DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

Course: MA642: Real Analysis - I
Instructor: Rajesh Srivastava
Duration: 2.0 hours

MidSem
Date: September 16, 2025
Maximum Marks: 30

Note: Answers lacking rigorous justification will not be awarded marks.

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1. (a) Suppose $\limsup a_n = L$. Is it necessarily true that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $a_n < L + \epsilon$ for all $n \geq N$? **1**

   (b) Does there exist a metric d on $\mathbb{R}$ such that the sequence $x_n = n$ converges in $(\mathbb{R}, d)$? **1**

   (c) Is it possible to cover the set $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$ by finitely many open intervals of arbitrarily small length? **1**

   (d) Determine the cardinality of the set of all metrics on an arbitrary finite set. **1**

   (e) Let f be a continuous function on the Cantor set $C \subset [0, 1]$. Is it always possible to extend f continuously to the entire interval $[0, 1]$? **1**

   (f) Consider the set $\{f \in C[0, 1] : f(0) = 0\}$. Is this set closed in $(C[0, 1], \|\cdot\|_1)$? **1**

2. Let $a_1 = 1$ and $a_{n+1} = \sqrt[3]{1 + a_n^2}$. Using fixed point theory, prove that the sequence $\{a_n\}$ is convergent, and that its limit satisfies the equation $x^3 - x^2 - 1 = 0$. **4**

3. For any $f \in C[0, 1]$, prove that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

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4. Construct a seminorm p on ℓ^∞ such that p is discontinuous with respect to the norm topology induced by $\|\cdot\|_\infty$. **4**

5. Let $f : (1, \infty) \to (1, \infty)$ be a contraction mapping and define $g(x) = x^3 - f(x)$. Prove that $g : (1, \infty) \to \mathbb{R}$ is injective. Is g surjective as well? Justify your answer. **4**

6. For $f \in C^1[0, 1]$, define $\|f\| = \|f\|_1 + \|f\|_\infty$. Determine whether $(C^1[0, 1], \|\cdot\|)$ is a complete normed linear space. **4**

7. Investigate the uniform convergence of the sequence

$$f_n(t) = \frac{(n + 1)t + n^2t^2}{1 + n^2t^2}, \qquad t \in \mathbb{R}.$$

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