Topology Lecture Notes

MA549: Topology

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Preface

These notes were prepared for the course *MA549: Topology* (July–November 2023) at IIT Guwahati. They aim to give a coherent and largely self-contained account of the core material in point-set topology that is standard across most graduate and advanced undergraduate syllabi worldwide.

This revised version also incorporates the missing statements, proofs, and examples from the handwritten lecture notes, so that the present document can be read as a single consolidated set of course notes.

Scope and syllabus map. The exposition is organized around the following themes.

1. **Foundations:** topologies, open and closed sets, neighborhoods, bases and subbases, and standard examples.

2. **Set-theoretic operations:** closure, interior, boundary, limit points; continuity and homeomorphisms.

3. **Constructions:** subspace, product, and quotient topologies; initial and final topologies; standard universal properties.

4. **Structural axioms:** countability axioms (first/second countability, separability, Lindelöfness) and separation axioms (T_0–T_4, complete regularity).

5. **Global properties:** connectedness and compactness; compactness in products (Tychonoff); compactifications (one-point, Stone–Čech where appropriate).

6. **Cornerstone theorems:** Urysohn's lemma, Tietze extension, and Urysohn metrization.

7. **Universal supplements:** nets and filters (the correct convergence language in arbitrary spaces) and the Baire category theorem (a basic tool connecting topology with analysis).

Conventions. Unless stated otherwise, all topological spaces are assumed nonempty. We use $\overline{A}$ for closure, $\operatorname{Int}(A)$ for interior, and ∂A for boundary. For maps $f : X \to Y$, the *fiber* over $y \in Y$ is $f^{-1}(\{y\})$, and the *graph* is $\{(x, f(x)) : x \in X\}$.

References. The presentation is compatible with standard texts such as Munkres, Willard, and Kelley. Whenever proofs are included, the goal is to emphasize the underlying ideas and the reusable patterns of argument (working with bases, compactness via open covers, separation via continuous functions, etc.).

Corrections. Despite care, typographical mistakes may remain. Readers are encouraged to report them.

Contents

Chapter 1

Introduction

These notes provide a self-contained introduction to point-set topology, with an emphasis on the structural viewpoint: a topology is the data needed to speak meaningfully about continuity *and* convergence *without committing to a particular metric. We develop the basic language of open and closed sets, bases and subbases, closure/interior/boundary, and then study continuous maps, standard constructions (subspaces, products, quotients), countability and separation axioms, connectedness, and compactness. The final chapters present three classical cornerstones of the subject: Urysohn's lemma, Tietze's extension theorem, and Urysohn's metrization theorem.*

Topology may be viewed as a systematic abstraction of the notions that first arise in metric spaces. In a metric space (X, d), openness is encoded by open balls, and continuity can be expressed by the requirement that the inverse image of every open set is open. Both ideas make sense without a metric once we decide which subsets of X are to be declared open.

From this point of view, a *topological space* is a set together with a distinguished collection of subsets (the open sets) satisfying axioms modeled on the behavior of open subsets in metric spaces. One then studies invariants of spaces preserved under *homeomorphisms* (bijective continuous maps with continuous inverse), as well as the behavior of constructions such as subspaces, products, and quotients.

Prerequisites. Basic set theory and familiarity with metric spaces (open/closed sets, convergence, continuity) are assumed. Whenever a result is first proved in the metric setting, we emphasize the abstract reformulation that survives in general topological spaces.

Chapter 2

Topological spaces, bases, and subbases

We introduce topologies and the intuition of continuity coming from metric spaces. After basic examples (discrete, indiscrete, cofinite), we discuss comparison of topologies (finer/coarser) and develop the indispensable tools of bases and subbases for generating and working with topologies.

2.1 From metric spaces to topologies

Let (X, d) be a metric space. A subset $O \subseteq X$ is *open* if for every $x \in O$ there exists $r > 0$ such that the open ball

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

is contained in O. In particular, every open set is a union of open balls.

Let $\mathcal{T}_d$ denote the collection of all open sets in (X, d). Then:

(i) $\varnothing \in \mathcal{T}_d$ and $X \in \mathcal{T}_d$;

(ii) if $\{O_i\}_{i \in I} \subseteq \mathcal{T}_d$, then $\bigcup_{i \in I} O_i \in \mathcal{T}_d$;

(iii) if $O_1, \dots, O_n \in \mathcal{T}_d$, then $\bigcap_{k=1}^{n} O_k \in \mathcal{T}_d$.

These three axioms do not mention the metric d explicitly; they depend only on the distinguished family $\mathcal{T}_d \subseteq \mathcal{P}(X)$. This motivates the abstract notion of a topology.

Definition 2.1. Let X be a set and let $\mathcal{T} \subseteq \mathcal{P}(X)$. We say that $\mathcal{T}$ is a *topology* on X if:

(i) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;

(ii) for every family $\{U_i\}_{i \in I} \subseteq \mathcal{T}$, the union $\bigcup_{i \in I} U_i$ belongs to $\mathcal{T}$;

(iii) for every finite subfamily $U_1, \dots, U_n \in \mathcal{T}$, the intersection $\bigcap_{k=1}^{n} U_k$ belongs to $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *topological space*, and the members of $\mathcal{T}$ are called *open sets*.

Example 2.2. Two extremal examples occur on every nonempty set X:

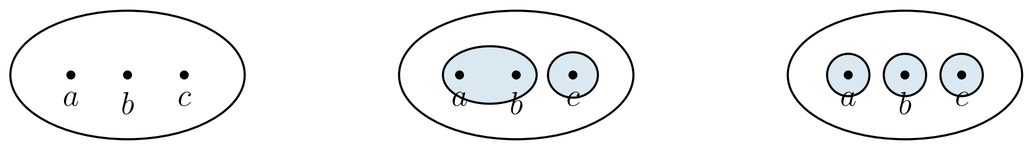
(a) $\mathcal{T}_{\text{ind}} = \{\varnothing, X\}$, the *indiscrete* topology;

(b) $\mathcal{T}_{\text{disc}} = \mathcal{P}(X)$, the *discrete* topology.

If $|X| \geq 3$, there are many other topologies in between these two extremes.

If $\mathcal{T}_1, \mathcal{T}_2$ are topologies on X, we say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$ (and $\mathcal{T}_2$ is *coarser* than $\mathcal{T}_1$) if $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Thus $\mathcal{T}_{\text{disc}}$ is the finest topology on X, and $\mathcal{T}_{\text{ind}}$ is the coarsest.

Example 2.3. Let $X = \{a, b, c\}$. Besides the indiscrete and discrete topologies, there are many intermediate topologies on X. We record a few typical examples. (Counting the number of distinct topologies on an n-point set is a difficult combinatorial problem; closed formulas are not known.)



$\tau_0 = \{\varnothing, X\}$ (indiscrete). $\tau_1 = \{\varnothing, X, \{a, b\}, \{c\}\}$. $\tau_{\text{disc}} = \mathcal{P}(X)$ (discrete).

Figure 2.1: Schematic depiction of some topologies on $X = \{a, b, c\}$.

It is also easy to write down collections that *fail* to be topologies. For instance,

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{b\}\} \qquad \text{is not a topology, since } \{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_1,$$

and

$$\mathcal{T}_2 = \{\varnothing, X, \{a, b\}, \{b, c\}\} \qquad \text{is not a topology, since } \{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T}_2.$$

Example 2.4 (Cofinite topology)**.** Let X be a set. The *cofinite topology* on X is

$$\mathcal{T}_{\text{cof}} = \{\varnothing\} \cup \{\, O \subseteq X : \ X \setminus O \text{ is finite} \,\}.$$

It is straightforward to check that $\mathcal{T}_{\text{cof}}$ is a topology:

- $\varnothing \in \mathcal{T}_{\text{cof}}$ by definition and $X \in \mathcal{T}_{\text{cof}}$ since $X \setminus X = \varnothing$ is finite;
- if $\{O_i\}_{i\in I} \subseteq \mathcal{T}_{\text{cof}}$ and $O = \bigcup_{i\in I} O_i \neq \varnothing$, then $X \setminus O = \bigcap_{i\in I}(X \setminus O_i)$, an intersection of finite sets, hence finite;
- if $O_1, \dots, O_m \in \mathcal{T}_{\text{cof}}$ are nonempty, then $X \setminus \bigcap_{k=1}^m O_k = \bigcup_{k=1}^m (X \setminus O_k)$, a finite union of finite sets, hence finite.

Example 2.5 (Cocountable topology)**.** Let X be a set. The *cocountable topology* on X is

$$\mathcal{T}_{\text{coc}} = \{\varnothing\} \cup \{\, O \subseteq X : \ X \setminus O \text{ is countable} \,\}.$$

The verification that $\mathcal{T}_{\text{coc}}$ is a topology is identical to the cofinite case, using that arbitrary intersections of countable sets are countable and finite unions of countable sets are countable.

Remark 2.6. If X is countable, then every subset of X has countable complement, hence $\mathcal{T}_{\text{coc}} = \mathcal{P}(X)$ is the discrete topology.

Definition 2.7 (Basis)**.** Let X be a set. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *basis* on X if

(B1) for every $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$;

(B2) whenever $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Proposition 2.8 (Topology generated by a basis)**.** *Given a basis $\mathcal{B}$ on X, define*

$$\mathcal{T}(\mathcal{B}) = \Big\{\, O \subseteq X : \ \textit{for every } x \in O \textit{ there exists } B \in \mathcal{B} \textit{ with } x \in B \subseteq O \,\Big\}.$$

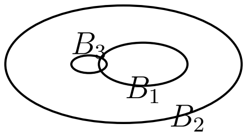
Then $\mathcal{T}(\mathcal{B})$ is a topology on X. Moreover, $\mathcal{B} \subseteq \mathcal{T}(\mathcal{B})$ and $\mathcal{T}(\mathcal{B})$ is the smallest topology on X containing $\mathcal{B}$.

Proof. By (B1), $X \in \mathcal{T}(\mathcal{B})$; and $\varnothing \in \mathcal{T}(\mathcal{B})$ holds vacuously. If $\{O_i\}_{i\in I} \subseteq \mathcal{T}(\mathcal{B})$ and $O = \bigcup_{i\in I} O_i$, then for $x \in O$ we have $x \in O_{i_0}$ for some i_0, hence $x \in B \subseteq O_{i_0} \subseteq O$ for some $B \in \mathcal{B}$; thus $O \in \mathcal{T}(\mathcal{B})$. If $O_1, \dots, O_n \in \mathcal{T}(\mathcal{B})$ and $x \in \bigcap_{k=1}^n O_k$, choose $B_k \in \mathcal{B}$ with $x \in B_k \subseteq O_k$. By repeated use of (B2) there exists $B \in \mathcal{B}$ with $x \in B \subseteq \bigcap_{k=1}^n B_k \subseteq \bigcap_{k=1}^n O_k$, proving $\bigcap_{k=1}^n O_k \in \mathcal{T}(\mathcal{B})$.

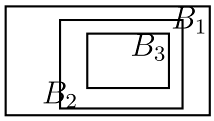
Finally, if $B \in \mathcal{B}$ and $x \in B$, then $x \in B \subseteq B$, so $B \in \mathcal{T}(\mathcal{B})$. If $\mathcal{T}$ is any topology on X with $\mathcal{B} \subseteq \mathcal{T}$, then every $O \in \mathcal{T}(\mathcal{B})$ is a union of sets in $\mathcal{B}$, hence a union of open sets in $\mathcal{T}$, so $O \in \mathcal{T}$. This proves minimality. $\square$

Remark 2.9. Equivalently, $\mathcal{T}(\mathcal{B})$ consists of all unions of basis elements: $O \subseteq X$ is open if and only if it can be written as $O = \bigcup_{\alpha \in A} B_\alpha$ with $B_\alpha \in \mathcal{B}$.

Example 2.10. Let $\mathcal{B}$ be the collection of all open discs in $\mathbb{R}^2$. Then $\mathcal{B}$ is a basis for the usual topology on $\mathbb{R}^2$.



Example 2.11. Let $\mathcal{B}'$ be the collection of all open rectangles in the plane $\mathbb{R}^2$. Then $\mathcal{B}'$ is a basis. In fact, the second condition is satisfied trivially.



We shall see later that both $\mathcal{B}$ and $\mathcal{B}'$ generate the same topology on $\mathbb{R}^2$, the usual topology.

Lemma 2.12. *Let X be a set, and let $\mathcal{B} \subset \mathcal{P}(X)$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is equal to all unions of members of $\mathcal{B}$.*

Proof: *Note that $\mathcal{T} = \mathcal{T}(\mathcal{B})$*

$\Rightarrow \mathcal{B} \subset \mathcal{T}$,

Since $\mathcal{T}$ is a topology, the union of members of $\mathcal{B}$ is in $\mathcal{T}$.

On the other hand, let $O \in \mathcal{T}$. Then for each $x \in O$, there exists $B_x \in \mathcal{B} \subset \mathcal{T}$ such that $x \in B_x \subset O$. But then

$$O = \bigcup_{x \in O} B_x.$$

However, this decomposition need not be unique.

Notice that any subcollection $\mathcal{B} \subset \mathcal{P}(X)$ is a basis if for all $x = x_1$:
there exists $B \in \mathcal{B}$ such that $x \in B$,
and if $x \in B_1 \cap B_2 \Rightarrow x \in B_3 \subset B_1 \cap B_2$.

Remark 2.13. Basis of an independent family satisfies these two conditions. However, it generates a topology via unions of its members.

Sometimes we need to go in the reverse direction, that is, to obtain a basis for a given topology.

Lemma 2.14. *Let $(X, \mathcal{T})$ be a topological space, let $\mathcal{C} \subset \mathcal{T}$ be such that for each open set $O \in \mathcal{T}$, for each $x \in O$, there exists $C \in \mathcal{C}$ such that $x \in C \subset O$. Then $\mathcal{C}$ is a basis for $\mathcal{T}$.*

Proof: *(1) Claim $\mathcal{C}$ is a basis. For*

Let $x \in X$, since X is open and $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C \subset X$.

Let $G_1, G_2 \in \mathcal{T}$ and $x \in G_1 \cap G_2$.

Since G_1, G_2 are open, by hypothesis there exists $G_3 \in \mathcal{G}$ such that $x \in G_3 \subset G_1 \cap G_2$.

Suppose $\mathcal{T}' = \mathcal{T}(\mathcal{G})$, the topology generated by $\mathcal{G}$.

***Claim:** $\mathcal{T} = \mathcal{T}'$.*

If $O \in \mathcal{T}$, then for each $x \in O$, there exists $G_\alpha \in \mathcal{G}$ such that $x \in G_\alpha \subset O$.

$$\Rightarrow O = \bigcup_{x \in O} G_\alpha$$

Since $\mathcal{G}$ is a basis, it follows that $O \in \mathcal{T}'$.

Conversely, if $W \in \mathcal{T}'$, then

$$W = \bigcup_{i \in I} G_i$$

but then $W \in \mathcal{T}$, as $G_i \in \mathcal{T}$.

