote the space of essentially bounded measurable functions on $\mathbb{T}$. Prove the following:

   (i) If $f \in H^2 \cap L^\infty$, then $fH^2 \subset H^2$.

   (ii) If $f \in H^2 \cap L^\infty$ with $\|f\|_\infty < 1$, then $1 + f$ is an outer function.

   (iii) If $f \in H^2 \cap L^\infty$, then $e^f \in H^2$ is an outer function.

10. Show that $z - \lambda$ is an outer function if and only if $|\lambda| \geq 1$. Hence, deduce that a polynomial $p$ is outer if and only if $p$ has no zero in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

11. Let $\mu$ be a finite measure on $\mathbb{T}$. If $H^2(\mu)$ is a proper subspace of $L^2(\mu)$, show that

$$\operatorname{dist}(1, H_0^2(\mu)) > 0.$$

12. If $f \in H^2$ is an outer function, prove that

$$\overline{\operatorname{span}}\{z^n f : n \geq 1\} = zH^2.$$

13. Let $\mu$ be a finite Borel measure on $\mathbb{T}$ and define

$$H_0^2(\mu) = \overline{\operatorname{span}}\{z^n : n \geq 1\} \subset L^2(\mu).$$

   For $f \in L^2(\mu)$, compute $\operatorname{dist}(f, H_0^2(\mu))$.

14. Let $f \in H^1(\mathbb{T}, m) \cap L^\infty(\mathbb{T}, m)$. Show that there exist $f_j \in L^2(\mathbb{T}, m)$ $(j = 1, 2)$ such that

$$E_{f^2} = f_1 E_{f_2},$$

   where $E_g := \overline{\operatorname{span}}\{z^n g : n \geq 0\}$.

15. Let $f(z) = e^z$ and suppose $g \in H^2(\mathbb{T}, m)$ satisfies $f * g = 1$. Show that $g$ must be constant.

## 9.2 Problem Set II

1. Determine whether each of the following statements is **TRUE** or **FALSE**, providing rigorous justification in each case.

   (a) An infinite Blaschke product has only finitely many repeated factors.

   (b) For functions in $H^p(\mathbb{D})$ with $0 < p < 1$, non-tangential limits coincide with radial limits.

   (c) Can a non-zero function $f \in H^p(\mathbb{T})$, $0 < p < 1$, vanish on a set of positive measure?

   (d) If $f \in H^1(\mathbb{D})$ is outer, then necessarily $\log|f| \in L^1(\mathbb{T})$.

   (e) If $f \in L^\infty(\mathbb{T})$, then there exist inner functions $\theta_1, \theta_2$ and a sequence of polynomials $P_n$ such that $P_n(\overline{\theta}_1\theta_2) \to f$ uniformly.

   (f) For $p > 0$, let $f \in H^p(\mathbb{D})$ with $f \not\equiv 0$. Does this imply that $\log|f| \in L^1(\mathbb{T})$?

   (g) Let $f \in \mathrm{Hol}(\mathbb{D})$. Does the existence of non-tangential limits of $f$ at a.e. $\xi \in \mathbb{T}$ imply the existence of radial limits at a.e. $\xi \in \mathbb{T}$?

   (h) If $\Theta$ is an inner function in $H^2(\mathbb{T}, m)$ such that $\Theta H^2(m) = H^2(m)$, does it follow that $\Theta$ is constant a.e. with respect to $m$?

   (i) Suppose $f, g \in H^2(\mathbb{T}, m)$ are two non-zero functions with $\hat{g}(0) = 0$. Does it follow that $\widehat{(fg)}(0) = 0$?

   (j) Let $f \in H^2(\mathbb{T})$ satisfy $\frac{1}{f} \in H^\infty(\mathbb{T})$. Does it follow that $\frac{1}{f} \in E_f$?

   (k) For $f \in H^\infty(\mathbb{D})$, define

   $$f_{(r)}(z) = f(rz), \quad |z| < \tfrac{1}{r}, \quad 0 \le r < 1.$$

   Does it follow that

   $$\lim_{r\to 1} \|f_{(r)}\|_\infty = \|f\|_{H^\infty(\mathbb{D})}?$$

2. Let $S_1 = \{z \in \mathbb{D} : |z - 1| \le c(1 - |z|)\}$. For $z = re^{i\tau}$, $|\tau| \le \pi$, $0 < r < 1$, show that $\frac{|\tau|}{1-r}$ is uniformly bounded on $S_1$.

3. Prove that $\mathbb{P}_+^0$ is dense in $H^p$ for $1 \le p < \infty$, and also dense in $H^\infty \cap C(\overline{\mathbb{D}})$.

