DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA550: Measure Theory	Mid Semester Exam
Instructor: Rajesh Srivastava	September 20, 2021
Time duration: Two hours	Maximum Marks: 25

N.B. Answer without proper justification will attract zero mark.

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1. (a) Recall that the Cantor can be expressed as $\cap_{n=1}^{\infty} C_n$, where C_n is the union of 2^n disjoint closed intervals each of length 3^{-n}. Whether diameter$(C_n) \to 0$? **[1]**

   (b) Let $A = \cup_{n=1}^{\infty} I_n$, where I_n's are open intervals. If there exists $\epsilon > 0$ such that $d(I_n, I_m) \geq \epsilon$ for $m \neq n$. Does it imply that $m(\text{boundary}(A)) = 0$? **[1]**

2. Let C be the Cantor set in $[0,1]$. Show that $C + (a, \infty)$ is a Lebesgue measurable subset of $\mathbb{R}$ for every choice of $a > 0$. **[2]**

3. Let $\mathcal{A}$ be a σ-algebra of on $\mathbb{R}$. Write $\bar{\mathcal{A}} = \{E \cup N : E \in \mathcal{A} \text{ and } N \subseteq F \in \mathcal{A} \text{ with } m(F) = 0\}$. Show that $\bar{\mathcal{A}}$ is a σ-algebra. Further, deduce that $\overline{B(\mathbb{R})} = M(\mathbb{R})$. **[3]**

4. Let A be a non-empty bounded subset of $\mathbb{R}$. Define $A_n = \{x \in \mathbb{R} : \ d(x, A) < \frac{1}{n}\}$. Show that $\lim\limits_{n \to \infty} m(A_n) < \infty$. Whether A should be necessarily Lebesgue measurable for the above conclusion to hold? **[3]**

5. For Lebesgue measurable subsets A and B of $[0,1]$, define a function f on $[0,1]$ by $f(x) = d(x, A + B)$. If $m(A)m(B) > \frac{1}{4}$, then show that $f(1) = 0$. **[3]**

6. Let A be a subset of $\mathbb{R}$ such that $m^*(A \cup B) = m^*(A) + m^*(B)$ for every subset B of $\mathbb{R}$. Show that A is Lebesgue measurable. Further, if $m(A) < \infty$, then show that $m(A) = 0$. **[2]**

7. Let A be a subset of $\mathbb{R}$ such that $m^*(A) < \infty$. Show that for each $\epsilon > 0$ there exists a compact set $K \subset \mathbb{R}$ such that $m^*(A \setminus K) < \epsilon$. **[3]**

8. Let (X, S, μ) be a finite measure space. For a sequence of sets $A_n \in S$, if we define $\overline{\lim} A_n = \cap_{k \geq 1}(\cup_{n \geq k} A_n)$, then show that $\mu(\overline{\lim} A_n) \geq \overline{\lim}\, \mu(A_n)$. **[3]**

9. Let (X, τ) be a topological space. Let $\mathcal{B}(X)$ be the σ-algebra generated by τ. Let μ^* be the outer measure generated by a σ-finite pre-measure μ_o on $\mathcal{B}(X)$. Show that $E \in M_{\mu^*}$ if and only if there exists $G \in \mathcal{B}(X)$ such that $\mu^*(G \setminus E) = 0$. **[4]**

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