Assignment 1: Metric and Normed Linear Spaces.

1. State TRUE or FALSE giving proper justification for each of the following statements.

   (a) It is impossible to define a metric d on $\mathbb{R}$ such that only finitely many subsets of $\mathbb{R}$ are open in $(\mathbb{R}, d)$.

   (b) If A and B are open (closed) subsets of a normed vector space X, then $A + B = \{a + b : a \in A, b \in B\}$ is open (closed) in X.

   (c) If A and B are closed subsets of $[0, \infty)$ (with the usual metric), then $A + B$ is closed in $[0, \infty)$.

   (d) It is possible to define a metric d on $\mathbb{R}$ such that the sequence $(1, 0, 1, 0, ...)$ converges in $(\mathbb{R}, d)$.

   (e) It is possible to define a metric d on $\mathbb{R}^2$ such that $((\frac{1}{n}, \frac{n}{n+1}))$ is not a Cauchy sequence in $(\mathbb{R}^2, d)$.

   (f) It is possible to define a metric d on $\mathbb{R}^2$ such that in $(\mathbb{R}^2, d)$, the sequence $((\frac{1}{n}, 0))$ converges but the sequence $((\frac{1}{n}, \frac{1}{n}))$ does not converge.

   (g) If (x_n) is a sequence in a complete normed vector space X such that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, then (x_n) must converge in X.

   (h) If (f_n) is a sequence in $C[0, 1]$ such that $|f_{n+1}(x) - f_n(x)| \le \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$, then there must exist $f \in C[0, 1]$ such that $\int_0^1 |f_n(x) - f(x)|\, dx \to 0$ as $n \to \infty$.

   (i) If (x_n) is a Cauchy sequence in a normed vector space, then $\lim\limits_{n\to\infty} \|x_n\|$ must exist.

   (j) $\{f \in C[0, 1] : \|f\|_1 \le 1\}$ is a bounded subset of the normed vector space $(C[0, 1], \|\cdot\|_\infty)$.

   (k) The space $(C^1[0, 1], \|\ .\ \|)$, where $\|f\| = (\|f\|_2^2 + \|f'\|_2^2)^{\frac{1}{2}}$ is a Banach space.

   (l) Let $f \in C^1[0, 1]$ and $\|f\| = \|f'\|_2 + \|f\|_\infty$. Then $(C^1[0, 1], \|\ .\ \|)$ is a Banach space.

   (m) Let $f \in C^1[0, 1]$. Then $\|f\| = \min(\|f'\|_2,\ \|f\|_\infty)$ defines a norm on $C^1[0, 1]$.

   (n) Let $X = \{f \in C^1[0, 1] :\ f(0) = 0\}$. Then $\|f\| = \|f'\|_2$ is a norm on $C^1[0, 1]$ but not complete.

2. Examine whether (X, d) is a metric space, where

   (a) $X = \mathbb{R}$ and $d(x, y) = \frac{|x-y|}{1+|xy|}$ for all $x, y \in \mathbb{R}$.

   (b) $X = \mathbb{R}$ and $d(x, y) = |x - y|^p$ for all $x, y \in \mathbb{R}$ $(0 < p < 1)$.

   (c) $X = \mathbb{R}^n$ and $d(x, y) = [\sum_{i=1}^n \frac{1}{i}(x_i - y_i)^2]^{\frac{1}{2}}$ for all $x = (x_1, ..., x_n),\ y = (y_1, ..., y_n) \in \mathbb{R}^n$.

   (d) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z, w) = \begin{cases} |z - w| & \text{if } \frac{z}{|z|} = \frac{w}{|w|}, \\ |z| + |w| & \text{otherwise.} \end{cases}$

(e) $X =$ The class of all finite subsets of a nonempty set and $d(A, B) =$ The number of elements of the set $A \triangle B$ (the symmetric difference of A and B).

3. Examine whether $\|\cdot\|$ is a norm on $\mathbb{R}^2$, where for each $(x, y) \in \mathbb{R}^2$,

   (a) $\|(x, y)\| = (|x|^p + |y|^p)^{\frac{1}{p}}$, where $0 < p < 1$.

   (b) $\|(x, y)\| = \sqrt{\frac{x^2}{9} + \frac{y^2}{4}}$.

