DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA550: Measure Theory — End Semester Exam
Instructor: Rajesh Srivastava — November 26, 2014
Time duration: Three hours — Maximum Marks: 45

N.B. Answer without proper justification will attract zero mark.

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1. (a) Does there exist an unbounded open set having finite Lebesgue measure ? **[1]**
   (b) Whether $L^1(X, S, \mu)$ has an almost non-zero function for every (X, S, μ) ? **[1]**
   (c) Does $L^\infty(X, S, \mu)$ contain an almost non-zero function for every (X, S, μ) ? **[1]**
   (d) Let $C_c^\infty(\mathbb{R})$ denotes the space of all compactly supported infinitely differentiable functions on $\mathbb{R}$. Whether $C_c^\infty(\mathbb{R})$ is a dense subspace of $L^1(\mathbb{R})$? **[1]**

2. Let $m^*(A) > 0$. Then show that there exists at least one open interval $I \subset \mathbb{R}$ with $m(I) < \infty$ such that $A \cap I \neq \emptyset$. **[2]**

3. Let F be a closed subset of $\mathbb{R}$ with $m(F) = 0$. Then for any $A \subset F$, show that $m^*\{x \in \mathbb{R} : d(x, A) = 0\} = 0$. **[2]**

4. Let A be a bounded subset of $\mathbb{R}$. Then show that $m\left(\overline{A}\right) < \infty$. **[2]**

5. Let K be a compact subset of $\mathbb{R}$ and $O_n = \left\{x \in \mathbb{R} : d(x, K) < \frac{1}{n}\right\}$. Then show that $\lim\limits_{n\to\infty} m(O_n) = m(K)$. **[4]**

6. Let $f : X \to \mathbb{R}$ and $D \subset \mathbb{R}$ be such that $\overline{D} = \mathbb{R}$. Then show that f is measurable if and only if $\{x \in X : f(x) > r\}$ is measurable for all $r \in D$. **[3]**

7. Let $f, g : X \to \mathbb{R}$. Define $\varphi(x) = (f(x), g(x))$. Then show that f and g are measurable if and only if φ is measurable. **[4]**

8. Let $1 \leq p < \infty$ and $f \in L^+(X, S, \mu) \cap L^p(X, S, \mu)$. Define $f_n(x) = \min\{n, f(x)\}$. Then show that f_n increases to f point wise a.e. and $\lim\limits_{n\to\infty} \int\limits_X |f_n - f|^p d\mu = 0$. **[4]**

9. Let $\{E_n\}$ be a sequence of disjoint measurable subsets of X such that $\mu(E_n) = \dfrac{1}{n^3}$. Let $f = \sum\limits_{n=1}^{\infty} n\, \chi_{E_n}$. Then show that $f \notin L^p(X, S, \mu)$, for any $p \geq 2$. **[3]**

10. Let $f \in L^p(X, S, \mu)$ and $g \in L^\infty(X, S, \mu)$. Then show that the inequality $\|fg\|_p \leq \|f\|_p \|g\|_\infty$, holds for each p with $1 \leq p < \infty$. **[2]**

11. Let (X, S, μ) be a σ-finite measure space. Then show that $\|f\|_\infty = \sup\limits_{\|g\|_1 = 1} \left| \int\limits_X fg d\mu \right|$. **[4]**

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12. Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 < 1\}$. Show that $\mathbb{D}$ is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable. Compute $m \times m(\mathbb{D})$, using product measure technique. **4**

13. Let (X, S, μ) be a finite measure space and $f : X \to [1, \infty]$ be a measurable function. Compute $\mu \times m \{(x, y) \in X \times \mathbb{R} : \; y < f(x)\}$. **3**

14. Let $E, F \in M(\mathbb{R})$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \chi_E(x)\chi_F(x - y)$. Then show that f is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable and $\int_{\mathbb{R}^2} f(x, y) d(m \times m)(x, y) = m(E)m(F)$. **4**

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