If we know the bases for some topologies, then it is useful to have criteria in terms of bases for comparing them.

Lemma 2.15. *Let $\mathcal{B}, \mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively on X. Then the following statements are equivalent:*

1. *$\mathcal{T}'$ is finer than $\mathcal{T}$*
   (i.e., $\mathcal{T}' \supset \mathcal{T}$)

2. *For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. (Finer topology has smaller size open sets)*

*$(ii) \Rightarrow (i)$: **Claim:** $\mathcal{T} \subset \mathcal{T}'$. Let $O \in \mathcal{T}$. We show that $O \in \mathcal{T}'$.*
Let $x \in O$, then there exists $B \in \mathcal{B}$ such that $x \in B \subset O$.
By (ii), there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset O$.

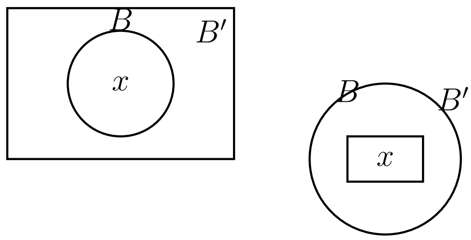
$$\Rightarrow \text{for every } x \in O, \text{ there exists } B \in \mathcal{B} \text{ such that } x \in B \subset O.$$

Thus, $O \in \mathcal{T}'$.
$(i) \Rightarrow (ii)$: Let $x \in X$ and let $B \in \mathcal{B}$ be such that $x \in B$. Since $B \in \mathcal{T}$, by definition and $\mathcal{T} \subset \mathcal{T}'$ by condition (i),

$$\Rightarrow B \in \mathcal{T}'$$

Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.
(i.e. $B = \bigcup_{x \in B} B_x$, $B_x \in \mathcal{B}'$.)

Now, it follows from the above lemma (theorem 2.15) that topology generated by open discs and open rectangles are same.



i.e., $\mathcal{T}' = \mathcal{T}$(open rectangles) and $\mathcal{T} = \mathcal{T}$ (open discs) $\Rightarrow \mathcal{T}' = \mathcal{T}$.

Example 2.16. Let

$$\mathcal{B} = \{(a,b) : a < b\}, \qquad \mathcal{B}' = \{[a,b) : a < b\},$$

and let

$$K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}, \qquad \mathcal{B}'' = \mathcal{B} \cup \{(c,d) \setminus K : c < d\}.$$

Denote by $\mathcal{T} = \mathcal{T}(\mathcal{B})$ the usual topology on $\mathbb{R}$, by $\mathcal{T}' = \mathcal{T}(\mathcal{B}')$ the lower limit topology, and by $\mathcal{T}'' = \mathcal{T}(\mathcal{B}'')$ the K-topology. Then $\mathcal{T}'$ and $\mathcal{T}''$ are both strictly finer than $\mathcal{T}$, but $\mathcal{T}'$ and $\mathcal{T}''$ are not comparable.

Indeed, every Euclidean open interval belongs to $\mathcal{T}'$ because

$$(a,b) = \bigcup_{x \in (a,b)} [x,b).$$

Hence $\mathcal{T} \subseteq \mathcal{T}'$. The inclusion is strict because $[a,b) \in \mathcal{T}'$ whereas $[a,b) \notin \mathcal{T}$.

Likewise $\mathcal{T} \subseteq \mathcal{T}''$ because $\mathcal{B} \subseteq \mathcal{B}''$. The inclusion is strict since $(-1,1) \setminus K \in \mathcal{T}''$, but this set is not Euclidean open: it contains 0, and every Euclidean open interval about 0 contains some $1/n \in K$.

Finally, the two finer topologies are not comparable. The set $[0,1)$ is open in $\mathcal{T}'$ but not in $\mathcal{T}''$, because every $\mathcal{T}''$-basic neighborhood of 0 contains either negative numbers or points of K. On the other hand, $(-1,1) \setminus K$ is open in $\mathcal{T}''$ but not in $\mathcal{T}'$, since every $\mathcal{T}'$-basic neighborhood $[0,\varepsilon)$ of 0 contains points of K.

More generally, if $\mathcal{S} \subseteq \mathcal{P}(X)$, let $\mathcal{B}(\mathcal{S})$ be the collection of all finite intersections of members of $\mathcal{S}$, together with X (viewed as the intersection of the empty family). Then $\mathcal{B}(\mathcal{S})$ is a basis, and the topology generated by this basis is called the topology generated by the subbasis $\mathcal{S}$. In particular, when $\mathcal{S} = \varnothing$, the generated topology is the indiscrete topology $\{\varnothing, X\}$.

Example 2.17. We know that

$$\mathcal{B} = \{(a, b) : a < b,\ a, b \in \mathbb{R}\}$$

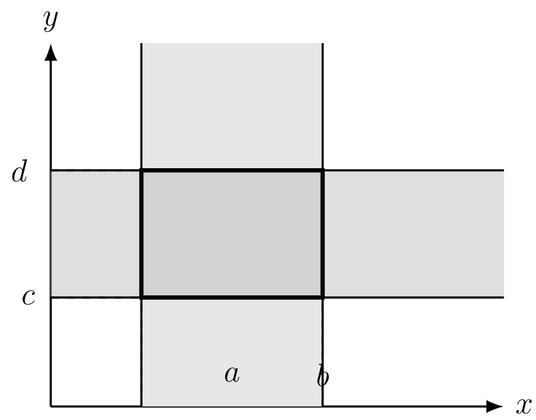
is a basis for the usual topology on $\mathbb{R}$.

$$\mathcal{J} = \{(a, +\infty),\ (-\infty, b) : a < b,\ a, b \in \mathbb{R}\}$$

is an open sub-basis for the usual topology $\mathcal{U}$ on $\mathbb{R}$.

Example 2.18. We know that open rectangles are a basis for the usual topology on $\mathbb{R}^2$.

But any open rectangle can be written as intersection of two strips.



Thus, open strips form a sub-basis for the usual topology on $\mathbb{R}^2$.

$$(a, b) \times \mathbb{R} \cap \mathbb{R} \times (c, d) = (a, b) \times (c, d).$$

$$\mathcal{J} = \{(a, b) \times \mathbb{R} : a, b \in \mathbb{R}\} \cup \{\mathbb{R} \times (c, d) : c, d \in \mathbb{R}\}$$

is a sub-basis for $\mathcal{T}(\mathbb{R}^2)$.

Exercises

Exercise 2.19. Let $\mathcal{B} \subseteq \mathcal{P}(X)$. Prove that $\mathcal{B}$ is a basis for some topology on X if and only if (i) $\bigcup \mathcal{B} = X$ and (ii) for any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

Exercise 2.20. Let $\mathcal{S} \subseteq \mathcal{P}(X)$. Describe explicitly the topology generated by $\mathcal{S}$ as a subbasis, and prove that it is the smallest topology containing $\mathcal{S}$.

Exercise 2.21. On an infinite set X, show that the cofinite topology is T_1 but not Hausdorff.

Exercise 2.22. On an uncountable set X, show that the cocountable topology is T_1 but not Hausdorff. What changes if X is countable?

Exercise 2.23. Let $\mathcal{T}_1 \subseteq \mathcal{T}_2$ be topologies on X. Prove that the identity map $\mathrm{id}_X : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous, and that the converse holds.

Exercise 2.24. Let (X, d) be a metric space. Show that the family of all open balls with rational radii is a basis for $\mathcal{T}_d$.

Chapter 3

Closed sets, subspaces, and set-theoretic operations

We study closed sets and the subspace topology, and then develop the fundamental set-theoretic operations in a topological space: closure, interior, boundary, and limit points. These notions govern density and nowhere density and provide a convenient language for formulating many later arguments.

3.1 Closed sets and the subspace topology

Closed set

A set A in a topological space X is said to be *closed* if its complement $X \setminus A$ is open.

Example 3.1. $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, +\infty)$, which is open.

Example 3.2. In the cofinite topological space X, the closed sets are X and all finite sets.

Example 3.3. In the discrete topological space, every set is open and closed.

Example 3.4. Let $(X, \mathcal{T})$ be a topological space. Then the family of closed sets satisfies the axioms dual to those of open sets:

(i) $\varnothing$ and X are closed;

(ii) finite unions of closed sets are closed;

(iii) arbitrary intersections of closed sets are closed.

Now let $Y \subseteq X$. The collection

$$\mathcal{T}_Y = \{O \cap Y : O \in \mathcal{T}\}$$

defines a topology on Y, called the *subspace topology.* Thus the open sets of the subspace Y are precisely the traces on Y of open sets in the ambient space X.

Remark 3.5. The subspace topology is the natural mechanism for restricting a topology to a subset. Many arguments in topology proceed by passing from a space to a subspace and comparing the corresponding notions of openness, closedness, closure, and continuity. Some properties are hereditary (for example, second countability), whereas others need not pass to arbitrary subspaces without additional hypotheses.

Example 3.6. Let $Y = [0, 1] \cup (2, 3)$.

(i) $[0, 1] = \left(-\frac{1}{2}, \frac{3}{2}\right) \cap ([0, 1] \cup (2, 3))$
$\Rightarrow [0, 1]$ is open in the subspace topology $(Y, \mathcal{T}_Y)$.

Similarly, $(2, 3)$ is open in Y, in fact open in $\mathbb{R}$.

Since $[0, 1]$ and $(2, 3)$ are complements of each other, both of them are open and closed in Y.
(Note: Complementation is taken in Y.)

Example 3.7. Let Y be a subset of the topological space X. Then A is closed in Y if and only if $A = C \cap Y$ for some closed set C in X.

Let A be closed in Y. Then $Y \setminus A$ is open in Y.

$$Y \setminus A = Y \cap O$$

where O is open in X.

Since $X \setminus O$ is closed in X,

$$A = Y \setminus (Y \cap O) = Y \cap (X \setminus O) = Y \cap C, \quad C = X \setminus O.$$

On the other hand, let

$$A = Y \cap C, \quad C \text{ closed in } X.$$

Since $X \setminus C$ is open in X,

$$(Y \setminus C) \cap Y \text{ is open in } Y$$

$$= Y \setminus A$$

$\Rightarrow$ A is closed in Y.

3.2 Closure of a set

The closure of a subset $A \subseteq X$ is the smallest closed subset of X containing A; equivalently,

$$\overline{A} = \bigcap \{F \subseteq X : F \text{ is closed and } A \subseteq F\}.$$

The next theorem explains how closure behaves when one passes to a subspace.

Theorem 3.8. *Let Y be a subspace of X, and let $A \subseteq Y$. If $\overline{A}^X$ denotes the closure of A in X and $\overline{A}^Y$ denotes the closure of A in Y, then*

$$\overline{A}^Y = \overline{A}^X \cap Y.$$

Proof. Since $\overline{A}^X$ is closed in X, the set $\overline{A}^X \cap Y$ is closed in the subspace Y. Moreover, because $A \subseteq \overline{A}^X$ and $A \subseteq Y$, we have

$$A \subseteq \overline{A}^X \cap Y.$$

Hence $\overline{A}^Y \subseteq \overline{A}^X \cap Y$, by the minimality of $\overline{A}^Y$ among closed subsets of Y containing A.

Conversely, let $B \subseteq Y$ be any closed subset of Y with $A \subseteq B$. By the definition of the subspace topology, there exists a closed set $C \subseteq X$ such that $B = C \cap Y$. Since $A \subseteq B \subseteq C$, the minimality of $\overline{A}^X$ gives $\overline{A}^X \subseteq C$. Intersecting with Y, we obtain

$$\overline{A}^X \cap Y \subseteq C \cap Y = B.$$

This inclusion holds for every closed subset B of Y containing A; in particular, it holds for $B = \overline{A}^Y$. Therefore $\overline{A}^X \cap Y \subseteq \overline{A}^Y$, and the proof is complete. □

3.3 Interior of a set

The interior of a subset $A \subseteq X$ is the largest open subset of X contained in A. Equivalently,

$$A^\circ = \bigcup \{O \subseteq X : O \text{ is open and } O \subseteq A\}.$$

Example 3.9. $\mathbb{Q}^\circ = \varnothing$ in $(\mathbb{R}, \mathcal{U})$

Example 3.10. Interior of the Cantor set is empty.

Example 3.11. $(\mathbb{R} \setminus \mathbb{Q}) \cup [0, 1]$ has interior $([0, 1])^\circ = (0, 1)$, etc.

3.4 Limit points, boundary, and related notions

The set-theoretic formulas for closure and interior are conceptually important, but they are often inconvenient for computation. The next results recast these notions in terms of neighborhoods and basis elements, which is the form most frequently used in practice.

Theorem 3.12. *Let A be a subset of a topological space (X, τ). Then:*

1. *$x \in \overline{A}$ if and only if every open set $O \ni x$ intersects A.*
2. *If $\mathcal{B}$ is a basis for τ, then $x \in \overline{A}$ if and only if every basis element $B \in \mathcal{B}$ with $x \in B$ intersects A.*

Proof. For (1), note that $x \notin \overline{A}$ if and only if $x \in X \setminus \overline{A}$. Since $X \setminus \overline{A}$ is open, this is equivalent to the existence of an open set $O \ni x$ such that $O \cap A = \varnothing$. Taking complements gives the stated criterion.

Statement (2) is immediate from (1) in one direction, because every basis element is open. Conversely, assume every basis element containing x intersects A, and let O be any open set containing x. Since $\mathcal{B}$ is a basis, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq O$. By hypothesis, $B \cap A \neq \varnothing$, hence also $O \cap A \neq \varnothing$. Therefore $x \in \overline{A}$ by (1). □

Example 3.13. Let A be a subset of a metric space (X, d). Then

$$x \in \overline{A} \iff B_\varepsilon(x) \cap A \neq \varnothing \text{ for every } \varepsilon > 0.$$

This is exactly theorem 3.12(2) applied to the standard basis of open balls. In first-countable spaces, this is also equivalent to the existence of a sequence $(x_n) \subseteq A$ with $x_n \to x$.

Corollary 3.14. *Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if every neighborhood of x intersects A.*

A point $x \in X$ is called a limit point (or accumulation point) of A if every neighborhood of x meets $A \setminus \{x\}$. The set of all limit points of A is denoted by A' and is called the derived set of A.

Example 3.15. For $A \subseteq X$ and $x \in X$,

$$x \in A' \quad \Longleftrightarrow \quad x \in \overline{A \setminus \{x\}}.$$

Proof. This is a direct reformulation of the definition: every neighborhood of x meets $A \setminus \{x\}$ if and only if every neighborhood of x intersects $A \setminus \{x\}$, which is equivalent to $x \in \overline{A \setminus \{x\}}$ by theorem 3.14. □

Theorem 3.16. *Let $A \subseteq X$. Then*

$$\overline{A} = A \cup A'.$$

Proof. If $x \in A$, then certainly $x \in \overline{A}$, so $A \subseteq \overline{A}$. If $x \in A'$, every neighborhood of x meets $A \setminus \{x\} \subseteq A$; hence $x \in \overline{A}$ by theorem 3.14. Thus $A \cup A' \subseteq \overline{A}$.

