DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA550: Measure Theory	End Semester Exam
Instructor: Rajesh Srivastava	November 20, 2021
Time duration: Three hours	Maximum Marks: 35

N.B. Answer without proper justification will attract zero mark.

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1. (a) Let $F(x,y) = f(x)f(y)$, where $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. Does it imply that F is finite a.e. $m \times m$? **[1]**

   (b) $f(x) = \min\left\{1, \frac{1}{x^2}\right\}$. Whether $f \in L^1(\mathbb{R})$? **[1]**

   (c) Suppose $f_1, f_1 : \mathbb{R} \to [0,\infty)$ are such that $\text{supp} f_1 \cap \text{supp} f_2 = \emptyset$. Does it imply that $\max\{f_1, f_2\} = f_1 + f_2$? **[1]**

   (d) Whether $\{(x,y) \in \mathbb{R}^2 : \ y = \sin\frac{1}{x}\}$ is belonging to $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$? **[1]**

2. For $E, F \in M(\mathbb{R})$, define $h(y) = \int\limits_{\mathbb{R}} \chi_E(x)\chi_F(x-y)dx$. Show that h is a Borel measurable function on $\mathbb{R}$. **[3]**

3. Let $T : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ be defined by $T(f)(x) = \int\limits_{\mathbb{R}} \frac{f(x+y)}{1+y^2}dy$. Show that T is bounded and $\|T\| = \pi$. **[3]**

4. Suppose $f \to f$ in $L^p(\mathbb{R})$ for $1 \le p < \infty$. Let $g_n \in L^\infty(\mathbb{R})$ and $\|g_n\| \le 1$. If g_n converges to g uniformly a.e., then $f_n g_n \to fg$ in $L^p(\mathbb{R})$. **[3]**

5. Let $|f_n| \le g \in L^1(\mathbb{R})$. Let f_{n_k} be subsequence of f_n such that $f_{n_k} \to f$ point wise a.e. on $\mathbb{R}$. If $\lim\limits_{k\to\infty} \|f_{n_k} - f\| = \overline{\lim\limits_{n}}\|f_n - f\|_1 < \infty$. Show that $f_n \to f$ in $L^1(\mathbb{R})$. **[4]**

6. Let $f_n = 1 + n\chi_{\left(\frac{1}{n+1},\frac{1}{n}\right)}$ and $f = 1$ a.e. on $(0,1)$. Show that $\int\limits_{(0,1)} f_n \to \int\limits_{0,1)} f$. Further, show that there does not exist $g \in L^1([0,1])$ such that $f_n \le g$ for every $n \in \mathbb{N}$. **[4]**

7. Let $f_n : (0,1) \to \mathbb{R}$ be defined by $f_n(x) = \sqrt{n|\sin\frac{1}{nx}|}$. Evaluate $\lim\limits_{n\to\infty} \int\limits_{(0,1)} f_n(x)dx$. **[4]**

8. Let $X = Y = [0,1]$. Construct a sequence of functions f_n such that $\text{supp} f_n \subseteq \left(\frac{1}{n+1}, \frac{1}{n}\right)$ and $\int_{(0,1)} f_n(x)ds = 1$. Let $f(x,y) = \int_0^1 \sum_{n=1}^\infty [f_n(x) - f_{n+1(x)}]f_n(y)dx$. Show that

$$\int_0^1 \int_0^1 f(x,y)dxdy \neq \int_0^1 \int_0^1 f(x,y)dydx.$$

   Does $f \in L^1([0,1]\times[0,1], M \otimes M, m \times m)$? **[6]**

9. Let $f(x) = \frac{1}{\sqrt{|x|(1+\log^2|x|}}$, if $x \neq 0$. Show that $f \in L^2(\mathbb{R})$. Whether $f \in L^1(\mathbb{R})$? **[4]**

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