Assignment 2

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1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) If H is a Hilbert space and $T \in \mathcal{B}(H)$ is self-adjoint, then it is necessary that $T \geq 0$ or $T \leq 0$.
   (b) If H is a Hilbert space and $T \in \mathcal{B}(H)$ is self-adjoint such that range(T) is dense in H, then T must be one-one.
   (c) If H is a Hilbert space and if $T \in \mathcal{B}(H)$ is self-adjoint, then there must exist $\alpha \in \mathbb{R}$ with $\alpha > 0$ such that $\alpha T \leq I$.
   (d) If X is an infinite dimensional normed linear space, $p(t)$ is a polynomial in t with coefficients in $\mathbb{K}$ and $T \in \mathcal{B}_0(X)$, then $p(T) \in \mathcal{B}_0(X)$ iff $p(0) = 0$.
   (e) If H is a Hilbert space and $T, S \in \mathcal{B}(H)$ such that TS is compact, then at least one of T and S must be compact.
   (f) Every $T \in \mathcal{B}_0(\ell^2) \setminus \mathcal{B}_{00}(\ell^2)$ is normal.
   (g) Every bounded linear operator from $(c_0, \|\cdot\|_\infty)$ to $(\ell^2, \|\cdot\|_2)$ is compact.
   (h) If H is a Hilbert space and $T \in \mathcal{B}_0(H)$, then $T(B_H)$ must be a compact subset of H.
   (i) If $T \in \mathcal{B}_0(H)$ such that $Tx = x$ for some $x \in H$ with $\|x\| = 1$, then there must exist $y \in H$ such that $\|y\| = 1$ and $T^*y = y$.
   (j) If H is a Hilbert space, $T \in \mathcal{B}(H)$ and (T_n) is a sequence in $\mathcal{B}_0(H)$ such that $T_n x \to Tx$ for each $x \in H$, then T must be compact.
   (k) If H is a Hilbert space and $T, S \in \mathcal{B}(H)$ such that $TT^* + SS^* = 0$, then it is necessary that $T = S = 0$.
   (l) If H is a Hilbert space and if $T \in \mathcal{B}(H)$ is bounded below, then T cannot be compact.
   (m) If H is a Hilbert space and $T \in \mathcal{B}(H)$ is normal such that T is bounded below, then T must be invertible in $\mathcal{B}(H)$.

2. Let X, Y be normed linear spaces and let $T \in \mathcal{B}(X, Y)$. Show that $T^* : Y^* \to X^*$ is one-one iff $T(X)$ is dense in Y.

3. Let H be a Hilbert space and let $y \in H$. If $f(x) = \langle x, y \rangle$ for all $x \in H$, then determine the adjoint operator f^*.

4. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be such that dim(range(T)) $= 1$. Show that there exist $y, z \in H$ such that $Tx = \langle x, y \rangle z$ for all $x \in H$. Also, find T^*.

5. Consider $T \in \mathcal{B}(\ell^2)$, defined by $T((x_n)) = (0, 3x_1, x_2, 3x_3, x_4, ...)$ for all $(x_n) \in \ell^2$. Determine T^*.

6. Let (α_n) be a sequence in $\mathbb{K}$ and let $T((x_n)) = (\alpha_n x_n)$ for all $(x_n) \in \ell^2$. Prove that $T \in \mathcal{B}(\ell^2)$ iff $(\alpha_n) \in \ell^\infty$ and in such case determine $\|T\|$ and T^*. Also, prove that
   (a) T is self-adjoint iff $\alpha_n \in \mathbb{R}$ for each $n \in \mathbb{N}$.
   (b) $T \geq 0$ iff $\alpha_n \geq 0$ for each $n \in \mathbb{N}$.
   (c) T is unitary iff $|\alpha_n| = 1$ for all $n \in \mathbb{N}$.

7. Let $\{u_n : n \in \mathbb{N}\}$ be an (countably infinite) orthonormal basis of a Hilbert space H. Consider the bounded linear operator $T : \ell^2 \to H$, defined by $T((\alpha_n)) = \sum_{n=1}^{\infty} \alpha_n u_n$ for all $(\alpha_n) \in \ell^2$.

Determine the adjoint operator T^*.

8. Let H be a Hilbert space and let $T \in B(H)$. Prove that
(a) $\ker(T) = (\text{range}(T^*))^\perp = \ker(T^*T)$.
(b) $\ker(T^*) = (\text{range}(T))^\perp = \ker(TT^*)$.
(c) $(\ker(T))^\perp = \overline{\text{range}(T^*)} = \overline{\text{range}(T^*T)}$.
(d) $(\ker(T^*))^\perp = \overline{\text{range}(T)} = \overline{\text{range}(TT^*)}$.