Conversely, let $x \in \overline{A}$. If $x \in A$, there is nothing to prove. If $x \notin A$, then every neighborhood of x meets A, and since $x \notin A$ this is equivalent to meeting $A \setminus \{x\}$. Hence $x \in A'$. Therefore $\overline{A} \subseteq A \cup A'$, and the result follows. □

Example 3.17. If $A, B \subset X$ (topological space), then show that

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

Notice that $A \cup B \subset \overline{A \cup B}$.
Since $\overline{A \cup B}$ is a closed set containing $A \cup B$, it follows that

$$\overline{A \cup B} \subset \overline{A} \cup \overline{B}$$

Also, $A \subset A \cup B \Rightarrow \overline{A} \subset \overline{A \cup B}$
$B \subset A \cup B \Rightarrow \overline{B} \subset \overline{A \cup B}$
$\Rightarrow \overline{A} \cup \overline{B} \subset \overline{A \cup B}$
Hence,

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

However, the inclusion

$$\overline{A \cap B} \subset \overline{A} \cap \overline{B}$$

may be strict. For example,

$$A = [0, 1), \quad B = [1, 2]$$

$$A \cap B = \varnothing, \qquad \overline{A} \cap \overline{B} = \{1\}$$

Notice that the set-theoretic equality $\overline{A} = A \cup A'$ need not be disjoint. However, we can define a new set, called the set of isolated points, to get a disjoint union.

Definition 3.18. Let $A \subset X$ (topological space). A point $x \in X$ is called an *isolated point* of A if some neighborhood N_x exists which intersects A only at x, i.e.,

$$N_x \cap A = \{x\}$$

The set of all isolated points is denoted by $\mathrm{iso}(A)$.

Theorem 3.19. *Let X be a topological space and let $A \subseteq X$. Then*

$$\overline{A} = \mathrm{iso}(A) \sqcup A',$$

where $\mathrm{iso}(A)$ *is the set of isolated points of A and A' is the derived set (the set of limit points of A).*

Proof. First note that $\mathrm{iso}(A) \subseteq A \subseteq \overline{A}$, and every limit point of A lies in $\overline{A}$, so $A' \subseteq \overline{A}$.

Conversely, let $x \in \overline{A}$. If x is a limit point of A, then $x \in A'$. Otherwise $x \notin A'$, so there exists a neighborhood U of x such that $U \cap (A \setminus \{x\}) = \varnothing$. Since $x \in \overline{A}$, we also have $U \cap A \neq \varnothing$, and therefore $x \in A$ and $U \cap A = \{x\}$. Hence $x \in \mathrm{iso}(A)$. This shows $\overline{A} \subseteq \mathrm{iso}(A) \cup A'$.

Finally, $\mathrm{iso}(A) \cap A' = \varnothing$: if $x \in \mathrm{iso}(A)$, then some neighborhood U of x satisfies $U \cap (A \setminus \{x\}) = \varnothing$, so $x \notin A'$. □

Definition 3.20 (Boundary)**.** For $A \subseteq X$, the *boundary* of A is

$$\partial(A) = \overline{A} \cap \overline{X \setminus A}.$$

Proposition 3.21. *The boundary $\partial(A)$ is closed. Moreover, a point $x \in X$ lies in $\partial(A)$ if and only if every neighborhood of x meets both A and $X \setminus A$.*

Proof. Closedness is immediate since $\partial(A)$ is an intersection of closed sets. For the neighborhood characterization, note that $x \in \overline{A}$ means every neighborhood of x intersects A, and $x \in \overline{X \setminus A}$ means every neighborhood of x intersects $X \setminus A$. Combining the two conditions gives the claim. □

Example 3.22. In $(\mathbb{R},\mathcal{U})$ with the usual topology,

$$\partial(\mathbb{N}) = \mathbb{N}, \qquad \partial(\mathbb{Q}) = \mathbb{R}, \qquad \partial(\mathbb{R}) = \varnothing.$$

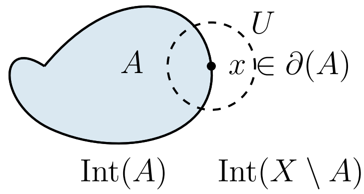


Figure 3.1: A boundary point x is characterized by neighborhoods U that meet both A and $X \setminus A$.

Thus the boundary may be strictly larger than A (for example, $A = \mathbb{Q}$).

Example 3.23.

$$\partial(\{0\}) = \overline{\{0\}} \cap \overline{\mathbb{R} \setminus \{0\}} = \{0\} \cap \mathbb{R} = \{0\}.$$

Example 3.24.

$$\partial(\{0, 1\}) = \{0, 1\}.$$

Example 3.25. Show that $A \subseteq X$ is closed if and only if $A \supseteq \partial(A)$.

Example 3.26. Show that $\partial(A) = \emptyset$ if A is both closed and open.

Example 3.27. If $A \subseteq X$ (topological space), then

$$\overline{A} = A^\circ \cup \partial(A).$$

Definition 3.28. A set $A \subseteq X$ (topological space) is said to be *perfect* if $A = A'$.

Example: Cantor's set is a perfect set.

Example: $A = [0, 1]'$ in $(\mathbb{R}, \mathcal{U})$. (As any neighborhood of each point intersects A other than the point.)

Definition 3.29. A set $A \subseteq X$ (topological space) is said to be *dense* if $\overline{A} = X$, i.e., every point $x \in X$ has N_α which intersects A, i.e. $N_\alpha \cap A \neq \emptyset$.

Example 3.30. Show that

$$\text{Int}(A^c) = (\overline{A})^c$$

Definition 3.31. A subset A of topological space X is said to be *nowhere dense* if

$$(\overline{A})^\circ = \emptyset$$

(i.e., closure has no interior.)

Example: Cantor's set is nowhere dense in $[0, 1]$.

$$(\overline{C})^\circ = C^\circ = \emptyset$$

Example 3.32. Let $A \subseteq X$ (topological space) be closed. Then A is nowhere dense if $\overline{A^c} = X$.

Example 3.33. Show that boundary of a closed set is nowhere dense. Is this true for an arbitrary set?

Example 3.34. If $A \subseteq X$ (metric space), then $\overline{A}$ is the set of points of X which have zero distance from A.

$$\overline{A} = \{x \in X : d(x, A) = 0\}$$

And

$$\partial(A) = \{x \in X : d(x, A) = 0 \text{ and } d(x, X \setminus A) = 0\}$$

Example 3.35.

$$\partial(A \cup (X \setminus A)) = \overline{X} \cap \overline{\emptyset} = \emptyset,$$
$$X \setminus A^\circ = \overline{X \setminus A}.$$

Claim: $x \in \partial(A) \Rightarrow$ for every neighborhood N_α, $N_\alpha \cap A \neq \emptyset$ and $N_\alpha \cap (X \setminus A) \neq \emptyset \Rightarrow x \notin X \setminus A$.

Example 3.36. Let $\partial(A) = \overline{A} \cap \overline{A^c}$. Then

(i) $\overline{A} = A^\circ \cup \partial(A)$,

(ii) $A^\circ = A \setminus \partial(A)$,

(iii) $X = A^\circ \cup \partial(A) \cup (A^c)^\circ$.

Exercises

Exercise 3.37. Prove that $\overline{A}$ is the smallest closed subset of X containing A, and that $\text{Int}(A)$ is the largest open subset of X contained in A.

Exercise 3.38. Show that for every $A \subseteq X$,

$$\partial A = \overline{A} \setminus \text{Int}(A) = \overline{A} \cap \overline{X \setminus A}.$$

Exercise 3.39. Let $Y \subseteq X$ with the subspace topology. Prove that for any $A \subseteq Y$,

$$\overline{A}^{Y} = \overline{A}^{X} \cap Y \quad \text{and} \quad \mathrm{Int}_Y(A) = Y \cap \mathrm{Int}_X(A).$$

Exercise 3.40. Let $A \subseteq X$. Prove that A is dense in X if and only if every nonempty open subset of X meets A.

Exercise 3.41. Show that $A \subseteq X$ is nowhere dense if and only if $\mathrm{Int}(\overline{A}) = \varnothing$.

Exercise 3.42. Let X be a T_1 space. Show that the derived set A' is closed for every $A \subseteq X$.

Chapter 4

Continuity and homeomorphisms

We define continuity for maps between topological spaces and collect standard equivalent characterizations (via open sets, closed sets, neighborhoods, and bases/subbases). We then discuss homeomorphisms and basic tools such as the pasting lemma and restrictions to subspaces.

4.1 Continuous maps

Continuity is the central organizing principle of topology. In metric spaces it is first encountered through the ε–δ definition; however, that formulation is only one manifestation of a more structural fact, namely that continuity is exactly the preservation of openness under inverse images. This observation survives unchanged in arbitrary topological spaces and therefore becomes the correct abstract definition.

Thus, for topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a map

$$f : X \to Y$$

is continuous if and only if $f^{-1}(O)$ is open in X for every open set $O \subseteq Y$. Equivalently, f is continuous at $x \in X$ if every neighborhood of $f(x)$ has inverse image containing a neighborhood of x. In symbols, for each neighborhood $N_{f(x)}$ of $f(x)$ there exists a neighborhood N_x of x such that

$$f(N_x) \subseteq N_{f(x)}, \qquad \text{or equivalently} \qquad N_x \subseteq f^{-1}(N_{f(x)}).$$

A major advantage of the topological definition is that one need not test continuity on all open sets. As with bases and subbases for topologies, it is enough to check inverse images of basis elements, or even of subbasic elements. The following discussion makes this precise.

Let $\mathcal{B}_Y$ be a basis for Y. Then every open set $O \in \mathcal{T}_Y$ can be represented as

$$O = \bigcup_{i \in I} B_i, \quad B_i \in \mathcal{B}_Y$$

(union of members of $\mathcal{B}_Y$)

$$\Rightarrow f^{-1}(O) = \bigcup_{i \in I} f^{-1}(B_i)$$

Hence, $f^{-1}(O)$ is open for each $O \in \mathcal{T}_Y$ if and only if $f^{-1}(B_i)$ is open for each $i \in I$.

Further, if $\mathcal{S}_Y$ is a subbasis for $\mathcal{T}_Y$, then sets of the form

$$B = \bigcap_{i=1}^{n} S_i = (\text{intersection of finitely many } S_i \in \mathcal{S}_Y)$$

are a basis for $\mathcal{T}_Y$, and

$$f^{-1}(B) = \bigcap_{i=1}^{n} f^{-1}(S_i)$$

Hence, $f^{-1}(O)$ is open if and only if $f^{-1}(S_i)$ is open for each $i = 1, 2, \ldots, n$. (Note that the choice of O is arbitrary.)

Theorem 4.1. *Let $f : X \to Y$, where $X = (X, \mathcal{T}_X)$ and $Y = (Y, \mathcal{T}_Y)$ are topological spaces. The following are equivalent:*

1. *f is continuous.*
2. *For every basis $\mathcal{B}$ of Y and every $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X.*
3. *For every subbasis $\mathcal{S}$ of Y and every $S \in \mathcal{S}$, the set $f^{-1}(S)$ is open in X.*
4. *For every closed set $F \subseteq Y$, the preimage $f^{-1}(F)$ is closed in X.*
5. *For every $x \in X$ and every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$.*
6. *For every $A \subseteq X$, one has $f(\overline{A}) \subseteq \overline{f(A)}$.*
7. *For every $B \subseteq Y$, one has $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.*

Proof. The equivalence of (1), (2), and (3) is the usual basis/subbasis criterion: every open set is a union of basis elements, and every basis element generated by a subbasis is a finite intersection of subbasic sets.

The equivalence of (1) and (4) is immediate from complements.

To see that (1) implies (5), let V be a neighborhood of $f(x)$. Choose an open set $O \subseteq V$ with $f(x) \in O$. Then $f^{-1}(O)$ is an open neighborhood of x, and $f\big(f^{-1}(O)\big) \subseteq O \subseteq V$. Conversely, assume (5), and let $O \subseteq Y$ be open. For each $x \in f^{-1}(O)$, apply (5) with $V = O$ to obtain a neighborhood U_x of x such that $f(U_x) \subseteq O$. Then $U_x \subseteq f^{-1}(O)$, so $f^{-1}(O)$ is open. Thus (5) implies (1).

Assume (1), and let $x \in \overline{A}$. If W is any neighborhood of $f(x)$, then $f^{-1}(W)$ is a neighborhood of x, so $f^{-1}(W) \cap A \neq \varnothing$. Hence $W \cap f(A) \neq \varnothing$, which shows that $f(x) \in \overline{f(A)}$. Therefore (1) implies (6).

Assume (6), and let $B \subseteq Y$. Apply (6) to $A = f^{-1}(B)$:

$$f\Big(\overline{f^{-1}(B)}\Big) \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B}.$$

Taking preimages under f yields

$$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}),$$

so (6) implies (7).

Finally, if (7) holds and $F \subseteq Y$ is closed, then

$$\overline{f^{-1}(F)} \subseteq f^{-1}(\overline{F}) = f^{-1}(F).$$

Hence $f^{-1}(F)$ is closed, so (4) holds. Therefore all seven statements are equivalent. □

Lemma 4.2. *If $f : X \to Y$ is continuous and $A \subseteq X$, then the restriction $f|_A : A \to Y$ is continuous with respect to the subspace topology on A.*

Proof. Let $O \subseteq Y$ be open. Then

$$(f|_A)^{-1}(O) = A \cap f^{-1}(O),$$

which is open in the subspace topology on A because $f^{-1}(O)$ is open in X. □

Theorem 4.3 (Pasting Lemma)**.** *Let $X = A \cup B$, where either both A and B are open in X, or both are closed in X. Suppose $f : X \to Y$ satisfies that $f|_A$ and $f|_B$ are continuous and agree on $A \cap B$. Then f is continuous.*

Proof. The agreement on $A \cap B$ ensures that f is well defined. Let $O \subseteq Y$ be open. Then

$$f^{-1}(O) = \Big((f|_A)^{-1}(O)\Big) \cup \Big((f|_B)^{-1}(O)\Big).$$

If A and B are open, each term on the right is open in the corresponding subspace and therefore open in X; hence $f^{-1}(O)$ is open. If A and B are closed, let $F \subseteq Y$ be closed. Then

$$f^{-1}(F) = \Big((f|_A)^{-1}(F)\Big) \cup \Big((f|_B)^{-1}(F)\Big)$$

is a union of closed subsets of the closed sets A and B, hence is closed in X. Thus f is continuous in either case. □

Theorem 4.4. *Let X, Y, and Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

Proof. Let $O \subseteq Z$ be open. Since g is continuous, $g^{-1}(O)$ is open in Y. Since f is continuous, its preimage under f is open in X. But

$$(g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O)),$$

so $g \circ f$ is continuous. □

Definition 4.5. A family $\{A_i : i \in I\}$ of subsets of X is called *locally finite* if for every $x \in X$ there exists a neighborhood U_x of x that meets only finitely many of the sets A_i.

