Assignment 3

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1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) If H is a complex Hilbert space, then for each $T \in \mathcal{B}(H)$, there exist $A, B \in \mathcal{B}(H)$ such that A and B are invertible in $\mathcal{B}(H)$ and $T = A + B$.
   (b) If H is a complex Hilbert space and $T \in \mathcal{B}(H)$ such that $3T^3 + 4I = 2T^2$, then $\sigma(T)$ must be a finite set.
   (c) If H is a complex Hilbert space and $T \in \mathcal{B}(H)$ such that $r_\sigma(T) = \|T\|$, then it is necessary that $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$.
   (d) If H is a complex Hilbert space and if $T \in \mathcal{B}(H)$ is self-adjoint, then $I + iT$ must be invertible in $\mathcal{B}(H)$.
   (e) If H is an infinite dimensional complex Hilbert space and $T \in \mathcal{B}(H)$ such that T^2 is compact, then it is necessary that $0 \in \sigma(T)$.
   (f) If X is a nonzero Banach space over $\mathbb{C}$ and $T \in \mathcal{B}_0(X)$, then it is necessary that $\sigma(T) = \sigma_{ap}(T)$.
   (g) There does not exist any nonzero proper reducing subspace in the Hilbert space $\mathbb{C}^2$ for $T \in \mathcal{B}(\mathbb{C}^2)$, where $T(x, y) = (y, 0)$ for all $(x, y) \in \mathbb{C}^2$.
   (h) If H is an infinite dimensional separable Hilbert space over $\mathbb{C}$ and $T \in \mathcal{B}(H)$ is normal, then $\sigma_p(T)$ must be countable.
   (i) If H is a complex Hilbert space and $T \in \mathcal{B}_0(H)$ such that $I + T$ is one-one, then $I + T$ must be invertible in $\mathcal{B}(H)$.

2. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be normal. If $\lambda \in \mathbb{K}$, then show that $\ker(T - \lambda I)$ is a reducing subspace in H for T.

3. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be normal. Show that $\{x \in H : \|Tx\| = \|T\|\|x\|\}$ is a reducing subspace in H for T.

4. Let H be an infinite dimensional Hilbert space and let $T, S \in \mathcal{B}(H)$ such that $S \neq 0, I$ and $STS = TS$. Show that there exists a nonzero proper invariant subspace in H under T.

5. Let H be a Hilbert space and $T \in \mathcal{B}_0(H)$. If M is a closed subspace of H which is invariant under T and if $S(x + M) = Tx + M$ for all $x \in H$, then show that $S \in \mathcal{B}_0(H/M)$.

6. If H is a non-separable Hilbert space and if $T \in \mathcal{B}(H)$, then show that H contains a nonzero proper invariant subspace for T.

7. Show that there is no nonzero proper reducing subspace of the right shift operator on ℓ^2.

8. Let X be a Banach space and let $T \in \mathcal{B}(X)$ such that $\|T^m\| < 1$ for some $m \in \mathbb{N}$. Show that $I - T$ is invertible in $\mathcal{B}(X)$ and that $(I - T)^{-1} = \sum\limits_{n=0}^{\infty} T^n$.

9. Let X be a Banach space and $T \in B(X)$. Prove that $\exp(T) = \sum_0^\infty \frac{T^n}{n!}$ is invertible and $\sigma(\exp T) = \exp(\sigma(T))$.

10. Let H be a Hilbert space. Let (T_n) and (S_n) be sequences in $\mathcal{B}(H)$ and let $T, S \in \mathcal{B}(H)$. If $T_n \to T$ (in norm) and $S_n \xrightarrow{WOT} S$, then show that $T_nS_n \xrightarrow{WOT} TS$.

11. Let H be a Hilbert space. Let (T_n) be a sequence in $\mathcal{B}(H)$ and let $T \in \mathcal{B}(H)$. If for each $x \in H$, $\|T_nx\| \to \|Tx\|$ and $\langle T_nx, x\rangle \to \langle Tx, x\rangle$ as $n \to \infty$, then show that $T_n \xrightarrow{SOT} T$.

12. Let H be a Hilbert space. Let $T_n \in \mathcal{B}(H)$ be normal for each $n \in \mathbb{N}$ and let $T \in \mathcal{B}(H)$ be normal. If $T_n \xrightarrow{SOT} T$, then show that $T_n^* \xrightarrow{SOT} T^*$.

13. Let X be a nonzero Banach space over $\mathbb{C}$ and let $T \in \mathcal{B}(X)$. If E is the set of all eigenvectors of T and if $E^0 \neq \emptyset$, then show that there exists $\lambda \in \mathbb{C}$ such that $T = \lambda I$.

14. Let S be the left shift operator on ℓ^2. Show that there does not exist any $T \in \mathcal{B}(\ell^2)$ such that $T^2 = S$.