9. Let H be a Hilbert space and let $T, S \in \mathcal{B}(H)$. Show that
$(\text{range}(T) + \text{range}(S))^\perp = \ker(T^*) \cap \ker(S^*)$.

10. Let H be a Hilbert space. Show that $T \in \mathcal{B}(H)$ is invertible in $\mathcal{B}(H)$ iff both T and T^* are bounded below.

11. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ with $\|T\| = 1$. If $x \in H$ such that $Tx = x$, then show that $T^*x = x$.

12. Let H be a Hilbert space and let $x \in H$, $T \in \mathcal{B}(H)$. Show that $T^*Tx = \|T\|^2 x$ iff $\|Tx\| = \|T\|\|x\|$.

13. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ such that $T \geq 0$. If $x \in H$ such that $\langle Tx, x\rangle = 0$, then show that $x \in \ker(T)$.

14. Let M, N be closed subspaces of a Hilbert space H and let $T \in \mathcal{B}(H)$. Show that $T(M) \subset N$ iff $T^*(N^\perp) \subset M^\perp$.

15. Let $\{u_n : n \in \mathbb{N}\}$ be an orthonormal basis of a Hilbert space H and let $T \in \mathcal{B}(H)$. Show that $\left\|\sum_{n=1}^{\infty} \langle Tx, u_n\rangle T u_n\right\|^2 = \sum_{n=1}^{\infty} |\langle Tx, T^*u_n\rangle|^2$.

16. Let H be a Hilbert space and let $T, S \in \mathcal{B}(H)$ such that T is self-adjoint. Show that $TS = 0$ iff $\text{range}(T) \perp \text{range}(S)$.

17. Let H be a Hilbert space. If $T \in \mathcal{B}(H)$ is self-adjoint and if $T \neq 0$, then show that $T^n \neq 0$ for each $n \in \mathbb{N}$.

18. Let (T_n) be a sequence of (bounded) self-adjoint operators on a Hilbert space H and let $T \in \mathcal{B}(H)$. If $T_n x \xrightarrow{w} Tx$ for each $x \in H$, then show that T is self-adjoint.

19. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be self-adjoint. If $\|x\| = d(x, \text{range}(T))$ for all $x \in \ker(T)$, then show that $\|\cdot\|$ is a norm on $\ker(T)$.

20. If H is a Hilbert space and if $T \in \mathcal{B}(H)$ is positive, then show that
(a) $|\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle$ for all $x, y \in H$.
(b) $\|Tx\|^2 \leq \|T\|\langle Tx, x\rangle$ for all $x \in H$.

21. If H is a Hilbert space and if $T \in \mathcal{B}(H)$ is positive, then show that $I+T : H \to H$ is invertible.

22. If H is a Hilbert space and if $T \in \mathcal{B}(H)$, then show that both $I + T^*T : H \to H$ and $I + TT^* : H \to H$ are invertible.

23. Let H be a Hilbert space. If $T, S \in \mathcal{B}(H)$ are self-adjoint and satisfy $T \geq S$, is it necessary that $T^2 \geq S^2$?

24. If H is a Hilbert space and if $T \in \mathcal{B}(H)$ such that $\|T\| \leq 1$, then show that $I - T^*T \geq 0$.

25. Let H be a Hilbert space. If $T \in \mathcal{B}(H)$ is positive and invertible, then show that $T^{-1} \geq 0$.

26. Let H be a Hilbert space. If $T \in \mathcal{B}(H)$ such that $0 \leq T \leq I$, then show that $T^2 \leq T$.

27. Let H be a Hilbert space and let (T_n) be a sequence in $\mathcal{B}(H)$ such that $T_n \geq 0$ for all $n \in \mathbb{N}$. If $T_n \xrightarrow{WOT} 0$, then show that $T_n \xrightarrow{SOT} 0$.

28. Let H be a Hilbert space. If $T \in \mathcal{B}(H)$ and if $\alpha, \beta \in \mathbb{K}$ such that $|\alpha| = |\beta|$, then show that $\alpha T + \beta T^*$ is normal.

29. Let H be a Hilbert space. Show that $T \in \mathcal{B}(H)$ is normal iff $\langle Tx, Ty \rangle = \langle T^*x, T^*y \rangle$ for all $x, y \in H$.