Lemma 4.6. *Let $\{A_i\}_{i\in I}$ be a locally finite family of subsets of X.*

1. *For every $B \subseteq X$, the family $\{A_i \cap B\}_{i\in I}$ is locally finite in the subspace B.*
2. *If each A_i is closed in X, then $\bigcup_{i\in I} A_i$ is closed in X.*

Proof. For (1), if $x \in B$, choose a neighborhood U_x of x in X meeting only finitely many A_i. Then $U_x \cap B$ is a neighborhood of x in B meeting only finitely many $A_i \cap B$.

For (2), let $x \notin \bigcup_{i\in I} A_i$. Choose a neighborhood U_x meeting only $A_{i_1}, \dots, A_{i_n}$. Since each A_{i_k} is closed and does not contain x, the set

$$V_x = U_x \cap \bigcap_{k=1}^{n} (X \setminus A_{i_k})$$

is an open neighborhood of x disjoint from every A_i. Thus every point of the complement has an open neighborhood contained in the complement, so the union is closed. □

Theorem 4.7. *Let $\{A_i\}_{i\in I}$ be a cover of X. Assume either:*

1. *each A_i is open, or*

2. *each A_i is closed and the family is locally finite.*

Then a set $B \subseteq X$ is open (respectively closed) in X if and only if each $B \cap A_i$ is open (respectively closed) in the subspace A_i.

Proof. If B is open (respectively closed) in X, then each $B \cap A_i$ is open (respectively closed) in the subspace topology on A_i.

Conversely, suppose each $B \cap A_i$ is open in A_i. If the A_i are open, then each $B \cap A_i$ is open in X, and hence

$$B = \bigcup_{i \in I} (B \cap A_i)$$

is open in X.

Now suppose each $B \cap A_i$ is closed in A_i, with the family $\{A_i\}$ locally finite and closed. Then each $B \cap A_i$ is closed in X, and by theorem 4.6 the family $\{B \cap A_i\}$ is locally finite. Therefore

$$B = \bigcup_{i \in I} (B \cap A_i)$$

is closed in X. □

Theorem 4.8. *Let $\{A_\lambda\}_{\lambda \in I}$ be a cover of a topological space X such that either all A_λ are open, or all A_λ are closed and the family is locally finite. Suppose $f_\lambda : A_\lambda \to Y$ is continuous for each $\lambda \in I$, and that the maps agree on overlaps:*

$$f_\lambda|_{A_\lambda \cap A_\mu} = f_\mu|_{A_\lambda \cap A_\mu} \qquad (\lambda, \mu \in I).$$

Then there exists a unique continuous map $f : X \to Y$ such that $f|_{A_\lambda} = f_\lambda$ for every λ.

Proof. Define $f(x) = f_\lambda(x)$ whenever $x \in A_\lambda$. This is well defined because the maps agree on overlaps, and uniqueness is immediate from the definition.

Let $O \subseteq Y$ be open. For each λ,

$$f^{-1}(O) \cap A_\lambda = f_\lambda^{-1}(O),$$

which is open in the subspace A_λ because f_λ is continuous. By theorem 4.7, it follows that $f^{-1}(O)$ is open in X. Hence f is continuous. □

Theorem 4.9. *Let $f : X \to Y$ be a bijection. The following are equivalent:*

1. *f is a homeomorphism.*

2. *f is continuous and open.*

3. *f is continuous and closed.*

4. *For every $A \subseteq X$, one has $f(\overline{A}) = \overline{f(A)}$.*

Proof. Clearly (1) implies both (2) and (3), because if f^{-1} is continuous then images of open (respectively closed) sets under f are open (respectively closed). Conversely, if f is continuous and open, then for every open set $O \subseteq X$ we have

$$(f^{-1})^{-1}(O) = f(O),$$

which is open in Y; hence f^{-1} is continuous. This proves (1) $\Leftrightarrow$ (2). The proof of (1) $\Leftrightarrow$ (3) is identical, using closed sets instead of open sets.

Assume (1). Since continuous maps satisfy $f(\overline{A}) \subseteq \overline{f(A)}$, it remains to prove the reverse inclusion. Apply the same inclusion to the continuous map f^{-1} and the set $f(A)$:

$$f^{-1}(\overline{f(A)}) \subseteq \overline{f^{-1}(f(A))} = \overline{A}.$$

Applying f gives $\overline{f(A)} \subseteq f(\overline{A})$, and therefore $f(\overline{A}) = \overline{f(A)}$.

Finally, assume (4). If $F \subseteq X$ is closed, then

$$f(F) = f(\overline{F}) = \overline{f(F)},$$

so $f(F)$ is closed in Y. Thus f is closed. Since (3) implies (1), we conclude that (4) also implies (1). Hence all four statements are equivalent. □

Theorem 4.10. *Let $f : X \to Y$. The following are equivalent:*

1. *f is an open map.*

2. *For every $A \subseteq X$, one has $f(A^\circ) \subseteq (f(A))^\circ$.*

3. *If $\mathcal{B}$ is a basis for X, then $f(B)$ is open in Y for every $B \in \mathcal{B}$.*

4. *For every $x \in X$ and every neighborhood N_x of x, there exists a neighborhood W of $f(x)$ such that $W \subseteq f(N_x)$.*

Proof. (1) $\Rightarrow$ (2): Since A° is open and $A^\circ \subseteq A$, the set $f(A^\circ)$ is open and contained in $f(A)$, hence $f(A^\circ) \subseteq (f(A))^\circ$.

(2) $\Rightarrow$ (3): If $B \in \mathcal{B}$, then $B = B^\circ$, so

$$f(B) = f(B^\circ) \subseteq (f(B))^\circ \subseteq f(B).$$

Thus $f(B) = (f(B))^\circ$ is open.

(3) $\Rightarrow$ (4): Let N_x be a neighborhood of x. Choose $B \in \mathcal{B}$ with $x \in B \subseteq N_x$. Then $f(B)$ is an open neighborhood of $f(x)$ contained in $f(N_x)$.

(4) $\Rightarrow$ (1): Let $O \subseteq X$ be open and let $y \in f(O)$. Choose $x \in O$ with $f(x) = y$. Applying (4) to the neighborhood O of x, we obtain a neighborhood W_y of y such that $W_y \subseteq f(O)$. Therefore

$$f(O) = \bigcup_{y \in f(O)} W_y,$$

so $f(O)$ is open.

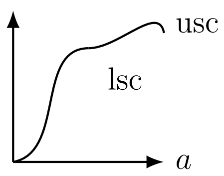
For maps into $\mathbb{R}$, it is often enough to test inverse images of rays. Indeed,

$$(a,b) = (-\infty, b) \cap (a, \infty), \qquad (-\infty, b) = \bigcup_{n=1}^{\infty} (-n, b), \qquad (a, \infty) = \bigcup_{n=1}^{\infty} (a, n).$$

Hence a map $f : X \to \mathbb{R}$ is continuous if and only if the sets $\{x : f(x) < b\}$ and $\{x : f(x) > a\}$ are open for all $a, b \in \mathbb{R}$. This leads naturally to lower and upper semicontinuity. □

Definition 4.11. $f : X \to \mathbb{R}$ is said to be *lower semicontinuous* if $\{x : f(x) > a\}$ is open for each $a \in \mathbb{R}$. Similarly, *upper semicontinuous* if $\{x : f(x) < a\}$ is open for each $a \in \mathbb{R}$.

Hence, f is continuous if and only if f is both lower semicontinuous (lsc) and upper semicontinuous (usc).



Example 4.12. Show that $f : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous if and only if

$$f(x) \le \liminf_{n\to\infty} f(x_n)$$

for all $x_n \to x$.

Example 4.13. Show that $f : \mathbb{R} \to \mathbb{R}$ is upper semicontinuous if and only if

$$\limsup_{n\to\infty} f(x_n) \le f(x), \quad \text{for every } x_n \to x.$$

Notice that it follows from the above two characterizations that f is continuous if and only if for all $x_n \to x$

$$\Rightarrow \lim_{n\to\infty} f(x_n) = f(x).$$

Also,

$$\{x : f(x) > a\} = \{x : -f(x) < -a\}$$

Hence f is continuous if and only if it is both lower semicontinuous and upper semicontinuous.

The following theorem is not the simplest, rather generalizes to other situations (for example, measure theory, measurable functions, etc.)

Theorem 4.14. *Let $f, g : X \to \mathbb{R}$ be continuous. Then:*

1. *$|f|$ is continuous;*
2. *$af + bg$ is continuous for all $a, b \in \mathbb{R}$;*
3. *fg is continuous;*
4. *if $g(x) \neq 0$ for all $x \in X$, then f/g is continuous (in particular, $1/f$ is continuous whenever f is nowhere zero).*

Proof. The maps $t \mapsto |t|$, $(s,t) \mapsto as + bt$, $(s,t) \mapsto st$, and $t \mapsto 1/t$ on $\mathbb{R} \setminus \{0\}$ are continuous. Therefore each assertion follows by composing these continuous maps with f and g. If one prefers explicit formulas, note that

$$fg = \frac{1}{4}\Big((f+g)^2 - (f-g)^2\Big), \qquad \frac{f}{g} = f \cdot \frac{1}{g}.$$

This proves the result. □

Example 4.15. Let $f \in C'([0,1])$, and define

$$L(f) = \int_0^1 \sqrt{1 + (f'(t))^2}\, dt$$

is lower semicontinuous on $C'([0,1])$ to $\mathbb{R}$.

Example 4.16. Let $\{f_\alpha\}_{\alpha \in I}$ be a family of lower semicontinuous functions on X, for which, for each $x \in X$, the set $\{f_\alpha(x) : \alpha \in I\}$ has an upper bound.
Let

$$g(x) = \sup\{f_\alpha(x) : \alpha \in I\}.$$

Then g is lower semicontinuous.
Note that $g(x) > a$ if and only if there exists at least one α such that $f_\alpha(x) > a$. Hence,

$$\{x : g(x) > a\} = \bigcup_{\alpha \in I} \{x : f_\alpha(x) > a\}$$

so g is lower semicontinuous.

Example 4.17. A similar result is true for upper semicontinuous functions.

Example 4.18. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous from the right, that is,

$$\lim_{x \to a^+} f(x) = f(a), \quad \text{for every } a \in \mathbb{R}.$$

Show that f is continuous, when considered as a function

$$f : (\mathbb{R}, \tau_r) \to (\mathbb{R}, \nu),$$

where τ_r is the topology of right open intervals and ν is the usual topology.

For a given $\varepsilon > 0$, there exists $\delta > 0$ such that for $x \in (a, a + \delta)$, we have $|f(x) - f(a)| < \varepsilon$.
$(*)$ i.e. $f\Big((a, a + \delta)\Big) \subset (f(a) - \varepsilon, f(a) + \varepsilon)$;
i.e., for every neighborhood of $f(a)$ in $(\mathbb{R}, \nu)$, there exists a neighborhood $(a, a + \delta)$ contained in it such that $(*)$ holds.

Example 4.19. Let $f : X \to Y$ be a map. The following are equivalent.

(i) f is continuous on X.

(ii) $f(A^\circ) \subset \Big(f(A)\Big)^\circ$ for all $A \subset X$.

(iii) $\partial f^{-1}(B) \subset f^{-1}\Big(\partial B\Big)$ for all $B \subset Y$.

Proof. (i) $\Rightarrow$ (ii):
$f(A \cap A^\circ) \subset f(A^\circ) \subset (f(A))^\circ$ ($\subseteq$ possibly strict) and previous result about the closure of continuous functions.

(ii) $\Rightarrow$ (iii):

Let $x \in \partial\Big(f^{-1}(B)\Big)$. Then $x \in f^{-1}(B) \cap \Big(X \setminus f^{-1}(B)\Big)'$.

(i) If $x \notin f^{-1}(B)$, then $x \in (X \setminus f^{-1}(B))'$. By (ii),

$$f(x) \in f\left(X \setminus f^{-1}(B)\right) \subset Y \setminus B$$

so $f(x) \notin B$ and so $x \notin f^{-1}(\overline{B})$.

If $x \notin f^{-1}(\overline{B})$ then $x \in X \setminus f^{-1}(\overline{B})$.
But $x \in X \setminus f^{-1}(B) \Rightarrow f(x) \in Y \setminus B$.
By (ii), $f(x) \in f(f^{-1}(B))' \subset f(f^{-1}(B))' \subset Y \setminus B$.
So $f(x) \in \overline{B}$, hence $x \in f^{-1}(\overline{B})$.
Therefore,

$$\partial\Big(f^{-1}(B)\Big) \subset f^{-1}\Big(\partial B\Big).$$

(iii) $\Rightarrow$ (i):
Let $O \subset X$ be open. Then $\partial\Big(f^{-1}(Y \setminus O)\Big) \subset f^{-1}\Big(\partial(Y \setminus O)\Big)$.
We use the fact that A is closed if and only if for every $A \subset C$, $A = \overline{A}$. This implies that $\partial(Y \setminus O) \subset Y \setminus O$.
Thus,

$$\partial\left(f^{-1}(Y \setminus O)\right) \subset f^{-1}\Big(\partial(Y \setminus O)\Big) \subset f^{-1}(Y \setminus O) = X \setminus f^{-1}(O).$$

Thus $X \setminus f^{-1}(O)$ is closed, and hence $f^{-1}(O)$ is open.

□

Example 4.20. Let $f, g : X \to \mathbb{R}$ be continuous.

(a) Show that $A = \{x : f(x) \leq g(x)\}$ is closed.

Hint:

$$A^c = \{x : f(x) - g(x) > 0\}.$$

But $f - g$ is continuous, so it is lower semicontinuous, which implies A^c is open.

(b) Let $h : X \to \mathbb{R}$ be defined by

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous. We can re-write:

$$h(x) = \chi_B(x)f(x) + \chi_D(x)g(x)$$

where χ_B and χ_D are characteristic functions of

$$B = \{x : f(x) \leq g(x)\},$$

and $D = \{x : f(x) > g(x)\}$ respectively.

If $x \in B \cap D$, then $f(x) = g(x)$.
Since B and D are closed by pasting lemma, f is unique.

$$K : X \times \mathbb{R} \longrightarrow \mathbb{R}$$

which is nothing but the identity due to uniqueness.

Exercises

Exercise 4.21. Let $f : X \to Y$ be a map between topological spaces. Prove that f is continuous if and only if $f^{-1}(C)$ is closed in X whenever C is closed in Y.

Exercise 4.22. Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Prove that $g \circ f$ is continuous.

Exercise 4.23. Let $f : X \to Y$ be a bijection. Prove that f is a homeomorphism if and only if f is continuous and open (equivalently, continuous and closed).

Exercise 4.24 (Pasting lemma)**.** Let $X = A \cup B$ where A, B are closed in X. If $f : A \to Y$ and $g : B \to Y$ are continuous and agree on $A \cap B$, show that the induced map $h : X \to Y$ is continuous.

Exercise 4.25. Give an example of a continuous bijection $f : X \to Y$ that is not a homeomorphism. (Hint: compare different topologies on the same set.)