   (c) $\|(x, y)\| = \begin{cases} \sqrt{x^2 + y^2} & \text{if } xy \geq 0, \\ \max\{|x|, |y|\} & \text{if } xy < 0. \end{cases}$

4. Let $\|f\| = \min\{\|f\|_\infty, 2\|f\|_1\}$ for all $f \in C[0, 1]$. Prove that $\|\cdot\|$ is not a norm on $C[0, 1]$.

5. If $\mathbf{x} \in \mathbb{R}^n$, then show that $\lim\limits_{p \to \infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$.

6. If $1 \leq p < q \leq \infty$, then show that $\|x\|_q \leq \|x\|_p$ for all $x \in \ell^p$.

7. Let d be a metric on a real vector space X satisfying the following two conditions:

   (i) $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$,

   (ii) $d(\alpha x, \alpha y) = |\alpha| d(x, y)$ for all $x, y \in X$ and for all $\alpha \in \mathbb{R}$.

   Show that there exists a norm $\|\cdot\|$ on X such that $d(x, y) = \|x - y\|$ for all $x, y \in X$.

8. Let $\mathbb{R}^\infty$ be the real vector space of all sequences in $\mathbb{R}$, where addition and scalar multiplication are defined componentwise. Let $d((x_n), (y_n)) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|}$ for all $(x_n), (y_n) \in \mathbb{R}^\infty$. Show that d is a metric on $\mathbb{R}^\infty$ but that no norm on $\mathbb{R}^\infty$ induces d.

9. Let $(X, \|\cdot\|)$ be a nonzero normed vector space. Consider the metrics d_1, d_2 and d_3 on X:

$$\begin{aligned} d_1(x, y) &:= \min\{1, \|x - y\|\}, \\ d_2(x, y) &:= \frac{\|x - y\|}{1 + \|x - y\|}, \\ d_3(x, y) &:= \begin{cases} 1 + \|x - y\| & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases} \end{aligned}$$

   for all $x, y \in X$. Prove that none of d_1, d_2 and d_3 is induced by any norm on X.

10. Let X be a normed vector space containing more than one point, let $x, y \in X$ and let $\varepsilon, \delta > 0$. If $B_\varepsilon[x] = B_\delta[y]$, show that $x = y$ and $\varepsilon = \delta$. Does the result remain true if X is assumed to be a metric space? Justify.

11. Let $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is a closed/an open subset of $\mathbb{R}^3$ with respect to the usual metric on $\mathbb{R}^3$.

12. For all $x, y \in \mathbb{R}$, let $d_1(x,y) = |x-y|$, $d_2(x,y) = \min\{1, |x-y|\}$ and $d_3(x,y) = \frac{|x-y|}{1+|x-y|}$. If G is an open set in any one of the three metric spaces $(\mathbb{R}, d_i)$ $(i = 1,2,3)$, then show that G is also open in the other two metric spaces.

13. Let X be a nonzero normed vector space. Show that $\{x \in X : \|x\| < 1\}$ is not closed in X and $\{x \in X : \|x\| \leq 1\}$ is not open in X.

14. Let X be a normed vector space and let $Y (\neq X)$ be a subspace of X. Show that Y is not open in X.

15. Let (x_n) and (y_n) be Cauchy sequences in a metric space (X,d). Show that the sequence $(d(x_n, y_n))$ is convergent.

16. Let (x_n) be a sequence in a complete metric space (X,d) such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. Show that (x_n) converges in (X,d).

17. Let (x_n) be a sequence in a metric space X such that each of the subsequences (x_{2n}), (x_{2n-1}) and (x_{3n}) converges in X. Show that (x_n) converges in X.

18. Show that the following are incomplete metric spaces.

    (a) $(\mathbb{N}, d)$, where $d(m,n) = |\frac{1}{m} - \frac{1}{n}|$ for all $m, n \in \mathbb{N}$
    (b) $((0,\infty), d)$, where $d(x,y) = |\frac{1}{x} - \frac{1}{y}|$ for all $x, y \in (0,\infty)$
    (c) $(\mathbb{R}, d)$, where $d(x,y) = |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$ for all $x, y \in \mathbb{R}$
    (d) $(\mathbb{R}, d)$, where $d(x,y) = |e^x - e^y|$ for all $x, y \in \mathbb{R}$

