DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA642: Real Analysis-I — Mid Semester Exam
Instructor: Rajesh Srivastava — February 28, 2017
Time duration: Two hours — Maximum Marks: 30

N.B. Answer without proper justification will attract zero mark.

---

1. (a) Is $\left\{\left(n, \sin\frac{1}{n}\right) : \ n \in \mathbb{N}\right\}$ a closed set w.r.t. usual metric on $\mathbb{R}^2$? **[1]**

   (b) Can the function $f(x) = \frac{x}{|x|}, \ x \neq 0$ be extended continuously on $\mathbb{R}$? **[1]**

   (c) Let (X, d) be a metric space. Then $d(x, y) = 0 \ \forall \ x, y \in X$ if and only if .... **[1]**

   (d) Does there exist an unbounded set $A \subset \mathbb{R}$ such that $\lambda(A) = 0$, but $\lambda(\bar{A}) = 1$? **[1]**

   (e) Does it possible that any metric d on an infinite set X satisfying that every closed and bounded set is compact? **[1]**

2. For $x, y \in (l^\infty, \|\cdot\|_\infty)$ define $d(x, y) = \sharp\{n \in \mathbb{N} : \ x_n \neq y_n\}$. Prove or disprove that d is a metric on l^∞. **[2]**

3. Let $(X, \|\cdot\|)$ be a normed linear space. Show that $\|x\| = \sup\{|\alpha| : \ |\alpha| < \|x\|\}$. **[2]**

4. Let f be a non-negative function on a linear space X such that $f(\alpha x) = |\alpha| f(x)$ for all $\alpha \in \mathbb{C}$. Show that f is norm on X if and only if f is a convex map which can vanish at most at one point. **[2+2]**

5. For $x \in (c_o, \|\cdot\|_\infty)$, define $f_n(x) = \frac{1}{n+1} \sum_{j=1}^{n} x_j$. Show that $\lim_{n \to \infty} f_n(x) = 0$. Further, prove that $\sup_{\|x\|_\infty = 1} |f_n(x)| = 1$. **[2+3]**

6. Show that $\{f \in C[0, 1] : \ \|f\|_2 < 1\}$ is an unbounded subset of the normed linear space $(C[0, 1], \|\cdot\|_\infty)$. **[3]**

7. Let $X = \{f \in C[-1, 1] : \ f(0) = 0\}$ and $M = \left\{f \in X : \ \int_{-1}^{1} f(t)dt = 0\right\}$. Show that M is an infinite dimensional closed subspace of the normed linear space $(X, \|\cdot\|_\infty)$. Does there exist a unit vector $f \in (X, \|\cdot\|_\infty)$ such that $\|f + M\| > 1$? **[3+1]**

8. Let M be a closed subspace of normed linear space X. Define $\pi : X \to X/M$ by $\pi(x) = x + M$. Show that $\pi^{-1}(\pi(E)) = E + M$. Further, deduce that π is open. **[3+2]**

END