Chapter 5

Convergence: sequences, nets, and filters

In metric spaces, many topological notions can be phrased using sequences. In general topological spaces, however, sequences may fail to detect closure and continuity. The correct replacement is the language of nets *(and, equivalently,* filters*). We develop these notions and record the standard characterizations: closure via nets, continuity via nets, and compactness via ultrafilters. We also explain how the countability axioms recover the sequential picture.*

5.1 Why sequences are not enough

In an arbitrary topological space, the set of limits of sequences from $A \subseteq X$ may be strictly smaller than the closure $\overline{A}$. This motivates the introduction of generalized sequences indexed by directed sets.

Definition 5.1 (Directed set)**.** A *directed set* is a nonempty set D equipped with a preorder $\preceq$ such that for any $d_1, d_2 \in D$ there exists $d \in D$ with $d_1 \preceq d$ and $d_2 \preceq d$.

Definition 5.2 (Net)**.** Let X be a set and D a directed set. A *net* in X is a function $x_\bullet : D \to X$, written $d \mapsto x_d$.

Definition 5.3 (Convergence of a net)**.** Let X be a topological space and $x_\bullet = (x_d)_{d \in D}$ a net in X. We say $x_d \to x$ (or $x_\bullet \to x$) if for every neighborhood U of x there exists $d_0 \in D$ such that $x_d \in U$ for all $d \succeq d_0$.

Remark 5.4. Every sequence $(x_n)_{n \in \mathbb{N}}$ is a net by taking $D = \mathbb{N}$ with the usual order. Thus nets extend sequences without altering the familiar convergence notion.

5.2 Closure and cluster points via nets

Theorem 5.5 (Closure via nets). *Let X be a topological space, $A \subseteq X$, and $x \in X$. Then $x \in \overline{A}$ if and only if there exists a net $(a_d)_{d \in D}$ in A such that $a_d \to x$.*

Proof. If $a_d \to x$ with $a_d \in A$, then every neighborhood U of x contains some a_d, hence intersects A; this is exactly $x \in \overline{A}$.

Conversely, assume $x \in \overline{A}$. Let D be the set of all neighborhoods of x, directed by reverse inclusion: $U \preceq V$ if and only if $U \supseteq V$. For each $U \in D$ choose $a_U \in A \cap U$ (possible since $U \cap A \neq \varnothing$). Then $(a_U)_{U \in D}$ is a net in A, and by construction $a_U \to x$. □

Definition 5.6 (Cluster point of a net). Let $x_\bullet = (x_d)_{d \in D}$ be a net in X. A point $x \in X$ is a *cluster point* of $x_\bullet$ if for every neighborhood U of x and every $d_0 \in D$ there exists $d \succeq d_0$ such that $x_d \in U$.

Definition 5.7 (Subnet). Let $x_\bullet : D \to X$ be a net. A *subnet* is a net $x_{\phi(\bullet)} : E \to X$ of the form $e \mapsto x_{\phi(e)}$, where E is a directed set and $\phi : E \to D$ is cofinal: for every $d_0 \in D$ there exists $e_0 \in E$ such that $\phi(e) \succeq d_0$ for all $e \succeq e_0$.

Theorem 5.8 (Subnets and cluster points). *A point $x \in X$ is a cluster point of a net $x_\bullet$ if and only if some subnet of $x_\bullet$ converges to x.*

Proof. ($\Rightarrow$) Standard diagonal/cofinal construction: consider the directed set of pairs (d, U) where $d \in D$ and U is a neighborhood of x, ordered by $(d_1, U_1) \preceq (d_2, U_2)$ if and only if $d_1 \preceq d_2$ and $U_1 \supseteq U_2$. Using the cluster property, pick $d' \succeq d$ with $x_{d'} \in U$ and define $\phi(d, U) = d'$. Then the induced subnet converges to x.

(is implied by) If a subnet converges to x, then by definition x is a cluster point of the original net. □

5.3 Continuity in terms of nets

Theorem 5.9 (Continuity preserves limits of nets). *Let $f : X \to Y$ be a map between topological spaces. Then f is continuous if and only if for every net $x_\bullet$ in X and every $x \in X$,*

$$x_\bullet \to x \quad \Rightarrow \quad f(x_\bullet) \to f(x).$$

Proof. If f is continuous and $x_\bullet \to x$, let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x, so eventually $x_d \in f^{-1}(V)$, hence eventually $f(x_d) \in V$.

Conversely, assume the net condition. To prove continuity, let $U \subseteq Y$ be open and take $x \in f^{-1}(U)$. If $f^{-1}(U)$ were not a neighborhood of x, then $x \in \overline{X \setminus f^{-1}(U)}$.

By Theorem 5.5 there exists a net $x_\bullet$ in $X \setminus f^{-1}(U)$ with $x_\bullet \to x$. Then $f(x_\bullet) \to f(x) \in U$, so eventually $f(x_\bullet) \in U$, contradicting $x_\bullet \subseteq X \setminus f^{-1}(U)$. Thus $f^{-1}(U)$ is a neighborhood of each of its points, hence open. □

5.4 Filters and ultrafilters (optional but standard)

Definition 5.10 (Filter)**.** Let X be a set. A *filter* on X is a nonempty family $\mathcal{F} \subseteq \mathcal{P}(X)$ such that:

(i) $\varnothing \notin \mathcal{F}$;

(ii) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;

(iii) if $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.

A filter $\mathcal{U}$ is an *ultrafilter* if it is maximal among filters with respect to inclusion.

Definition 5.11 (Filter convergence)**.** Let X be a topological space, $x \in X$, and $\mathcal{F}$ a filter on X. We write $\mathcal{F} \to x$ if every neighborhood of x belongs to $\mathcal{F}$.

Theorem 5.12 (Compactness via ultrafilters)**.** *A topological space X is compact if and only if every ultrafilter on X converges to at least one point of X.*

Proof. ($\Rightarrow$) Let $\mathcal{U}$ be an ultrafilter. If $\mathcal{U}$ does not converge, then for each $x \in X$ there exists a neighborhood $U_x \ni x$ with $U_x \notin \mathcal{U}$. By maximality, $X \setminus U_x \in \mathcal{U}$. The open cover $\{U_x : x \in X\}$ has a finite subcover $U_{x_1}, \dots, U_{x_n}$. Then $\bigcap_{k=1}^{n}(X \setminus U_{x_k}) = \varnothing$ belongs to $\mathcal{U}$, a contradiction.

(is implied by) Suppose X is not compact. Then some open cover has no finite subcover. Equivalently, the family of complements of finite unions of members of the cover has the finite intersection property. Extend this family to an ultrafilter $\mathcal{U}$. By assumption $\mathcal{U} \to x$ for some $x \in X$; then some member of the cover is a neighborhood of x and must lie in $\mathcal{U}$, contradicting the construction. □

5.5 Countability and the sequential viewpoint

The first countability axiom guarantees that nets can be replaced by sequences in many arguments.

Proposition 5.13. *Let X be first countable, $A \subseteq X$, and $x \in X$. Then $x \in \overline{A}$ if and only if there exists a sequence (a_n) in A with $a_n \to x$.*

Proof. The forward implication follows by choosing, for a countable local base $(U_n)_{n\in\mathbb{N}}$ at x, points $a_n \in A \cap U_n$ and noting that $(a_n) \to x$. The reverse implication holds in every space: if $a_n \to x$ with $a_n \in A$, then every neighborhood of x meets A. □

Exercises

Exercise 5.14. Give an example of a topological space X and a subset $A \subseteq X$ such that $\overline{A}$ is strictly larger than the set of limits of sequences in A. (One classical example uses the first uncountable ordinal with the order topology.)

Exercise 5.15. Let $f : X \to Y$ be a map. Prove Theorem 5.9 directly from the open-set definition of continuity.

Exercise 5.16. Show that a space X is compact if and only if every net in X has a convergent subnet.

Chapter 6

Subspace, product, and quotient topologies

We discuss the three central ways to build new spaces from old ones: subspaces, products, and quotients. For products we emphasize the role of projections and contrast the product topology with the box topology in infinite products. For quotients we highlight the universal property and the role of quotient maps in constructing identification spaces.

6.1 Finite products: the product topology on $X \times Y$

Alongside the subspace construction, the product construction is one of the two most basic mechanisms for producing new topological spaces from old ones. The guiding principle is that the coordinate projections should be continuous and that neighborhoods in the product should record information from finitely many coordinates at a time. For two spaces, this leads to the familiar basis of open rectangles.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces, and let

$$\mathcal{B} = \mathcal{B}(X \times Y) = \{O \times W : O \in \mathcal{T}_X,\, W \in \mathcal{T}_Y\}.$$

We can see that $\mathcal{B}$ is a basis on $X \times Y$:

(i) If $(x, y) \in X \times Y \Rightarrow x \in O$, $y \in W$

$$\Rightarrow (x, y) \in O \times W.$$

(ii)
$$(O_1 \times W_1) \cap (O_2 \times W_2) = (O_1 \cap O_2) \times (W_1 \cap W_2) \in \mathcal{B}.$$

$$\Rightarrow \mathcal{B} \text{ is a basis.}$$

The topology generated by $\mathcal{B}$ on $X \times Y$ is called the *product topology* on $X \times Y$.

It is sometimes useful to express product topology in terms of sub-basis.

$$\begin{cases} \pi_1 : X \times Y \longrightarrow X, & \pi_1(x, y) = x \\ \pi_2 : X \times Y \longrightarrow Y, & \pi_2(x, y) = y \end{cases}$$

are called projections of $X \times Y$ onto X and Y, respectively.

Notice that the maps are onto unless one of X or Y is empty. In that case, $X \times Y$ is empty.

Observe that
$$\pi_1^{-1}(O) = O \times Y, \quad \text{for every } O \in \mathcal{T}_X,$$
and
$$\pi_2^{-1}(W) = X \times W, \quad \text{for every } W \in \mathcal{T}_Y.$$

Also,
$$\pi_1^{-1}(O) \cap \pi_2^{-1}(W) = O \times W.$$

Theorem 6.1. *Let*

$$\mathcal{J} = \left\{\pi_1^{-1}(O) : O \in \mathcal{T}_X\right\} \cup \left\{\pi_2^{-1}(W) : W \in \mathcal{T}_Y\right\}.$$

Then $\mathcal{J}$ is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$, and let $\mathcal{T}'$ be the topology generated by the subbasis $\mathcal{J}$. Since each member of $\mathcal{J}$ is open in the product topology, we immediately obtain $\mathcal{T}' \subseteq \mathcal{T}$.

Conversely, every basic open rectangle in the product topology has the form

$$O \times W = \pi_1^{-1}(O) \cap \pi_2^{-1}(W), \qquad O \in \mathcal{T}_X,\ W \in \mathcal{T}_Y.$$

Thus each basis rectangle is a finite intersection of members of $\mathcal{J}$ and therefore belongs to $\mathcal{T}'$. Since the product topology is generated by such rectangles, it follows that $\mathcal{T} \subseteq \mathcal{T}'$. Hence $\mathcal{T} = \mathcal{T}'$, proving that $\mathcal{J}$ is indeed a subbasis.

□

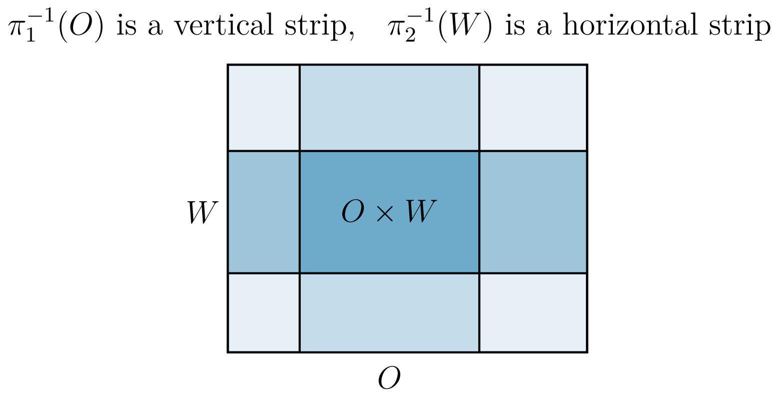


Figure 6.1: A basis rectangle $O \times W$ as the intersection of two subbasic sets.

Example 6.2. Show that projections are open maps, sending open sets to open sets.

6.2 Products, box topology, and the universal property

Return to product topology

We know that product topology of topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ can be defined through a basis

$$\mathcal{B} = \{O_X \times O_Y : O_X \in \mathcal{T}_X,\ O_Y \in \mathcal{T}_Y\} \subset \mathcal{T}_X \times \mathcal{T}_Y.$$

Also, we can generate the same topology on $X \times Y$ via projections
let $\pi_1 : X \times Y \to X,\ \pi_1((x, y)) = x$
and $\pi_2 : X \times Y \to Y,\ \pi_2((x, y)) = y$.
Then

$$\mathcal{J} = \{\pi_1^{-1}(O_X),\ \pi_2^{-1}(O_Y) : O_X \in \mathcal{T}_X,\ O_Y \in \mathcal{T}_Y\}$$

defines a subbasis for the product topology.

Note that $\pi_1^{-1}(O_Y) = O_X \times Y$ and $\pi_2^{-1}(O_Y) = X \times O_Y$ and this implies that

$$\pi_1^{-1}(O_X) \cap \pi_2^{-1}(O_Y) = O_X \times O_Y \in \mathcal{B}$$

Hence, $\mathcal{J}$ and $\mathcal{B}$ generate the same topology on $X \times Y$.

The above conclusion is true for finite products of topological spaces.

Let

$$X = X_1 \times X_2 \times \cdots \times X_n = \prod_{i=1}^{n} X_i.$$

Then

$$\mathcal{B}_n = \left\{ \prod_{i=1}^{n} O_i : O_i \subset \mathcal{T}_i \right\}$$

forms a basis for a topology on X, and

$$\bigcap_{j=1}^{n} \pi_j^{-1}(O_j) = \prod_{i=1}^{n} O_i \in \mathcal{B}_n.$$

Let

$$\mathcal{J}_n = \left\{ \pi_i^{-1}(O_i) : i = 1, 2, \ldots, n;\ O_i \in \mathcal{T}_i \right\}$$

Then $\mathcal{B}_n$ and $\mathcal{J}_n$ generate the same topology on X.

However, in general, the topology generated by $\mathcal{B}_n$ is called the box topology on $\prod_{i \in I} X_i$, whereas the topology produced by $\mathcal{J}_n$ is known as the product topology on $\prod_{i \in I} X_i$.

Let $X = \prod_{i \in I} X_i$, and

$$\mathcal{J} = \left\{ \pi_i^{-1}(O_i) : i \in I \right\}$$

Then $\mathcal{J}$ is a sub-basis for some topology on X, and it can be realized through basis

$$\mathcal{B}_p = \left\{ \bigcap_{i \in F}^{\text{finite}} \pi_i^{-1}(O_i) : F \subset I \text{ finite} \right\}$$

that is, all such intersections of (finite) n-tuples in I.