15. Let R be the right shift operator on l^2. Prove that
(a) resolvent set $\rho(R) = \{\lambda \in \mathbb{C} : |\lambda| > 1\}$,
(b) point spectrum set $\sigma_p(R) = \emptyset$,
(c) continuous spectrum set $\sigma_c(R) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$,
(d) residual spectrum set $\sigma_r(R) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

16. Let $T : (C[0,1], \|.\|_\infty) \to (C[0,1], \|.\|_\infty)$ be a linear map defined by $Tf(t) = f\left(\frac{t}{3}\right)$. Find the spectral radius of T and show that $0 \in \sigma(T)$. Is T compact?

17. Let $T : L^2[0,1] \to L^2[0,1]$ be a linear map defined by $T(f)(x) = \int\limits_0^x f(t)dt$. Show that spectral radius $r(T) = 0$ and $0 \in \sigma_c(T)$, continuous spectrum.

18. Let $g \in C[0,1]$ and $T : L^2[0,1] \to L^2[0,1]$ be a linear map defined by $T(f)(t) = g(t)f(t)$. Find the spectrum $\sigma(T)$ and deduce that T is not compact.

19. Let $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$. For $x = (x_k)_{-\infty}^{\infty} \in l^2(\mathbb{Z})$, define $T(x) = (x_{k-1})_{-\infty}^{\infty}$ (right shift operator). Show that
(a) the point spectrum $\sigma_p(T) = \emptyset$,
(b) $\text{Im}(\lambda I - T) = l^2(\mathbb{Z})$ if $|\lambda| \neq 1$,
(c) spectrum $\sigma(T) = \{\lambda : |\lambda| = 1\}$.

20. Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$. For $x \in l^2(\mathbb{N})$, define $T(x) = (x_2, x_3, \ldots)$. Prove that
(a) $\rho(T) = \{\lambda \in \mathbb{C} : |\lambda| > 1\}$.
(b) $\sigma_c((T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.
(c) $\sigma_p((T) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.
(d) $\sigma_r(T) = \emptyset$.

21. Let g be a continuous and bounded function on $\mathbb{R}$. Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be linear map defined by $T(f)(t) = g(t)f(t)$. Show that T is bounded and spectrum $\sigma(T) = \overline{\{g(x) : x \in \mathbb{R}\}}$.

22. Let P be an orthogonal projection on a Hilbert space H. Show that $\sigma(P) = \sigma_p(P) = \{0, 1\}$. Further, derive that the resolvent function for P is given by $R_P(\lambda) = (\lambda I - P)^{-1} = \frac{I}{\lambda} + \frac{1}{\lambda(1-\lambda)} P$.

23. A bounded linear operator T on a separable Hilbert space H is called Hilbert-Schmidt operator if there exists an orthonormal basis $\{e_n : \ n \in \mathbb{N}\}$ such that $\sum \|Te_n\|^2 < \infty$. Write $\|T\|_{\text{H.S.}} = \left(\sum \|Te_n\|^2\right)^{1/2}$. Show that
(a) T is a compact operator.
(b) $\|T^*\|_{\text{H.S.}} = \|T\|_{\text{H.S.}}$.
(c) Hilbert-Schmidt norm is independent of choice of orthonormal basis.

24. Let $y \in C([0,1], \mathbb{R})$ and let $T : (C([0,1], \mathbb{R}), \|\cdot\|_\infty) \to (C([0,1], \mathbb{R}), \|\cdot\|_\infty)$ be defined by $Tx(t) = x(t)y(t)$ for all $t \in [0,1]$. Determine $\sigma(T)$.

25. If X is a complex Banach space and if $T \in \mathcal{B}(X)$ such that $T^n = 0$ for some $n \in \mathbb{N}$, then show that $\sigma(T) = \{0\}$.

26. If X is a complex Banach space and if $T \in \mathcal{B}(X)$ is invertible in $\mathcal{B}(X)$, then show that $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.

27. If X is a complex Banach space and if $T, S \in \mathcal{B}(X)$, then show that $\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$, although it is not necessary that $\sigma(TS) = \sigma(ST)$.

28. Let X be a nonzero Banach space over $\mathbb{C}$ and let $T, S \in \mathcal{B}(X)$. If T is invertible in $\mathcal{B}(X)$, then show that $\sigma(TS) = \sigma(ST)$.

29. Let $T : (\ell^\infty, \|\cdot\|_\infty) \to (\ell^\infty, \|\cdot\|_\infty)$ be defined by $T((x_n)) = (x_2, x_3, ...)$ for all $(x_n) \in \ell^\infty$. Determine $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$.

30. Let $T : \ell^2 \to \ell^2$ be defined by $T((x_n)) = \left(\frac{n}{n+1} x_n\right)$ for all $(x_n) \in \ell^2$. Determine $\sigma(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$, $\sigma_{ap}(T)$ and $\sigma_{cp}(T)$.

