DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA550: Measure Theory | End Semester Make-up Exam
Instructor: Rajesh Srivastava | December 7, 2014
Time duration: Three hours | Maximum Marks: 45

N.B. Answer without proper justification will attract zero mark.

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1. (a) Does there exist an unbounded closed set having finite Lebesgue measure ? **[1]**
   (b) Give an example of a set $A \subset \mathbb{R}$ such that $m\{\text{Interior}(A)\} = 0$ but $m(\bar{A}) = 1$. **[1]**
   (c) Let $\mathcal{S}(\mathbb{R})$ denotes the space of all continuous functions on $\mathbb{R}$ such that $|x|^\alpha f(x)$ is bounded, for any $\alpha \in \mathbb{N}$. Whether $\mathcal{S}(\mathbb{R})$ is a dense subspace of $L^2(\mathbb{R})$? **[1]**
   (d) Give an example of non-zero function in $\mathcal{S}(\mathbb{R})$. **[1]**
   (e) Let $C_c(\mathbb{R})$ denotes the space of all compactly supported continuous functions on $\mathbb{R}$. Give an example of non-zero function in $C_c(\mathbb{R})$. **[1]**
2. Let $m^*(A) > 0$. Then show that there exists at least one closed set $F \subset \mathbb{R}$ with $m(F) < \infty$ such that $A \cap F \neq \emptyset$. **[2]**
3. Let μ be a finite measure on $M(\mathbb{R})$. Suppose for each closed set $F \subset \mathbb{R}$ with $m(F) < \infty$, implies $\mu(F) = 0$. Then show that $\mu = 0$. **[2]**
4. Let E be a measurable subset of $\mathbb{R}$ with $m(E) < \infty$ and $m\{E \cap (n, n+1)\} < \frac{1}{2^{|n|+2}} m(E)$, for all $n \in \mathbb{Z}$. Show that $m(E) = 0$. **[2]**
5. Let K be a compact subset of $\mathbb{R}$. Show that $\{x \in \mathbb{R} : d(x, K) < 1\}$ is Lebesgue measurable and $m\{x \in \mathbb{R} : d(x, K) < 1\} \geq 2$. **[4]**
6. Let (X, S, μ) be a measurable space and $f : X \to [0, 1]$. Then show that f is measurable if and only if $\left\{x \in X : f(x) > \frac{k}{2^n}\right\}$ is measurable $\forall k = 0, 1, 2, 3, \ldots, 2^n$ and $\forall n \in \mathbb{N}$. **[3]**
7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x, .)$ is measurable and $f(., y)$ is continous. Show that f is Lebesgue measurable. **[4]**
8. Let $f_n = \chi_{\left[\frac{1}{n}, \frac{1}{n+1}\right]}$. Construct an increasing sequence g_n of measurable functions in terms of f_n such that $\lim\limits_{n\to\infty} \int\limits_{\mathbb{R}} g_n(x) dm(x) < \infty$. **[4]**
9. Let $f(x) = \frac{1}{\sqrt[3]{x}} \chi_{(0,1]}(x)$ and $f_n(x) = f(|x - n|)$. Write $g(x) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} f_n(x)$. Show that $g \in L^2(\mathbb{R}, M, m)$ but $g \notin L^p(\mathbb{R}, M, m)$, for any $p \geq 3$. **[3]**
10. Let $1 \leq p < \infty$. If $L^\infty(X, S, \mu) \subset L^p(X, S, \mu)$. Show that μ is a finite measure. **[2]**
11. Let T be a bounded linear functional on $C_c(\mathbb{R})$. Construct a bounded linear functional on $L^1(\mathbb{R})$ such that $\widetilde{T}$ restricted to $C_c(\mathbb{R})$ is T and $\|\widetilde{T}\| = \|T\|$. **[4]**

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12. Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : \; y \geq x^2 \text{ and } y \leq 1\}$. Show that $\mathbb{D}$ is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable. Compute $m \times m(\mathbb{D})$, using product measure technique. **4**

13. Let $P(x, y)$ be a polynomial on $\mathbb{R}^2$. Show that the set $G_P = \{(x, y) \in \mathbb{R}^2 : P(x, y) = 1\}$ is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable. Compute $m \times m\,(G_P)$. **3**

14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \frac{x^2}{1+(x-y)^2}\chi_{[-1,1]}(x)$. Then show that f is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable. Compute $\int\limits_{\mathbb{R}^2} f(x, y) d(m \times m)(x, y)$. **4**

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