Note that a typical member of $\mathcal{B}_p$ looks like

$$B = \prod_{i \in I} O_i,$$

where all but finitely many $O_i = X_i$.

Now, let

$$\mathcal{B}_b = \left\{ \prod_{i \in I} O_i : O_i \in \mathcal{T}_i \right\}$$

Then $\mathcal{B}_p \subset \mathcal{B}_b$, thus box topology on $\prod_{i \in I} X_i$ is finer than product topology on $\prod_{i \in I} X_i$.

Notice that a typical point of $\prod_{i \in I} X_i$ is represented by

$$x = (x_i)_{i \in I} \quad \text{rather } x_i \in X_i.$$

Once again, we know that

$$\mathcal{T}(\mathcal{B}_p) \subset \mathcal{T}(\mathcal{B}_b)$$

Hence, a function f continuous on $(X, \mathcal{T}(\mathcal{B}_b))$ need not be continuous on $(X, \mathcal{T}(\mathcal{B}_p))$.

Note that box topology is not as interesting as the product topology, on which we try to generalize finite dimensional results. Besides this, there is the clear distinction between product topology and box topology as long as continuity and many more, to be discussed in the due course of time.

Theorem 6.3. *Let $f : A \to \prod_{\alpha \in I} X_\alpha$ be given by $f(a) = (f_\alpha(a))_{\alpha \in I}$, where $f_\alpha : A \to X_\alpha$. Let $\prod X_\alpha$ be given the product topology. Then f is continuous if and only if each f_α is continuous.*

Proof. Suppose f is continuous on A.

Notice that each projection π_α is continuous. For this, let $Q_\alpha \in \mathcal{U}_\alpha$. The set $\pi_\alpha^{-1}(Q_\alpha)$ is a subbasis element for the product topology on $\prod X_\alpha$. Hence $f^{-1}(\pi_\alpha^{-1}(Q_\alpha))$ is open in A.

Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by

$$f(t) = (t, t, t, \ldots).$$

Then each coordinate function $f_n(t) = t$ is continuous, so f is continuous in the product topology on $\mathbb{R}^\omega$. However, f is not continuous when $\mathbb{R}^\omega$ is endowed with box topology. For this, let us consider a basis open set

$$B = (-1, 1) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times \cdots \times (-\tfrac{1}{n}, \tfrac{1}{n}) \times \cdots$$

for the box topology on $\mathbb{R}^\omega$. Then $f^{-1}(B)$ is not open in $\mathbb{R}$.

If $f^{-1}(B)$ is open in $\mathbb{R}$, then as $0 \in f^{-1}(B)$, there exists a small interval (open) in $f^{-1}(B)$, i.e.,

$$(-\varepsilon, \varepsilon) \subset f^{-1}(B).$$

Therefore,

$$A_\varepsilon \subset B$$

where

$$A_\varepsilon = f((-\varepsilon, \varepsilon)).$$

Thus,

$$\pi_n(A_\varepsilon) \subset \pi_n(B)$$

which gives

$$(-\varepsilon,\varepsilon) \subset (-\tfrac{1}{n},\tfrac{1}{n}), \quad \text{for all } n \in \mathbb{N}.$$

Which is not possible for large n.

□

Example 6.4. Let $(a_1, a_2, \ldots), (b_1, b_2, \ldots) \in \mathbb{R}^\omega$ and define

$$h : \mathbb{R}^\omega \to \mathbb{R}^\omega$$

by

$$h(x_1, x_2, \ldots) = (a_1x_1 + b_1, a_2x_2 + b_2, \ldots).$$

Then show that h is a homeomorphism on $\mathbb{R}^\omega$ onto $\mathbb{R}^\omega$ with product topology on $\mathbb{R}^\omega$.

Proof. The nth coordinate of $h(x_1, x_2, \ldots)$ is $a_nx_n + b_n$. Therefore, each projection h_n is continuous, hence h is continuous.

Also,

$$h^{-1}(y_1, y_2, \ldots) = \left(\frac{y_1 - b_1}{a_1}, \frac{y_2 - b_2}{a_2}, \ldots\right)$$

so h^{-1} is continuous.

Question. What if $\mathbb{R}^\omega$ is given box topology?

(Hint: it is enough to show that

$$I : \mathbb{R}^\omega \to \mathbb{R}^\omega, \quad I(x_1, x_2, \ldots) = (x_1, x_2, \ldots)$$

is a homeomorphism with respect to box topology or not.)

□

Theorem 6.5. *: Let $\{X_\alpha\}_{\alpha\in I}$ be a family of topological spaces, and let $A_\alpha \subset X_\alpha$. If $\prod X_\alpha$ is given either the product topology or the box topology, then*

$$\overline{\prod A_\alpha} = \prod \overline{A_\alpha}.$$

Proof. Let $x = (\alpha_\lambda)_{\lambda\in I} \in \prod \overline{A_\alpha}$, and $O = \bigcap_{\lambda\in I} O_\lambda$ be a basis element for either product topology or box topology containing x. Since $x_\lambda \in \overline{A_\lambda}$, choose $x_\lambda \in O_\lambda \cap A_\lambda$. Then $y = (y_\lambda) \in O$ and $y \in \prod A_\alpha$, i.e. $y \in O \cap \prod A_\alpha \neq \emptyset$. $\Rightarrow x \in \overline{\prod A_\alpha}$.

On the other hand, let $x = (x_\beta) \in \overline{\prod A_\alpha}$ (in either topology). Let O_β be an open set containing x_β. Since $\pi_\beta^{-1}(O_\beta)$ is open in either topology on $\prod X_\alpha$, it must contain a point $y = (y_\lambda)_{\lambda\in I} \in \prod A_\alpha$, so $y_\beta \in O_\beta \cap A_\beta$, $\Rightarrow x_\beta \in \overline{A_\beta}$ for all β.

(Notice that in either topology,

$$\pi_\beta^{-1}(O_\beta) = \prod_{i\in I} O_i,$$

where $O_i = X_i$ if $i \neq \beta$ and $O_\beta = O_\beta$.)

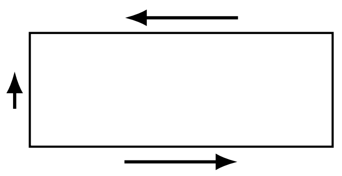
Quotient Spaces:

It is a way to construct complicated objects out of simple ones via a cut-paste technique.

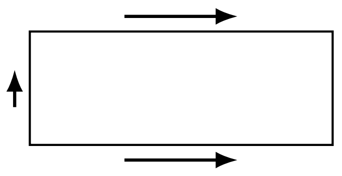
For example, a circle can be obtained by identifying two points of the real line as one, which differs by an integer (i.e. $x \sim y$ if $x - y \in \mathbb{Z}$).

- Rectangle to cylinder, by identifying two opposite sides.
- Torus, by identifying two ends of cylinder.
- Möbius strip, by identifying two a pair of opposite sides in opposite direction.

- Klein bottle, by identifying one pair of opposite sides in the same direction and the other in the opposite direction.



- Projective plane, by identifying each pair of sides in the opposite direction.



- Sphere, by identifying boundary of rectangle as one point. There are other prominent examples too.

The above basic examples, and many others, can be summarized into two methods for topologizing sets that play an increasingly important role in the topology.

The first uses a map of a space into a set to topologize the set, which makes precise numerous constructions as mentioned some of them. The second constructs a space by "pasting" given spaces together along pre-assigned subsets.

Quotient map:

Let X be a topological space and Y be a set.

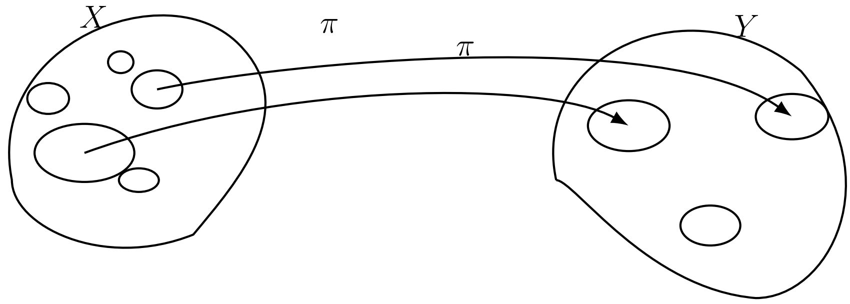
Let $\pi : X \to Y$ be a surjective map.

We construct a topology $\mathcal{T}_Y$ on Y by:

$$O \in \mathcal{T}_Y \text{ if and only if } \pi^{-1}(O) \in \mathcal{T}_X.$$

So,

$$\mathcal{T}_Y = \{O \subseteq Y : \pi^{-1}(O) \in \mathcal{T}_X\}?$$



(Eventually, identifying multiple open sets in X to one open set in Y.)

Notice that π has stronger properties than a simple continuity. Because if $f : X \to Y$ is continuous, there may be set $A \subseteq Y$ (could be not open), but $f^{-1}(A)$ is open.

Example: If $f : X \to Y$ is a continuous and open map, then f is a quotient map.

If $O \in \mathcal{T}_Y$, then $f^{-1}(O) \in \mathcal{T}_X$.

If $f^{-1}(O) = V$ for some set $O \subseteq Y$, then $f(V) = O$, and f is open, hence O is open.

Note that in the above discussion "open set" could be replaced with "closed set".

However, there are quotient maps which are neither open nor closed.

Example: $X = [0,1] \cup [2,3]$, $Y = [0,2]$,

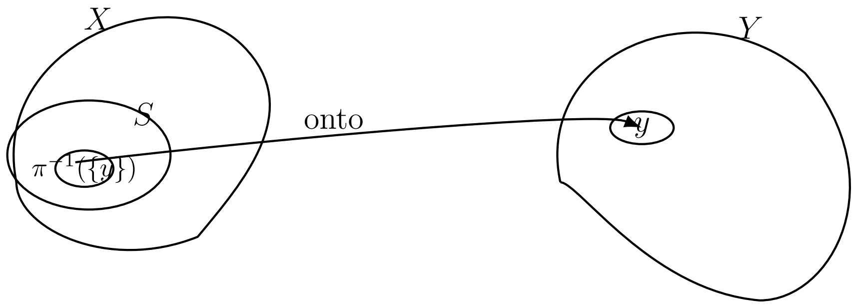
$$\pi : X \to Y \text{ by } \pi(x) = \begin{cases} x, & x \in [0,1] \\ x-1, & x \in [2,3] \end{cases}$$

π is a quotient map, but *not open* as $\pi([0,1]) = [0,1]$ is not open in Y (it is also not closed with same argument).

Note that a map being continuous and open is not too sufficient for it to be a quotient map, and a smaller restriction to spaces of maps can produce quotient maps.

□

Definition 6.6. Let $\pi : X \to Y$ be a surjective map. A set $S \subset X$ is said to be saturated if for any $y \in Y$ with $\pi^{-1}(\{y\}) \cap S \neq \emptyset \Rightarrow \pi^{-1}(\{y\}) \subseteq S$.



Since $\pi^{-1}(Y) = X \Rightarrow X = \pi^{-1}(T_1 \cup \pi^{-1}(T))$ for some $T \subseteq Y$. Thus, $S = \pi^{-1}(T)$, i.e., where we can assume that $\pi^{-1}(T) \cap S \neq \emptyset$ only. That is, S is a saturated subset of X with respect to π if and only if $S = \pi^{-1}(T)$ for some $T \subseteq Y$.

Proposition 6.7. *If $\pi : X \to Y$ is continuous, then π is a quotient map if and only if π is continuous and sends saturated open sets to open sets.*

Proof: Suppose π is a quotient map. Then we need to show that π sends saturated open sets to open sets. Let $S \subseteq X$ be open and saturated. Then $S = \pi^{-1}(T)$ for some $T \subseteq Y$. Since π is a quotient map, T must be open, that is, $\pi^{-1}(S) = T$.

On the other hand, suppose that π is continuous and sends saturated open sets to open sets. Let $V = \pi^{-1}(O)$ open. Then V is saturated and $\pi(V) = O$ is open.

Example 6.8. Let $X = [0,1] \cup [2,3]$, $\quad Y = [0,2]$.
Let $\beta : X \to Y$ by

$$\beta(x) = \begin{cases} x & \text{if } x \in [0,1], \\ x-1 & \text{if } x \in [2,3]. \end{cases}$$

Then β is a quotient map, but if
$A = [0,1) \cup [2,3]$ and
$g : A \to Y$, where $g = \beta|_A$

then β is continuous and onto but not a quotient map, because $[2,3]$ is open in A and saturated (i.e., $g^{-1}g([2,3]) = [2,3]$ is open in A and saturated), hence open, but $g([2,3]) = [1,2]$ is not open in Y.

Example 6.9. Let $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\pi(x,y) = x$.
Then π is open, continuous, and surjective; hence π is a quotient map. But
π is not a closed map, as
$\pi\left(\{(x,y) : xy = 1\}\right) = \mathbb{R} \setminus \{0\}$ is not closed.

Now, let $A = \{(x,y) : xy = 1\} \cup \{(0,0)\}$.
Then $g : A \to \mathbb{R}$, where $g = \pi|_A$, is continuous and surjective but not a quotient map.
Because $\{(0,0)\}$ is open and saturated in A, but $g(\{(0,0)\}) = \{0\}$ is not open in $\mathbb{R}$.

Now, we use quotient maps to construct a topology on a set.
Notice that if $A \subset X$ and
$f : X \to A$ is surjective, then there is exactly one topology on A relative to which f is a quotient map. The topology induced on A is called the *quotient topology*.
i.e. $\Sigma = \{O \subset A : f^{-1}(O) \text{ is open in } X\}$ is a topology.

Example 6.10. Let $A = \{a,b,c\}$ and $f : \mathbb{R} \to A$, where

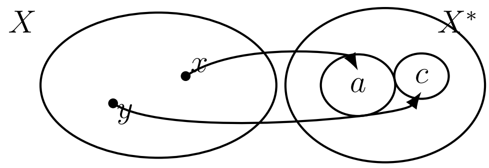
$$f(x) = \begin{cases} a & \text{if } x > 0, \\ b & \text{if } x < 0, \\ c & \text{if } x = 0. \end{cases}$$

Then the quotient topology on A generated by f is

$$\Sigma_A = \{\varnothing, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}.$$

There is one special situation in which quotient topologies occur frequently.

Let X be a topological space and X^* be a partition of X into disjoint subsets, i.e. $X = \bigcup_i X_i$.
Let $\varphi : X \to X^*$ be a surjective map which carries each point x in X to the partitioning set $X_i \in X^*$ containing x.



Note that $X \to X^*$ is an equivalence relation on X^*; for any x, y, $x, y \in X_i$ for exactly one i.

Topology of X^*.
A subset $O \subset X^*$ is said to be open if $f^{-1}(O) =$ union of equivalence classes $\subset X$ is open.
This topology is denoted by $\tau(f)$.
Thus, $(X^*, \tau(f))$ is a quotient space.

Now, we try to find a relationship between notions of quotient map and quotient space.