4. Prove that $H^\infty$ is not separable.

5. Show that $H^p \setminus H^q \neq \{0\}$ whenever $q < p$.

6. For $\xi \in \mathbb{D}$ and $1 \le p < \infty$, define

$$\varphi_\xi : H^p \to \mathbb{C}, \quad \varphi_\xi(f) = f(\xi).$$

Show that

$$\|\varphi_\xi/H^p\| = (1-|\xi|^2)^{-1/p}.$$

7. The *Nevanlinna class* is defined as

$$N(\mathbb{D}) = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \sup_{0<r<1} \int_{\mathbb{T}} \log^+ |f_r|\, dm < \infty \Big\},$$

where $\log^+ t = \max(0, \log t)$ for $t > 0$ and $f_r(z) = f(rz)$.

(i) Let $f \in N(\mathbb{D})$ with $f \neq 0$. Set $h_r(\xi) = \max(1, |f_r(\xi)|)$ for $\xi \in \mathbb{T}$, $0 < r < 1$, and define $\Phi_r = [h_r]$. Show that

$$\max(1, |f_r(z)|) \le |\Phi_r(z)| \quad (z \in \mathbb{D}), \quad \Phi_r(0) \le e^c,$$

where $c = \sup_{0<r<1} \int_{\mathbb{T}} \log^+ |f_r|\, dm$.

(ii) Deduce that $f_r = \psi_r/\varphi_r$, where $\varphi_r = 1/\Phi_r \in H^\infty$ with $|\psi_r| \le 1$, $\|\varphi_r\| \le 1$ in $\mathbb{D}$, and $|\varphi_r(0)| \ge e^{-c}$ for all $0 < r < 1$. Applying Montel's theorem, conclude that there exist $\varphi, \psi \in H^\infty$ with $f = \psi/\varphi$.

(iii) Show that

$$N(\mathbb{D}) = \{\psi/\varphi : \varphi, \psi \in H^\infty\} \cap \mathrm{Hol}(\mathbb{D}).$$

Hence, for every $f \in N(\mathbb{D})$, the non-tangential limits exist a.e., $\log|f| \in L^1$, and

$$f = \lambda B V_\mu[h], \quad (h = |f|),$$

where $V_\mu(z) = \exp\Big( \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, d\mu(\zeta) \Big)$ for $|z| < 1$ and $\mu$ is a singular measure on $\mathbb{T}$.

(iv) Conversely, $\lambda B V_\mu[h] \in N(\mathbb{D})$ for every $\lambda, B, V_\mu$, and every $h > 0$ with $\log h \in L^1$. Moreover, $H^p \subset L^p \cap N(\mathbb{D})$ for every $p > 0$, and $H^p = L^p \cap N_+$, where

$$N_+ = \{\lambda B V_\mu[h] \in N(\mathbb{D}) : \mu \ge 0\}.$$

(v) Let $f_k \in L^2(\mathbb{T})$ $(1 \le k \le n)$ and define

$$E = \overline{\mathrm{span}}\{z^m f_k : m \ge 0,\, 1 \le k \le n\}.$$

Show that $E$ is simply invariant (i.e., $zE \subsetneq E$) if and only if

(a) $\int_{\mathbb{T}} \log|f_k|\, dm > -\infty$ for all $k$, and

(b) $\theta \frac{f_j}{f_k} \in N(\mathbb{D})$ for all $j, k$, where $\theta$ is an inner function.

8. Let $f \in N(\mathbb{D})$ with $f(0) \neq 0$, and let $(\lambda_n)_{n\ge1} = Z(f)$ be its zero sequence. Suppose $\mu$

satisfies

$$V_\mu(z) = \exp\Big(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\Big).$$

(i) Show that

$$\log|f(0)| + \sum_{n\geq 1} \log\frac{1}{|\lambda_n|} + \mu(\mathbb{T}) = \int_{\mathbb{T}} \log|f|\, dm.$$

(ii) Let $f \in H^\infty$ with $|f(z)| \leq 1$ in $\mathbb{D}$ and $f(0) > 0$. Show that $f$ is a Blaschke product if and only if

$$\lim_{r\to 1} \int_{\mathbb{T}} \log|f_r|\, dm = 0.$$

(iii) Let $f \in \mathrm{Hol}(\mathbb{D})$ with $f(0) > 0$. Show that $f$ is a Blaschke product if and only if

$$\lim_{r\to 1} \int_{\mathbb{T}} \log|f_r|\, dm = 0.$$

(iv) Let $f \in \mathrm{Hol}(\mathbb{D}_R)$, $R > 0$, with zero set $(\lambda_n)_{n\geq 1}$ (counted with multiplicities). Define

$$n(s) = \mathrm{card}\{\lambda_k : |\lambda_k| \leq s\}, \quad s \geq 0.$$

(a) Assuming $f(0) \neq 0$, prove

$$\log|f(0)| + \int_0^r \frac{n(s)}{s}\, ds = \int_{\mathbb{T}} \log|f(r\xi)|\, dm(\xi), \quad r < R.$$

