DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA550: Measure Theory | Quiz II
Instructor: Rajesh Srivastava | November 7, 2014
Time duration: **01.5 hours** | Maximum Marks: 12

N.B. Answer without proper justification will attract zero mark.

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1. Let f be a non-negative function in $L^1(\mathbb{R}, M, m)$. Define $g(x) = \sum\limits_{n=1}^{\infty} f(2^n x + \frac{1}{n})$. Show that $g \in L^1(\mathbb{R}, M, m)$ and $\int\limits_{\mathbb{R}} g dm = \int\limits_{\mathbb{R}} f dm$. **2**

2. Let $f_n : [0, \infty) \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{n^2 x e^{-x^2}}{n^2 + x^2}$. Show that $f_n \in L^1([0, \infty))$, for each $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \int_{[0,\infty)} f_n dm = \frac{1}{2}$. **3**

3. Let $f : (X, S, \mu) \to \overline{\mathbb{R}}$ be an integrable function. Define a set function $\nu : S \to \mathbb{R}$ by $\nu(E) = \int\limits_E f \, d\mu$. Show that ν is countably additive on S. (**Hint:** Use $f = f^+ - f^-$.) **2**

4. Let $f : (X, S, \mu) \to [0, \infty]$ be such that $\|f\|_1 = 1$. Show that there exists at leat one $n \in \mathbb{N}$ such that $\mu\{x \in \mathbb{X} : |f(x)| < n\} > 0$. **3**

5. Let $f_n : X \to [0, \infty]$ be a sequence of measurable functions and $f_n \to f$ point wise. Suppose there exists $M > 0$ such that $\sup\limits_{n\geq 1} \int_X f_n \leq M$. Show that $f \in L^1(X, S, \mu)$. (**Hint:** Use Fatou's lemma.) **2**

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