We know that subspaces do not behave well, as restrictions of quotient maps need not be a quotient map.

However, one has the following result.

Theorem 6.11. *Let $p : X \to Y$ be a quotient map and $A \subset X$ be saturated with respect to p. Let $q : A \to p(A)$, where $q = p|_A$.*

(i) If A is open then q is a quotient map.

(ii) If p is open then q is a quotient map.

Proof. We verify first the following two equations:

(i) $q^{-1}(V) = p^{-1}(V)$, for every $V \subset p(A)$;

(ii) $p(U \cap A) = p(U) \cap p(A)$, for every $U \subset X$.

□

Proof of (i): Let $V \subset p(A)$. Let $y \in q^{-1}(V) \Rightarrow y \in A$ and $q(y) \in V \Rightarrow p(y) \in V$. Thus, $q^{-1}(V) \subset p^{-1}(V) \cap A$.

But A is saturated, so $p^{-1}(p(A)) = A$. Therefore, $p^{-1}(V) \subset A$ for all $V \subset p(A)$, so $p^{-1}(V) = q^{-1}(V)$.

Since $p(A) = q(A)$.

Let $x \in p^{-1}(V) \subset A$, then $p(x) \in V \subset p(A) = q(A)$, so $q(a) = p(a)$. Thus, $x \in q^{-1}(V)$.

Therefore, $q^{-1}(V) = p^{-1}(V)$.

Also, if $q(a) \in V \subset p(A) = q(A)$, then $x \in A$, so $q(a) = p(a) \in V$.

Thus, $q(a) \in V \subset p(A) = q(A)$ implies $x \in A$. Therefore, $q(a) = p(a) \in V$, so $x \in p^{-1}(V)$.

□

Proof of (ii): For $U, A \subset X$,

$$p(U \cap A) \subset p(U) \cap p(A)$$

For the reverse inclusion, let $y = p(u) = p(a)$, $u \in U, a \in A$. Since A is saturated and $a \in p^{-1}(p(a)) \subset A$, so $p^{-1}(p(a)) \subset A$, so $p(u) = p(a) \Rightarrow u \in A$. Thus, $y = p(u)$, $u \in U \cap A$.

Now, suppose A is open and p is open.

Let $q : A \to p(A)$.

Let $V' \subset p(A)$ and assume $q^{-1}(V')$ is open in A. We claim V' is open in $p(A)$.

(a) First case: If A is open and $q^{-1}(V')$ is open in A, it follows that $q^{-1}(V')$ is open in X.

Since $q^{-1}(V') = p^{-1}(V')$ is open in X, then V' is open in $p(A)$.

Now, suppose p is open, and $q^{-1}(V')$ is open in A. Then, $q^{-1}(V') = p^{-1}(V')$, so $p^{-1}(V')$ is open in X. Hence, it follows

$$p(p^{-1}(V')) = V' \cap p(A)$$

for some open set V in X.

Now, $p(p^{-1}(V')) = V'$, since p is onto. But then,

$$V' = p(U \cap A) = p(U) \cap p(A)$$

Hence, $p(U)$ is open, as p is an open map. Therefore, V' is open in $p(A)$.

□

Corollary 6.12. *The above result (theorem 6.11) can be imitated when p is closed and A is closed.*

Example 6.13. Composition of two quotient maps is a quotient map, because let $p_1 : Z \to X$, $p_2 : X \to Y$, quotient maps, then

$$p_2(p_1^{-1}(V)) = (p_2 \circ p_1)^{-1}(V).$$

However, product of two quotient maps need not be a quotient map. If both of them are open, then product is open, continuous and onto. Hence, quotient map.

One of the most important results in the study of quotient spaces is to construct continuous functions on a quotient space.

Theorem 6.14. *Let $\pi : X \to Y$ be a quotient map and $g : X \to Z$ be a map constant on each $\pi^{-1}(y)$ for $y \in Y$. Then g induces a map*

$$f : Y \to Z$$

such that $f \circ \pi = g$.

(i) f is continuous if and only if g is continuous.

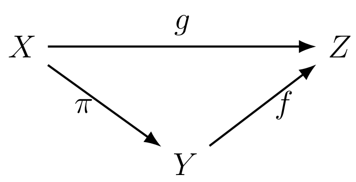
(ii) f is a quotient map if and only if g is a quotient map.

Proof. Since g is constant on fiber $\pi^{-1}(y)$, it implies

$$g\left(\pi^{-1}(\{y\})\right) = \text{singleton in } Z = \{g(x)\}.$$

We can define a map

$$f : Y \to Z \quad \text{via} \quad f(\pi(x)) = g(x) \Rightarrow f \circ \pi = g.$$



(i) If f is continuous, then g is continuous.

Conversely, let g be continuous and $V \subset Z$ open. Then $g^{-1}(V)$ is open in X. But

$$g^{-1}(V) = \pi^{-1}(f^{-1}(V));$$

since π is a quotient map, $f^{-1}(V)$ has to be open. $\Rightarrow f$ is continuous.

(ii) If f is a quotient map, then composition will be so. Hence g is a quotient map.

Conversely, suppose g is a quotient map. Since g is surjective, f is surjective.

Let $V \subset Z$. We claim that $f^{-1}(V)$ is open $\Rightarrow V$ is open.

Now, set $\pi^{-1}(f^{-1}(V)) = g^{-1}(V) =$ open in X as π is continuous.

But $\pi^{-1}(f^{-1}(V)) = g^{-1}(V)$ is open $\Rightarrow V$ is open as g is quotient.

Thus, f is a quotient map.

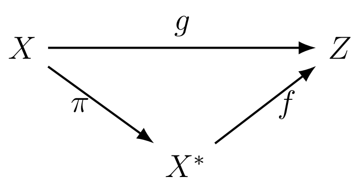
□

Corollary 6.15. *Let* $g : X \xrightarrow{\text{onto, cont}} Z$*, and*

$$X^* = \{g^{-1}(\{z\}) : z \in Z\}.$$

Give X^* *the quotient topology generated by* $g : X \to X^*$*. Then*

(i) g *induces a continuous bijection* $f : X^* \to Z$*, which is a homeomorphism if* g *is a quotient map.*



(ii) If Z *is Hausdorff, then so is* X^**.*

Proof. By previous theorem (theorem 6.14), g induces a map $f : X^* \to Z$, clearly f is one-to-one and onto, because $X^* = \{g^{-1}(z) : z \in Z\}$.

Suppose f is a homeomorphism. Then both f and π are quotient maps, and hence their composition $f \circ \pi = g$ is a quotient map.

Conversely, suppose g is a quotient map, then by previous theorem (theorem 6.14) f is a quotient map, and f is a bijection, so f is a homeomorphism.

(ii) Suppose Z is a Hausdorff space. Given distinct points of X^*, x_1, x_2,

$$f(x_1) \neq f(x_2) \text{ in } Z,$$

and hence there exist open disjoint sets U and V such that

$$f(x_1) \in U \quad \text{and} \quad f(x_2) \in V.$$

But $U \cap V = \varnothing \Rightarrow f^{-1}(U) \cap f^{-1}(V) = \varnothing$.

Thus $f(U)$ and $f(V)$ are disjoint and so are X_1 and X_2.

□

Remark 6.16. Some of properties, such as compactness, connectedness, and path-connectedness can pass to quotient space, but most of the other properties need not be passed onto quotient space. In fact, quotient spaces are simply not very tractable. It is important to consider several basic examples of quotient spaces rather than thinking of general theory about quotient spaces.

Exercises

Exercise 6.17. Let $X = \prod_{\lambda \in I} X_\lambda$ with the product topology. Show that each coordinate projection $\pi_\lambda : X \to X_\lambda$ is continuous, and that the product topology is the smallest topology making all π_λ continuous.

Exercise 6.18. Show that a subbasis for the product topology on $\prod_{\lambda \in I} X_\lambda$ is given by sets of the form $\pi_\lambda^{-1}(U)$ where $U \subseteq X_\lambda$ is open.

Exercise 6.19. Give an explicit example where the box topology on $\mathbb{R}^{\mathbb{N}}$ is strictly finer than the product topology.

Exercise 6.20. Let $q : X \to Y$ be a surjection. Prove that the quotient topology on Y is the largest topology making q continuous.

Exercise 6.21. Let $q : X \to Y$ be a quotient map and $f : Y \to Z$ be a map. Prove that f is continuous if and only if $f \circ q : X \to Z$ is continuous.

Exercise 6.22. Let X be a topological space and $A \subseteq X$. Show that collapsing A to a point produces a quotient space, and describe a neighborhood basis at the collapsed point.

Chapter 7

Countability and separation axioms

We study two families of structural hypotheses. The countability axioms (first and second countability, separability, Lindelöfness) control the size and complexity of the topology. The separation axioms (T_0, T_1, Hausdorff, regularity, normality) control how well points and closed sets can be separated by open neighborhoods.

7.1 Separation axioms

Definition 7.1. A topological space X is said to satisfy:

(i) T_0 if for any distinct $x, y \in X$ there exists an open set containing one of x, y but not the other;

(ii) T_1 if every singleton $\{x\}$ is closed (equivalently, for any distinct x, y there exists an open set containing x but not y);

(iii) T_2 (or *Hausdorff*) if for any distinct $x, y \in X$ there exist disjoint open sets $U \ni x$ and $V \ni y$.

Remark 7.2. In these notes we reserve the symbols T_0, T_1, T_2 for the separation axioms. Earlier examples of distinct *topologies* on a finite set are denoted by τ.

Definition 7.3. A T_1 space X is *regular* if for every closed set $F \subset X$ and every point $x \notin F$, there exist disjoint open sets U, V with $x \in U$ and $F \subset V$. A space X is *normal* if for any two disjoint closed sets $F, G \subset X$ there exist disjoint open sets U, V with $F \subset U$ and $G \subset V$.

Proposition 7.4. *Every metric space is Hausdorff and normal.*

Proof. Hausdorffness follows by taking disjoint open balls around distinct points. Normality follows by separating disjoint closed sets using the distance function; one standard argument uses the open sets

$$U = \{x \in X : d(x,F) < d(x,G)\}, \qquad V = \{x \in X : d(x,G) < d(x,F)\}.$$

These are open, disjoint, and contain F and G, respectively. □

Proposition 7.5. *A topological space X is Hausdorff if and only if the diagonal*

$$\Delta_X = \{(x,x) : x \in X\}$$

is closed in the product space $X \times X$.

Proof. Assume first that X is Hausdorff and let $(x,y) \in (X \times X) \setminus \Delta_X$. Then $x \neq y$, so there exist disjoint open sets $U \ni x$ and $V \ni y$. Hence $U \times V$ is an open neighborhood of (x,y) disjoint from Δ_X. Therefore $(X \times X) \setminus \Delta_X$ is open, so Δ_X is closed.

Conversely, assume Δ_X is closed. Let $x \neq y$. Then $(x,y) \notin \Delta_X$, so because $(X \times X) \setminus \Delta_X$ is open, there exist open sets $U \ni x$ and $V \ni y$ such that

$$(x,y) \in U \times V \subseteq (X \times X) \setminus \Delta_X.$$

If $U \cap V \neq \varnothing$, choose $z \in U \cap V$. Then $(z,z) \in U \times V \subseteq (X \times X) \setminus \Delta_X$, a contradiction. Thus $U \cap V = \varnothing$, and X is Hausdorff. □

Corollary 7.6. *Let X and Y be topological spaces, with Y Hausdorff, and let $f : X \to Y$ be continuous. Then the graph*

$$G_f = \{(x, f(x)) : x \in X\} \subseteq X \times Y$$

is closed.

Proof. Consider the map $\Phi : X \times Y \to Y \times Y$ given by $\Phi(x,y) = (f(x), y)$. This map is continuous. Since Y is Hausdorff, Δ_Y is closed in $Y \times Y$ by theorem 7.5. But

$$G_f = \Phi^{-1}(\Delta_Y),$$

hence G_f is closed in $X \times Y$. □

Proposition 7.7. *If X is compact and Hausdorff, then X is normal.*

Proof. Let $F, G \subset X$ be disjoint closed sets. For each $x \in F$ and $y \in G$, Hausdorffness yields disjoint open neighborhoods $U_{x,y} \ni x$ and $V_{x,y} \ni y$. Fix $x \in F$. Then $\{V_{x,y} : y \in G\}$ is an open cover of G, hence admits a finite subcover $V_{x,y_1}, \ldots, V_{x,y_n}$. Set $U_x = \bigcap_{k=1}^n U_{x,y_k}$ and $V_x = \bigcup_{k=1}^n V_{x,y_k}$. Then U_x is open, $x \in U_x$, and $U_x \cap V_x = \varnothing$ while $G \subset V_x$. Now $\{U_x : x \in F\}$ is an open cover of the compact set F; choose $x_1, \ldots, x_m$ with $F \subset \bigcup_{j=1}^m U_{x_j}$. Let $U = \bigcup_{j=1}^m U_{x_j}$ and $V = \bigcap_{j=1}^m V_{x_j}$. Then U, V are open, disjoint, $F \subset U$, and $G \subset V$, proving normality. □

7.2 Countability axioms

Definition 7.8. A space X is *first countable* if each $x \in X$ has a countable neighborhood basis. It is *second countable* if it has a countable base for the topology.

Remark 7.9. Second countability is a strong global restriction: it implies separability and the Lindelöf property (every open cover has a countable subcover), and it is a key hypothesis in metrization theorems.

In general, a basis is useful only if its sets are simple or few in number.

For instance, a space which has a countable basis has many pleasant properties, and such a space is known as second countable.

A central fact about second countable spaces is as follows:

Theorem 7.10. *Let $(X, \mathcal{T})$ be a second countable topological space. If a non-empty open set is represented by*

$$O = \bigcup_{i \in I} O_i, \quad O_i \in \mathcal{T},$$

then

$$O = \bigcup_{i=1}^{\infty} O_i$$

is a countable union.

This is known as Lindelöf's Theorem.

Proof. Let $\{B_n\}_{n=1}^{\infty}$ be a countable base for topology $\mathcal{T}$.

Let $x \in O$, then $x \in O_i$ for some $i \in I$.

Then by definition of base of topology, there exists B_n such that

$$x \in B_n \subseteq O_i \subseteq O$$

Thus,

$$O = \bigcup_{i=1}^{\infty} O_i$$

(We need as many O_i as there are B_n.)

□

Corollary 7.11. *Let X be a second countable topological space. Then any base can be reduced to a countable base.*

Proof. Let $\{B_n\}_{n=1}^{\infty}$ be a countable base for $(X, \mathcal{T})$, and $\{B_i : i \in I\}$ be a base for $(X, \mathcal{T})$.

Since each B_n is a union of B_i's,

$$B_n = \bigcup_{i \in J_n} B_i,$$

where J_n is a countable index set.

In this way, we obtain a countable family of countable unions of B_i's, and this family is a base for $\mathcal{T}$.

□

Corollary 7.12. *If topological space X has countable base $\{B_n\}$ then it has also a countable base subset.*

Proof. Let $\{x_n \in B_n : n = 1, 2, \ldots\} = A$.

