DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA550: Measure Theory — Mid Semester Exam
Instructor: Rajesh Srivastava — September 26, 2014
Time duration: Two hours — Maximum Marks: 30

N.B. Answer without proper justification will attract zero mark.

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1. (a) Give an example of almost everywhere vanishing Lebesgue measurable function which is not continuous. **1**

   (b) Whether the Lebesgue measure of $\{x \in \mathbb{R} : \inf_{x-1 \leq y \leq x} |y| = 1\}$ is zero? **1**

2. Let $\{E_n\}$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}$ such that $\sum_{n=1}^{\infty} m(E_n) < \infty$. Show that $m\left(\bigcap_{n=1}^{\infty} E_n\right) = 0$. **3**

3. Let $A \subset \mathbb{R}$ be a closed set with $m(A) = 0$. Show that A is nowhere dense in $\mathbb{R}$. But does this conclusion hold true when A is not closed? **2+1**

4. Let $[-1, 1] \cap \mathbb{Q} = \{r_1, r_2, \ldots\}$. For a Lebesgue measurable set $E \subset [0, 1]$ with $m(E) > 0$, define $E_n = E + r_n;\ n \in \mathbb{N}$. Show that all of E_n's can not be pairwise disjoint. **3**

5. Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) < \infty$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = m\{E \cap (-\infty, x^2)\}$. Show that f is differentiable at 0 and $f'(0) = 0$. **3**

6. Let E be a Lebesgue measurable subsets of $\mathbb{R}$ with $m(E) = \infty$. Show that there exists a sequence $\{E_n\}$ of pairwise disjoint measurable subsets of E such that $m(E_n) < \infty$, for all n and $E = \bigcup_{n=1}^{\infty} E_n$. **3**

7. Let $O \subset \mathbb{R}$ be an open set with $m(O) < \infty$. Show that for each $\epsilon > 0$ there exists a pair of disjoint open sets O_1 and O_2 such that $O = O_1 \cup O_2$ and $m(O_2) < \epsilon$. **3**

8. Let $\mathbb{Q}$ denotes set of rationals. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = \begin{cases} 1 & \text{if } x + y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$
   Show that f is Lebesgue measurable. **3**

9. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Show that $\{x \in \mathbb{R} : f \text{ is continuous at x }\}$ is Lebesgue measurable. **3**

10. Let C be the Cantor's ternary set. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in C \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$
    Show that f is Lebesgue measurable. By letting C has a non-Borel measurable subset, construct a Lebesgue measurable function which is not Borel measurable. **2+2**

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