30. If H is a Hilbert space and $T \in \mathcal{B}(H)$ is normal, then show that $\ker(T^2) = \ker(T)$.

31. If H is a Hilbert space and if $T \in \mathcal{B}(H)$ is normal, then show that $\|T^n\| = \|T\|^n$ for each $n \in \mathbb{N}$.

32. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. If $TT^* \leq T^*T$, then show that $\|T^n\|^2 \leq \|T^{n+1}\|\|T^{n-1}\|$ for all $n \in \mathbb{N}$.

33. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be normal. If $T^2 = T$, then show that T is an orthogonal projection.

34. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be self-adjoint such that $T^3 = T^2$. Show that T is an orthogonal projection.

35. If H is a Hilbert space, then show that every orthogonal projection $P \in \mathcal{B}(H)$ must satisfy $0 \leq P \leq I$.

36. Let H be a Hilbert space and let $P \in \mathcal{B}(H)$ such that $P \neq 0$ and $P^2 = P$. Show that P is an orthogonal projection iff $\|P\| = 1$.

37. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. If $T^2 = T$, $M = \text{range}(T)$ and $N = \ker(T)$, then show that $P_M - P_N$ is invertible in $\mathcal{B}(H)$ and that $(P_M - P_N)^{-1} = T + T^* - I$.

38. If M, N are closed subspaces of a Hilbert space H, then show that $\|P_M - P_N\| \leq 1$, where P_M and P_N are the orthogonal projection operators on M and N respectively.

39. If H is a Hilbert space and if $P, Q \in \mathcal{B}(H)$ are orthogonal projections, then prove that the following statements are equivalent.
(a) $P \leq Q$
(b) $\|Px\| \leq \|Qx\|$ for all $x \in H$
(c) $\ker(Q) \subset \ker(P)$
(d) $\text{range}(P) \subset \text{range}(Q)$
(e) $QP = P$
(f) $PQ = P$
(g) $QPQ = P$
(h) $Q - P$ is an orthogonal projection with $\text{range}(Q - P) = \text{range}(Q) \cap (\text{range}(P))^{\perp}$.

40. If H is a Hilbert space and if $P, Q \in \mathcal{B}(H)$ are orthogonal projections, then show that PQ is an orthogonal projection iff $PQ = QP$ and in such case $\text{range}(PQ) = \text{range}(P) \cap \text{range}(Q)$.

41. If H is a Hilbert space and if $P, Q \in \mathcal{B}(H)$ are orthogonal projections such that $PQ = QP$, then show that $P + Q - PQ$ is an orthogonal projection with
$\text{range}(P + Q - PQ) = \text{range}(P) + \text{range}(Q)$.

42. Let H be a Hilbert space. Show that $T \in \mathcal{B}(H)$ is self-adjoint and unitary iff $T = 2P - I$ for some orthogonal projection $P \in \mathcal{B}(H)$.

43. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ such that $T^*(T - I) = 0$. Show that T is an orthogonal projection.

44. Let H be a Hilbert space and let $A(\neq \emptyset) \subset H$. If $T \in \mathcal{B}(H)$ is unitary, then show that $T(A^{\perp}) = T(A)^{\perp}$.

45. Let $T((x_n)) = (x_1, x_1+x_2, x_1+x_2+x_3, ...)$ for all $(x_n) \in \ell^1$. Prove that $T : (\ell^1, \|\cdot\|_1) \to (\ell^\infty, \|\cdot\|_\infty)$ is linear and bounded but not compact.

46. If $Tx = x$ for all $x \in \ell^1$, then examine whether $T : (\ell^1, \| \cdot \|_1) \to (\ell^2, \| \cdot \|_2)$ is a compact linear operator.

47. Let $T((x_n)) = (x_2, x_1, \frac{1}{2}x_4, \frac{1}{2}x_3, ..., \frac{1}{n}x_{2n}, \frac{1}{n}x_{2n-1}, ...)$ for all $(x_n) \in \ell^2$. Show that
$T \in \mathcal{B}_0((\ell^2, \| \cdot \|_2))$.

48. Show that every bounded linear operator from $(\ell^2, \| \cdot \|_2)$ to $(\ell^1, \| \cdot \|_1)$ is compact.

49. Let $(Tx)(t) = x(t^2)$ for all $x \in C[0,1]$ and for all $t \in [0,1]$. Show that
$T : (C[0,1], \| \cdot \|_\infty) \to (C[0,1], \| \cdot \|_\infty)$ is linear and bounded but not compact.

50. Let X, Y be normed linear spaces and let $y \in Y$, $f \in X^*$. If $Tx = f(x)y$ for all $x \in X$, then show that $T \in \mathcal{B}_0(X, Y)$.