(b) Suppose $f(0) \neq 0$. For $0 \leq a < R$, show that

$$\int_a^r \frac{n(s)}{s}\, ds \leq \int_{\mathbb{T}} \log|f(r\xi)|\, dm(\xi) + C, \quad a < r < R,$$

where $C = C(f, a)$ depends only on $f$ and $a$.

9. Let $\mu$ be a finite Borel measure on $\mathbb{T}$ singular with respect to $m$. Define

$$f(z) = \exp\left(-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\mu(\xi)\right), \quad z \in \mathbb{D}.$$

Show that $|f| = 1$ a.e. on $\mathbb{T}$.

10. Let $f$ be holomorphic on $\mathbb{D}$ with $f(0) > 0$. If

$$\lim_{r\to 1} \int_{\mathbb{T}} |\log|f_r||\, dm = 0,$$

prove that $f$ is a Blaschke product.

11. Let $f \in \mathrm{Hol}(\mathbb{D})$. Show that there exists $g \in L^\infty(\mathbb{T})$ such that

$$\left|\frac{[g]}{f}\right| \le 1 \quad \text{a.e. on } \mathbb{T}.$$

12. Let $f \in H^\infty$. Show that there exists $g \in N(\mathbb{D})$ such that

$$Z(f) \cap \mathbb{D} = \{z \in \mathbb{D} : g(z) = 1\}.$$

13. Let $\{\Theta_i \in H^2 : i \in I\}$ be a family of inner functions. Show that

$$\operatorname{span}\{\Theta_i H^2 : i \in I\} = \Theta H^2,$$

where $\Theta = \gcd\{\Theta_i : i \in I\}$.

14. Show that a polynomial $p(z)$ is outer in $H^2(\mathbb{T})$ if and only if $Z(p) \subset \{z \in \mathbb{C} : |z| \ge 1\}$.

15. For $w \in L^1(\mathbb{T})$, define

$$E_w = \overline{\operatorname{span}}\{z^n w : n \ge 0\}\big|_{L^1(\mathbb{T})}.$$

Does there exist $w \in L^1(\mathbb{T})$ such that $\overline{z} \in E_w$? Determine all such $w$.

16. Let $M(\mathbb{T})$ denote the space of all complex Borel measures on $\mathbb{T}$, and define

$$W = \{\mu \in M(\mathbb{T}) : \hat{\mu}(k) = 0 \text{ for } k < 0\}.$$

Suppose $\mu_n \in W$ converges to $\mu \in M(\mathbb{T})$ in the weak* topology of $M(\mathbb{T})$. Show that there exists $h \in H^1(\mathbb{T})$ such that $\hat{\mu}(k) = \hat{h}(k)$ for all $k \in \mathbb{Z}$.

17. Let $f, g \in H^2(\mathbb{T}, m)$. Show that $fg \in H^1(\mathbb{T}, m)$. Does the same conclusion hold if $f \in L^2(\mathbb{T}, m)$?

18. Using the identification of $H^1(\mathbb{D})$ with $H^1(\mathbb{T})$, show that convergence in $H^1(\mathbb{T})$ implies uniform convergence on every disc in $\mathbb{D}$.

19. Let $f \in H^\infty(\mathbb{D})$. Show that $f_{(r)}$ converges to $\widetilde{f}$ in the weak* topology of $L^\infty(\mathbb{T})$.

## 9.3 Problem Set III

1. (a) Let $p > 0$ and suppose $f \in H^p(\mathbb{D})$ with $f \not\equiv 0$. Does it follow that $\log|f| \in L^1(\mathbb{T})$?

   (b) Let $f \in \mathrm{Hol}(\mathbb{D})$. Does the existence of non-tangential limits of $f$ at almost every $\xi \in \mathbb{T}$ imply the existence of radial limits of $f$ at almost every $\xi \in \mathbb{T}$?

2. Let $\mu$ be a finite Borel measure on $\mathbb{T}$, singular with respect to $m$. Define

$$f(z) = \exp\left(-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\mu(\xi)\right), \qquad z \in \mathbb{D}.$$

   Show that $|f| = 1$ almost everywhere on $\mathbb{T}$.