31. Let $T((x_n)) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...)$ and $S((x_n)) = (x_2, \frac{x_3}{2}, \frac{x_4}{3}, ...)$ for all $(x_n) \in \ell^2$. Prove the following:
(a) $T, S \in \mathcal{B}_0(\ell^2)$.
(b) $\sigma(T) = \{0\}$, $\sigma_p(T) = \emptyset$ and $\sigma(S) = \sigma_p(S) = \{0\}$.
(c) $\overline{\text{range}(T)} \neq \ell^2$ and $\overline{\text{range}(S)} = \ell^2$.

32. Let H be a complex Hilbert space and let $T \in \mathcal{B}(H)$ such that $|\langle Tx, x\rangle| \geq \langle x, x\rangle$ for all $x \in H$. Show that $0 \notin \sigma(T)$.

33. Let H be a nonzero Hilbert space over $\mathbb{C}$ and let $T \in \mathcal{B}(H)$. Show that $\sigma(T^*) = \sigma_{ap}(T^*) \cup \{\overline{\lambda} : \lambda \in \sigma_{ap}(T)\}$.

34. Let H be a complex Hilbert space and let $T \in \mathcal{B}(H)$. Show that $\{\lambda^n : \lambda \in \sigma_{ap}(T)\} \subset \sigma_{ap}(T^n)$ for all $n \in \mathbb{N}$.

35. If X is a complex Banach space and if $T \in \mathcal{B}(X)$, then show that $\sigma_{ap}(T)$ is a compact subset of $\mathbb{C}$.

36. Let X be a nonzero Banach space over $\mathbb{C}$. If (T_n) is a sequence in $\mathcal{G}(X)$ such that $T_n \xrightarrow{\|\cdot\|} T \in \mathcal{B}(X) \setminus \mathcal{G}(X)$, then show that $0 \in \sigma_{ap}(T)$.

37. If X is a complex Banach space and if $T \in \mathcal{B}(X)$, then show that $\partial\sigma(T) \subset \sigma_{ap}(T)$.

38. Let H be a complex Hilbert space and $y, z \in H$. If $Tx = \langle x, y \rangle z$ for all $x \in H$, then show that $\sigma(T) = \{0, \langle z, y \rangle\}$.

39. If H is a complex Hilbert space and if $T \in \mathcal{B}(H)$ is normal, then show that $\sigma_r(T) = \emptyset$, $\sigma_{ap}(T) = \sigma(T)$ and $r_\sigma(T) = \|T\|$.

40. If H is a complex Hilbert space and $T \in \mathcal{B}(H)$, then show that $\sigma_r(T) \subset \{\lambda \in \mathbb{C} : |\lambda| < \|T\|\}$.

41. Let H be a complex Hilbert space and let $T \in \mathcal{B}(H)$. If there is no nonzero proper invariant subspace in H under T, then show that $\sigma(T) = \sigma_c(T)$.

42. Let X be a complex Banach space and let $T(\neq 0, I) \in \mathcal{B}(X)$ such that $T^2 = T$. Show that $\sigma(T) = \sigma_p(T) = \{0, 1\}$.

43. Let H be a complex Hilbert space and let $T(\neq -I, I) \in \mathcal{B}(H)$ such that $T^2 = I$. Show that $\sigma(T) = \{-1, 1\}$.

44. Let X be a complex Banach space and let $T, T_n \in \mathcal{B}(X)$ for all $n \in \mathbb{N}$ such that $T_n \to T$. If $\lambda_n \in \sigma(T_n)$ for all $n \in \mathbb{N}$ and if $\lambda_n \to \lambda \in \mathbb{C}$, then show that $\lambda \in \sigma(T)$.

45. If H is a complex Hilbert space and if $T \in \mathcal{B}(H)$, then show that $\sigma_c(T^*) = \{\overline{\lambda} : \lambda \in \sigma_c(T)\}$.

46. If H is a non-separable Hilbert space and if $T \in \mathcal{B}_0(H)$, then show that $0 \in \sigma_p(T)$.

47. Let K be a nonempty compact subset of $\mathbb{C}$. Show that there exists $T \in \mathcal{B}((\ell^2, \|\cdot\|_2))$ such that $\sigma(T) = K$.

48. Let H be a complex Hilbert space. If $T \in \mathcal{B}(H)$ and $\lambda \in W(T)$ such that $|\lambda| = \|T\|$, then show that $\lambda \in \sigma_p(T)$.

49. Let H be an infinite dimensional Hilbert space over $\mathbb{C}$ and let $T \in \mathcal{B}(H)$. If $T^n \xrightarrow{WOT} 0$, then show that $\sigma_p(T) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

50. Let H be a complex Hilbert space. If $T \in \mathcal{B}(H)$ is normal and $\sigma(T) = \{0, 1\}$, then show that $T^2 = T$.