51. Let X, Y be a normed linear spaces and let $x(\neq 0) \in X$, $y \in Y$. Show that there exists $T \in \mathcal{B}_0(X, Y)$ such that $Tx = y$.

52. Let X, Y be normed linear spaces and let $T \in \mathcal{B}_0(X, Y)$. If $S(x + \ker(T)) = Tx$ for all $x \in X$, then show that $S \in \mathcal{B}_0(X/\ker(T), Y)$.

53. Let H be a Hilbert space and let $T \in \mathcal{B}_0(H)$. Show that there exists $x_0 \in H$ such that $\|x_0\| \leq 1$ and $\|Tx_0\| = \|T\|$.

54. Let X be a Banach space and let $T \in \mathcal{B}_0(X) \setminus \mathcal{B}_{00}(X)$. If $S = \{x \in X : \|x\| = 1\}$, then show that $0 \in \overline{T(S)}$.

55. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. If T^*T is compact, then show that T is compact.

56. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be normal. If T^2 is compact, then show that T is compact.

57. Let X be a normed linear space and let $T \in \mathcal{B}_0(X)$ such that $T^2 = T$. Show that T is a finite rank operator.

58. Let X, Y be Banach spaces and let $T \in \mathcal{B}_0(X, Y)$. If Y is infinite dimensional, then show that T is not onto.

59. Let X, Y be Banach spaces and let $T \in \mathcal{B}_0(X, Y)$. If range($T$) is infinite dimensional, then show that range(T) cannot be closed in Y.

60. Let X be a Banach space and let (T_n) be a sequence in $\mathcal{B}(X)$ such that for each $x \in X$, $T_n x \to 0$. If $S \in \mathcal{B}_0(X)$, then show that $\|T_n S\| \to 0$.

61. Let X and Y be Banach spaces. Show that the class of all completely continuous linear maps from X to Y is closed in $\mathcal{B}(X, Y)$.

62. Let $X \neq \{0\}$, Y be normed linear spaces such that $\mathcal{B}_0(X, Y)$ is a Banach spaces. Prove that Y is a Banach space.

63. Let H be a Hilbert space and let (T_n), (S_n) be sequences in $\mathcal{B}(H)$ such that $T_n \xrightarrow{SOT} T$ and $S_n \xrightarrow{SOT} S$, where $T, S \in \mathcal{B}(H)$. Show that $T_n S_n \xrightarrow{SOT} TS$.

64. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be normal. If $\lambda \in \mathbb{K}$, then show that $\ker(T - \lambda I)$ is a reducing subspace in H for T.

65. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be normal. Show that $\{x \in H : \|Tx\| = \|T\|\|x\|\}$ is a reducing subspace in H for T.

66. Let H be an infinite dimensional Hilbert space and let $T, S \in \mathcal{B}(H)$ such that $S \neq 0, I$ and $STS = TS$. Show that there exists a nonzero proper invariant subspace in H under T.

67. Let H be a Hilbert space and $T \in \mathcal{B}_0(H)$. If M is a closed subspace of H which is invariant under T and if $S(x + M) = Tx + M$ for all $x \in H$, then show that $S \in \mathcal{B}_0(H/M)$.

68. If H is a non-separable Hilbert space and if $T \in \mathcal{B}(H)$, then show that H contains a nonzero proper invariant subspace for T.

69. Show that there is no nonzero proper reducing subspace of the right shift operator on ℓ^2.

70. Let H be a Hilbert space. Let (T_n) and (S_n) be sequences in $\mathcal{B}(H)$ and let $T, S \in \mathcal{B}(H)$. If $T_n \to T$ (in norm) and $S_n \xrightarrow{WOT} S$, then show that $T_n S_n \xrightarrow{WOT} TS$.

71. Let H be a Hilbert space. Let (T_n) be a sequence in $\mathcal{B}(H)$ and let $T \in \mathcal{B}(H)$. If for each $x \in H$, $\|T_n x\| \to \|Tx\|$ and $\langle T_n x, x\rangle \to \langle Tx, x\rangle$ as $n \to \infty$, then show that $T_n \xrightarrow{SOT} T$.

72. Let H be a Hilbert space. Let $T_n \in \mathcal{B}(H)$ be normal for each $n \in \mathbb{N}$ and let $T \in \mathcal{B}(H)$ be normal. If $T_n \xrightarrow{SOT} T$, then show that $T_n^* \xrightarrow{SOT} T^*$.