3. Let $f$ be holomorphic on the open unit disc $\mathbb{D}$ with $f(0) > 0$. If

$$\lim_{r \to 1} \int_{\mathbb{T}} |\log |f_r||\, dm = 0,$$

   then show that $f$ is a Blaschke product.

4. Let $f \in \mathrm{Hol}(\mathbb{D})$. Show that there exists a function $g \in L^\infty(\mathbb{T})$ such that

$$\left|\frac{[g]}{f}\right| \leq 1 \quad \text{a.e. on } \mathbb{T}.$$

5. Let $f \in H^\infty$. Show that there exists a function $g \in \mathrm{Nev}(\mathbb{D})$ such that

$$Z(f) \cap \mathbb{D} = \{z \in \mathbb{D} : g(z) = 1\}.$$

**Additional Exercises.** The following exercises are from N. Nikolskii, *Operators, Functions, and Systems: An Easy Reading*, Vol. I.

Chapter 4, Exercises: 4.8.1–4.8.3

## 9.4 Problem Set IV

1. Determine whether the following statements are true or false, providing rigorous justification in each case:

   (a) Can a Blaschke product be an outer function?

   (b) Does the generalized Jensen inequality hold for $H^p$ when $0 < p < 1$?

   (c) Can an inner function arise as the uniform limit of Blaschke products with distinct zeros?

   (d) If $f \in \mathrm{Nev}(\mathbb{D})$ is outer on $\mathbb{D}$, does it follow that $f$ is outer on $\frac{1}{2}\mathbb{D}$?

   (e) If $u \in L^\infty(\mathbb{T})$ is real-valued, does this imply that its Hilbert transform $\widetilde{u}$ also belongs to $L^\infty(\mathbb{T})$?

2. Let $f \in \mathrm{Hol}(\mathbb{D})$. Suppose there exists a non-negative harmonic function $g$ on $\mathbb{D}$ such that $|f(z)| \le g(z)$ for all $z \in \mathbb{D}$. Show that $f \in H^1(\mathbb{T})$.

3. Prove that
$$\left\{ g \in L^\infty(\mathbb{T}) : \int_{\mathbb{T}} gf\, dm = 0 \text{ for all } f \in H^1_0 \right\} = H^\infty.$$

4. Show that the function $\dfrac{1}{\lambda - z}$ is outer in $\mathbb{D}$ whenever $|\lambda| > 1$.

5. Let $p, q, r \ge 1$ and let $f \in H^p(\mathbb{D})$. Suppose that for any $g \in H^q$, the condition $g/f \in L^r(\mathbb{T})$ implies $g/f \in H^r$. Prove that $f$ must be outer.

6. Let $\sigma \subset \mathbb{T}$ have positive Lebesgue measure. Define
$$f_n = n\chi_\sigma + \frac{1}{n}\chi_{\mathbb{T}\setminus\sigma}, \quad n \ge 2.$$
Show that $\frac{1}{n} < |f_n(z)| < n$ for all $z \in \mathbb{D}$ and that $|f_n|(\mathbb{T}) \subset \{\frac{1}{n}, n\}$.

7. Let
$$E = \overline{\mathrm{span}}\{z^m f_k : f_k \in L^2(\mathbb{T}),\, m \ge 0,\, 1 \le k \le n\}.$$
Show that if $zE \ne E$, then for some inner function $\theta$ we have $\theta\,\frac{f_j}{f_k} \in \mathrm{Nev}(\mathbb{D})$ for all $j, k$.

8. If $f \in H^1(\mathbb{C}_+)$ and $f \not\equiv 0$, show that
$$\int_{\mathbb{R}} \frac{|\log|f(x)||}{1+x^2}\, dx < \infty.$$

9. Let $f \in \mathrm{Hol}(\mathbb{D})$, $f \not\equiv 0$, and suppose $f = f_1/f_2$ with $f, f_2 \in H^1$. Show that there exist $g_1, g_2 \in H^\infty$ such that $f = g_1/g_2$.

10. Prove that
$$H^2(\mathbb{T}) = \overline{\mathrm{span}}_{L^2(\mathbb{T})}\left\{ \frac{1}{1-\bar{\lambda}z} : |\lambda| < 1 \right\}.$$

**Additional Exercises.** The following problems are taken from N. Nikolskii, *Operators, Functions, and Systems: An Easy Reading*, Vol. I:

- Chapter 5, Exercises 5.7.1–5.7.2
- Chapter 6, Exercises 6.6.1–6.6.3

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