Assignment 3: Functions of several variables - II.

1. Find the point on the surface $f(x, y, z) = 3x^2 - y^2 - z = 0$ at which the tangent plane is parallel to the plane $6x + 4y - z = 5$.

2. Find the equation of the surface generated by the normals to the surface $f(x, y, z) = x + 2yz + xyz^2 = 0$ at all points on the z-axis.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable and $g(t) = (x(t), y(t), z(t)),\ t \in \mathbb{R}$, be a differentiable curve. Suppose that $f(g(t))$ attains its minimum at some point t_o. Show that $\nabla f(g(t_o))$ is perpendicular to $g'(t_o)$.

4. Consider the surface $z = f(x, y) = x^2 - 2xy + 2y$. Find a point on the surface at which the surface has a horizontal tangent plane.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be function given by $z = f(x, y) = x^4 + y^4$. Find a point on the surface $z = f(x, y)$, where normal to the surface is perpendicular to the chord joining the points $(0, 0, f(0, 0))$ and $(1, 1, f(1, 1))$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a map defined by $f(x, y) = x^2 + y^2$. Find the maximum rate of change of f along the ellipse $R(t) = (a \cos t, b \sin t),\ t \in [0, 2\pi)$.

7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be function given by $f(x_1, x_2, \ldots, x_n) = \sin(x_1 + x_2 + \cdots, x_n)$. For the points $X,\ Y \in \mathbb{R}^n$, show that $|f(Y) - f(X)| \leq \sqrt{n}\ \|Y - X\|$.

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Show that f is convex if and only if $f(Y) \geq f(X) + f'(X).(Y - X)$, for all $X, Y \in \mathbb{R}^2$.

9. Let $f(x, y) = (x - y)(x - y^2)$. Examine the functions f for local maxima, local minima and saddle at $(0, 0)$.

10. Find the points of absolute maximum and absolute minimum of the function $f(x, y) = x^2 + y^2 - 2x + 2$ on the region $\{(x, y) : x^2 + y^2 \leq 4 \text{ with } y \geq 0\}$.

11. Let R be the region in $\mathbb{R}^2$ bounded by the straight lines $y = x,\ y = 3x$ and $x + y = 4$. Consider the transformation $T(u, v) = (u - v, u + v)$. Find the set S satisfying $T(S) = R$.

12. Evaluate the following integrals.
    (a) $\int\limits_0^1 \int\limits_0^{\sqrt{1-x^2}} \sqrt{1 - y^2}\, dydx$
    (b) $\int\limits_0^{\pi} \int\limits_x^{\pi} \frac{\sin y}{y} dydx$

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that $\int\limits_{y=0}^{x} \int\limits_{t=0}^{y} f(t) dtdy = \int\limits_{t=0}^{x} (x - t) f(t) dt.$

14. Let R be the region in $\mathbb{R}^2$ bounded by the curves $y = 2x^2$ and $y = 1 + x^2$. Evaluate the double integral $\iint\limits_R (2x^2 + y) dxdy$.

15. Evaluate $\int\limits_C T \cdot dR$, where C is the circle $x^2 + y^2 = 1$ and T is the unit tangent vector.

16. Let $\overrightarrow{N}$ the unit outward normal vector on the ellipse $x^2 + 2y^2 = 1$. Evaluate the line integral $\int\limits_C \overrightarrow{N}.\overrightarrow{dR}$ along the circle $C = \{(x, y) : \ x^2 + y^2 = 1\}$.

17. Let S be the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$ in the first octant. Evaluate $\iint\limits_S (z + 2xy)d\sigma$

18. Use Green's Theorem to compute $\int\limits_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ where C is the boundary of the region $\{(x, y) : x, y \geq 0 \ \& \ x^2 + y^2 \leq 1\}$.

19. Let C be the boundary of the cone $z = x^2 + y^2$ and $0 \leq z \leq 1$. Use Stoke's theorem to evaluate the line integral $\int_C \overrightarrow{F}.\overrightarrow{dR}$ where $\overrightarrow{F} = (y, xz, 1)$.

20. Let $\overrightarrow{F} = (xy, yz, zx)$ and S be the surface $z = 4 - x^2 - y^2$ with $2 \leq z \leq 4$. Use divergence theorem to find the surface integral $\iint\limits_S \overrightarrow{F}.\overrightarrow{n}\,dS$.