DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA642: Real Analysis-I	End Semester Exam
Instructor: Rajesh Srivastava	April 30, 2017
Time duration: Three hours	Maximum Marks: 45

N.B. Answer without proper justification will attract zero mark.

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1. (a) Does the space of polynomials $P[0,1]$ complete in any norm on it? **1**

   (b) Is it possible that $\mathbb{R}^2$ can be written as countable union of connected paths? **1**

   (c) What is the cardinality of set of all the polynomials on $\mathbb{R}$ such that complement of their zero set are connected? **1**

   (d) Does there exist a sequence (x_n) such that $x_n \to 0$ but not in any of the sequence space l^p for $1 \leq p < \infty$? **1**

   (e) For $f \in C([0,1] \times [0,1])$, define $f_n(x) = f(x, \frac{1}{n})$. Does the sequence (f_n) equicontinuous in $C[0,1]$? **1**

2. Let $f : (X, d) \to [0,1]$ be continuous map. Show that $f^{-1}(0)$ is a closed G_δ set. **2**

3. Show that $(l^1, \|.\|_1)$ is a proper open subspace of $(l^2, \|.\|_2)$. Does $(l^1, \|.\|_1)$ closed subspace of $(l^2, \|.\|_2)$? **3+1**

4. Let A be a bounded open convex and symmetric ($A = (-1)A$) set in $\mathbb{R}^2$ containing the origin. Show that $\|x\| = \inf\{\alpha > 0 : \ x \in \alpha A\}$ defines a norm on $\mathbb{R}^2$. Does any norm on $\mathbb{R}^2$ can be defined in this way? **3+1**

5. Let $W \subset (C[0,1], \|.\|_u)$ be such that every $f \in W$ satisfies $|f(x) - f(y)| \leq |x - y|$ and $\int_0^1 f(x)dx = 1$. Show that W is compact in $C[0,1]$. **3**

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{x\sqrt{x^2+y^2}}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$

   Show that f has all directional derivative at $(0,0)$ but not continuous at $(0,0)$. Further, show that the map $\varphi : S^1 \to \mathbb{R}$ defined by $\varphi(v) = D_v f(0,0)$ is onto. **2+1**

7. Let $f : \mathbb{R}^n \to [0,1]^n$ be a differetiable map whose all the partial derivatives are bounded by 1. Show that $\|f(x) - f(y)\| \leq \sqrt{n}\|x - y\|$. **3**

8. Let $A \in GL_n(\mathbb{R})$ and $\alpha \geq 2$. If $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\|\varphi(x) \leq |\|x\|^\alpha$. Show that the map $g = \varphi + A$ is a C^1- map at 0 and g is invertible in a neighborhood of 0. **3+1**

9. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous such that $f(x) > \|x\|$ for all $x \in \mathbb{R}^n$. Show that $f^{-1}(K)$ is compact, whenever K is compact in $\mathbb{R}$. **3**

10. Define a function $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(x) = d(x, \mathbb{Z})$. Show that $f_n(x) = \varphi(2^n x) \to 0$, however $\|f_n\|_\infty = \frac{1}{2}$. **1+2**

11. Let $T : \left(C\left[0, \frac{\pi}{2}\right], \|.\|_u\right) \to \left(C\left[0, \frac{\pi}{2}\right], \|.\|_u\right)$ be defined by $(Tf)(x) = \int_{s=0}^{x} f(s) \sin s ds$. Show that T is not a contraction but T^2 is a contraction. **1+2**

12. Use the method of Lagrange's multiplier to find the extremum values of the function $f(x, y) = xy$ on the ellipse $\frac{x^2}{2} + \frac{y^2}{8} = 1$. **3**

13. Let $\overrightarrow{N}$ the unit outward normal vector on the ellipse $x^2 + 2y^2 = 1$. Evaluate the line integral $\int_C \overrightarrow{N}.\overrightarrow{dg}$ along the circle $C = \{(x, y) : \ x^2 + y^2 = 1\}$. **2**

14. Show that the system of equations $xy - 1 = 0$ and $y^2 + z^2 - 1 = 0$ can be solved for x and y in terms of z near $(2, \frac{1}{2}, \frac{\sqrt{3}}{2})$ as $x = \varphi(z)$ and $y = \psi(z)$. Further, find the values of $\varphi'(\frac{\sqrt{3}}{2})$ and $\psi'(\frac{\sqrt{3}}{2})$. **2+1**

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