19. Examine whether the following metric spaces are complete.

    (a) $([0,1), d)$, where $d(x,y) = |\frac{x}{1-x} - \frac{y}{1-y}|$ for all $x, y \in [0,1)$
    (b) $((-1,1), d)$, where $d(x,y) = |\tan\frac{\pi x}{2} - \tan\frac{\pi y}{2}|$ for all $x, y \in (-1,1)$

20. For $X (\neq \emptyset) \subset \mathbb{R}$, let $d(x,y) = \frac{|x-y|}{1+|x-y|}$ for all $x, y \in X$. Examine the completeness of the metric space (X,d), where X is

    (a) $[0,1] \cap \mathbb{Q}$.
    (b) $[-1,0] \cup [1,\infty)$.
    (c) $\{n^2 : n \in \mathbb{N}\}$.

21. Examine whether the sequence (f_n) is convergent in $(C[0,1], d_\infty)$, where for all $n \in \mathbb{N}$ and for all $t \in [0,1]$,

    (a) $f_n(t) = \frac{nt^2}{1+nt}$.
    (b) $f_n(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}$.

(c) $f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ \frac{1}{nt} & \text{if } \frac{1}{n} < t \le 1. \end{cases}$

(d) $f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ \frac{n}{n-1}(1-t) & \text{if } \frac{1}{n} < t \le 1. \end{cases}$

22. Let X be the class of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that for each $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}$ such that $|f(x)| < \epsilon$, for all $x \in \mathbb{R} \setminus K$. Show that $(X, \ \| \, . \, \|_\infty)$ is a Banach space.

23. Let $X = C[0,1]$ be the space all the continuous functions on interval $[0,1]$. Prove that norms $\| \, . \, \|_\infty$ and $\| \, . \, \|_1$ on X are not equivalent.

24. Let $C^1[0,1]$ denote the space of all continuously differentiable functions on $[0,1]$. For $f \in C^1[0,1]$, define $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Show that space $(C^1[0,1], \| \, . \, \|)$ is a Banach space.

25. Let $1 \le p < \infty$. Let X_p be a class of all the Riemann integrable functions on $[0,1]$. Prove that $\|f\|_p = \left(\int_0^1 |f|^p \right)^{\frac{1}{p}} < \infty$. Prove that $(X_p, \ \| \, . \, \|_p)$ is a normed linear space but not complete.

26. Let $1 \le p < \infty$. Let $L^p[0,1] = \{ f : [0,1] \to \mathbb{C}, f \text{ is Lebesgue measurable } \}$ with $\|f\|_p = \left(\int_0^1 |f|^p \right)^{\frac{1}{p}} < \infty$. show that $L^p[0,1]$ is proper dense subspace of $L^1[0,1]$, whenever $1 < p < \infty$.

27. Let (x_n) be a sequence in a normed linear space X which converges to a non-zero vector $x \in X$. Show that
$$\frac{x_1 + \cdots + x_n}{n^\alpha} \to x$$
if and only if $\alpha = 1$. If the sequence $x_n \to 0$, prove that
$$\frac{x_1 + \cdots + x_n}{n^\alpha} \to 0, \ \text{for all } \alpha \ge 1.$$

28. Prove that $l^\infty = \{ x = (x_1, x_2, \ldots) : \ \|x\|_\infty = \sup_j |x_j| \}$ is a Banach space but not separable.

29. Let M be a subspace of a normed linear space X. Then show that M is closed if and only if $\{ y \in M : \ \|y\| \le 1 \}$ is closed in X.

30. Let $D = \{ z \in \mathbb{C} : \ |z| < 1 \}$. Let X be the class of all functions f which are analytic on D and continuous on $\bar{D}$. Define $\|f\|_\infty = \sup\{ |f(e^{it})| : \ 0 \le t \le 2\pi \}$. Prove that $(X, \ \| \, . \, \|_\infty)$ is a Banach space.

31. Let M be a closed subspace of a normed linear space X. Prove that projection $\pi : \ X \to X/M$ defined by $\pi(x) = \tilde{x}$ is a continuous map.

32. Let X be a normed linear space. Prove that norm of any $x \in X$, can be expressed as $\|x\| = \inf \{ |\alpha| : \ \alpha \in \mathbb{C} \setminus \{0\} \text{ with } \|x\| \le |\alpha| \}$.