Then A is countable. If $x \in X$, then there exists B_n such that $x \in B_n$ and $x_n \in B_n \cap A \neq \emptyset$.

Hence $\overline{A} = X$.

□

Definition 7.13. A topological space X is said to be *separable* if X has a countable dense set.

Example: (X, d), a metric space, if it is separable if there is a countable set $A \subseteq X$ such that

$$\bigcup_{x_i \in A} B_\epsilon(x_i) = X$$

for all $\epsilon > 0$.

For each $\epsilon > 0$, there exists $B_\epsilon(x_i)$ such that $X = \bigcup_i B_\epsilon(x_i)$.

(X, d) is separable if it is patched by countably many open balls of arbitrarily small radius.

Theorem 7.14. *Every separable metric space is second countable.*

Proof. Let A be a separable metric space, X, A a countable dense set in X.

Let
$$\mathcal{B} = \{B_{r_i}(x_j) : r_i \in \mathbb{Q}, x_j \in A\}$$

Claim: $\mathcal{B}$ is a base for (X, τ).

Let O be a non-empty open set and $x \in O$.

We need to find an open sphere $B_{r_i}(x_j)$ such that $x \in B_{r_i}(x_j) \subseteq O$.

Let $S_r(x) \subseteq O$. Since $\overline{A} = X$, there exists $a \in A$ such that

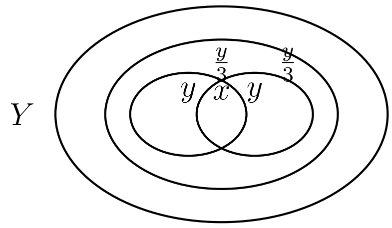
$$a \in S_{r/3}(x) \Rightarrow a \in S_r(x)$$

Let r_1 be a rational number such that

$$r/3 < r_1 < 2r/3.$$

Then $B_{r_1}(a) \subset S_\epsilon(x) \subset O$.

Thus $\mathcal{B}$ is a basis for $\mathcal{T}$, which is countable.



Question:

(i) Is it possible to find a sub-basis from a basis?

(ii) Is every separable topological space in some sense countable?

□

7.3 Baire category (a universal tool)

Definition 7.15 (Nowhere dense and meagre sets)**.** Let X be a topological space. A set $A \subseteq X$ is *nowhere dense* if $\text{Int}(\overline{A}) = \varnothing$. A set is *meagre* (or of *first category*) if it is a countable union of nowhere dense sets. A set is *comeagre* if its complement is meagre.

Theorem 7.16 (Baire category theorem)**.** *Every complete metric space is not meagre in itself. Equivalently, in a complete metric space X, the intersection of countably many dense open sets is dense.*

Proof. We prove the dense-intersection formulation. Let X be complete metric and let $G_1, G_2, \ldots$ be dense open subsets of X. Fix a nonempty open ball $B_0 \subseteq X$. Since G_1 is dense open, choose a nonempty open ball $B_1 \subseteq B_0 \cap G_1$ with radius $< 2^{-1}$. Inductively, having chosen B_{n-1}, pick a nonempty open ball $B_n \subseteq B_{n-1} \cap G_n$ with radius $< 2^{-n}$. Choose $x_n \in B_n$. Then (x_n) is Cauchy, hence converges to some $x \in X$ by completeness. Since the radii shrink to 0 and $B_n \subseteq B_{n-1}$, the limit x belongs to $\bigcap_{n\geq 1} \overline{B_n} \subseteq \bigcap_{n\geq 1} G_n$. In particular, $\bigcap_{n\geq 1} G_n$ meets B_0, so it is dense. □

Corollary 7.17. *In a complete metric space, every comeagre set contains a dense G_δ set (countable intersection of open sets), and every dense G_δ set is comeagre.*

Remark 7.18. Many spaces occurring in analysis are Baire spaces even without an explicit metric: for instance, every locally compact Hausdorff space is a Baire space. This principle frequently turns "countably many bad sets" into a "small" exceptional set.

Exercises

Exercise 7.19. Prove that second countability implies separability.

Exercise 7.20. Show that in a first countable space, sequential continuity (preserving limits of sequences) is equivalent to continuity.

Exercise 7.21. Prove that a subspace of a second countable space is second countable.

Exercise 7.22. Prove that every metric space is normal.

Exercise 7.23. Give an example of a T_1 space that is not Hausdorff.

Exercise 7.24. Show that if X is compact and Hausdorff, then it is normal.

Chapter 8

Connectedness

Connectedness formalizes the idea that a space cannot be decomposed into two disjoint nontrivial open pieces. We develop connectedness, components, and path connectedness, and briefly discuss local connectedness and totally disconnected spaces.

8.1 Connected spaces and components

Definition 8.1. A topological space X is *connected* if it cannot be written as $X = U \cup V$ where U, V are disjoint nonempty open sets. Such a decomposition (if it exists) is called a *separation* of X.

Proposition 8.2. *The following are equivalent for a space X:*

(i) X is connected.

(ii) The only subsets of X that are both open and closed are $\varnothing$ and X.

(iii) There does not exist a continuous surjection $f : X \to \{0, 1\}$ (with $\{0, 1\}$ discrete).

Proof. (i)⇒(ii): If $A \subset X$ is clopen and nontrivial, then $X = A \cup (X \setminus A)$ is a separation. (ii)⇒(iii): If f is a continuous surjection to a discrete two-point space, then $f^{-1}(\{0\})$ is nontrivial clopen. (iii)⇒(i): If $X = U \cup V$ is a separation, define $f(x) = 0$ on U and $f(x) = 1$ on V; this is continuous and surjective. □

Proposition 8.3. *The continuous image of a connected space is connected.*

Proof. Let $f : X \to Y$ be continuous and X connected. If $f(X) = U \cup V$ is a separation in Y, then $X = f^{-1}(U) \cup f^{-1}(V)$ is a separation of X, a contradiction. □

Proposition 8.4. *Let $\{C_i\}_{i\in I}$ be a family of connected subsets of X such that*

$$\bigcap_{i\in I} C_i \neq \varnothing.$$

Then $\bigcup_{i\in I} C_i$ is connected.

Proof. Suppose $\bigcup_{i\in I} C_i = U \cup V$ is a separation. Choose $p \in \bigcap_{i\in I} C_i$. Then $p \in U$ or $p \in V$; assume $p \in U$. Since each C_i is connected and meets U, it cannot meet V. Hence $C_i \subseteq U$ for every i, and therefore $\bigcup_{i\in I} C_i \subseteq U$, contradicting that $V \neq \varnothing$. □

Proposition 8.5. *If $A \subseteq X$ is connected, then its closure $\overline{A}$ is connected.*

Proof. Suppose $\overline{A} = U \cup V$ is a separation. Since $A \subseteq \overline{A}$, we have

$$A = (A \cap U) \cup (A \cap V),$$

where $A \cap U$ and $A \cap V$ are disjoint open subsets of A. Because A is connected, one of them must be empty; assume $A \cap V = \varnothing$. Then V is a nonempty open subset of $\overline{A}$ disjoint from A, contradicting the fact that every nonempty open subset of $\overline{A}$ meets A. Hence $\overline{A}$ is connected. □

Theorem 8.6. *Every interval in $\mathbb{R}$ is connected.*

Proof. Let $I \subseteq \mathbb{R}$ be an interval and suppose $I = U \cup V$ is a separation in the subspace topology. Choose $u \in U$ and $v \in V$ with $u < v$. Set

$$E = U \cap [u, v].$$

Then E is nonempty and bounded above by v, so $s = \sup E$ exists. Since I is an interval and $u, v \in I$, we have $[u, v] \subseteq I$, hence $s \in I$.

If $s \in U$, then because U is open in I, there exists $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \cap I \subseteq U$. By definition of supremum there exists $x \in E$ with $s - \varepsilon < x \leq s$, and then any point of I between x and $s + \varepsilon$ also lies in U. In particular one finds a point of E larger than s, a contradiction.

If $s \in V$, then because V is open in I, there exists $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \cap I \subseteq V$. Again by the definition of supremum, there exists $x \in E$ with $s - \varepsilon < x \leq s$. But then $x \in U \cap V$, impossible.

Both cases lead to contradictions. Therefore I is connected. □

Theorem 8.7. *A subset $A \subseteq \mathbb{R}$ is connected if and only if it is an interval.*

Proof. One implication is theorem 8.6. Conversely, suppose A is connected and let $a, b \in A$ with $a < b$. If there were $c \in (a, b) \setminus A$, then

$$A \cap (-\infty, c) \quad \text{and} \quad A \cap (c, \infty)$$

would be disjoint nonempty open subsets of A whose union is A, contradicting connectedness. Hence every point between a and b lies in A, so A is an interval. □

Corollary 8.8 (Intermediate value property)**.** *If X is connected and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ is an interval. In particular, if $a, b \in X$ and t lies between $f(a)$ and $f(b)$, there exists $x \in X$ such that $f(x) = t$.*

Proof. By theorem 8.3, $f(X)$ is connected, and by theorem 8.7 every connected subset of $\mathbb{R}$ is an interval. □

Definition 8.9. A *component* of a space X is a maximal connected subset of X.

Proposition 8.10. *Components form a partition of X, and each component is closed.*

Proof. By theorem 8.4, two connected sets with nonempty intersection have connected union, so maximality forces components to be either equal or disjoint; hence they partition X. If C is a component, then $\overline{C}$ is connected by theorem 8.5; maximality therefore gives $\overline{C} = C$, so C is closed. □

8.2 Path connectedness and local connectedness

Definition 8.11. A space X is *path connected* if for any $x, y \in X$ there exists a continuous map $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Proposition 8.12. *Path connected spaces are connected.*

Proof. If $X = U \cup V$ is a separation and $x \in U$, $y \in V$, any path γ from x to y yields a separation of $[0, 1]$ by $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$, contradicting connectedness of $[0, 1]$. □

Definition 8.13. A space X is *totally disconnected* if its connected components are singletons. It is *locally connected* if every point has a neighborhood basis consisting of connected open sets.

Example 8.14. The rational numbers $\mathbb{Q} \subset \mathbb{R}$ (with the subspace topology) are totally disconnected.

Proof. Given distinct $p < q \in \mathbb{Q}$, choose an irrational number $\alpha \in (p, q)$. Then

$$\mathbb{Q} \cap (-\infty, \alpha) \quad \text{and} \quad \mathbb{Q} \cap (\alpha, \infty)$$

form a separation of $\mathbb{Q}$ into disjoint nonempty open sets. □

Example 8.15. The Cantor set is compact, perfect, and totally disconnected.

Exercises

Exercise 8.16. Prove that the continuous image of a connected space is connected.

Exercise 8.17. Show that X is disconnected if and only if there exists a continuous surjection $f : X \to \{0, 1\}$ (with $\{0, 1\}$ discrete).

Exercise 8.18. Prove that if X is path connected then X is connected, and give an example of a connected space that is not path connected.

Exercise 8.19. Let $\{C_i\}_{i \in I}$ be a family of connected subsets of X with $\bigcap_i C_i \neq \varnothing$. Prove that $\bigcup_i C_i$ is connected.

Exercise 8.20. Show that the connected component of a point is closed, and that components form a partition of X.

Exercise 8.21. Prove that a space X is totally disconnected if and only if every connected subset of X is a singleton.

Chapter 9

Compactness and compactifications

Compactness is the topological analog of closed and bounded subsets in Euclidean spaces, and it controls finiteness phenomena such as the existence of convergent subsequences and finite subcovers. We develop basic permanence properties, the Tube Lemma and Tychonoff's theorem, and then discuss locally compact spaces and the one-point compactification.

9.1 Compact sets

Compactness is one of the central finiteness principles of topology. Finite spaces automatically enjoy many useful properties, and compactness is the correct topological substitute for that kind of finite behavior. In Euclidean spaces, the Heine–Borel theorem identifies compact sets with closed and bounded sets; in general topology, however, the defining feature is the finite subcover property for open covers. This formulation is robust, behaves well under continuous maps and products, and lies behind many of the deepest existence theorems in analysis and topology.

Definition 9.1. A topological space X is said to be *compact* if every open cover

$$\mathcal{G} = \{O_i : i \in I\}$$

has a finite subcover.

$$X = \bigcup_{i \in I} O_i \Rightarrow X = \bigcup_{j=1}^{n} O_{i_j}.$$

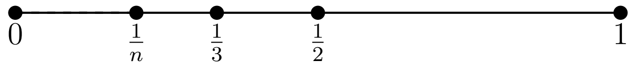
Example 9.2. $(\mathbb{R},\mathcal{U})$ is not compact because

$$\mathcal{G} = \{(n, n+2) : n \in \mathbb{Z}\}$$

has no finite subcover.

Example 9.3. $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset \mathbb{R}$.

$K =$



K is compact. For any cover, K has an open interval covering 0 which contains most of the points of K except finitely many.

Example 9.4. $A = (0, 1]$ is not compact, as the cover

$$\mathcal{G}_0 = \left\{\left(\frac{1}{n}, 1\right] : n \in \mathbb{N}\right\}$$

has no finite subcover of A.

Notice that A itself is not too large, but all can be stretched through homeo to $(-\infty, 1]$, however this is not possible for $[0, 1]$. Hence, we can guess that closedness of set is necessary for compactness.

In general, it takes a little effort to decide the compactness of a set.

We say $Y \subset X$ is covered by an open covering $\mathcal{G}$ in X if

$$Y \subset \bigcup \mathcal{G}.$$

Lemma 9.5. *Let Y be a subset of a topological space X. Then Y is compact if every cover of Y in X has a finite subcover of Y.*

Proof. Suppose Y is compact and $\mathcal{G} = \{O_i : i \in I\}$ be a cover of Y in X. Then

$$Y = \bigcup_{i \in I} (O_i \cap Y).$$

Hence Y is compact in its own right.

Conversely, suppose every open cover of Y in X has a finite subcover. Claim: Y is compact in its own right.

Let $Y = \bigcup_{i \in I} O_i'$, $O_i' \subset Y$. Then $O_i' = O_i \cap Y$ (by definition of subspace), so

$$Y \subset \bigcup_{i \in I} O_i$$

which gives $Y = \bigcup_{j=1}^n O'_{i_j}$. □

Theorem 9.6 (Closed subspaces of compact spaces)**.** *Every closed subspace of a compact space is compact.*

Proof. Let Y be a closed subspace of a compact space X. Let $\{U'_i : i \in I\}$ be an open cover of Y by sets open in Y. By definition of the subspace topology, for each i there exists an open set $U_i \subset X$ such that $U'_i = U_i \cap Y$. Then

$$X = (X \setminus Y) \ \cup \ \bigcup_{i \in I} U_i$$

is an open cover of X. Since X is compact, there exist indices $i_1, \dots, i_n$ with

$$X = (X \setminus Y) \ \cup \ \bigcup_{k=1}^n U_{i_k}.$$

Intersecting with Y gives

$$Y = \bigcup_{k=1}^n (U_{i_k} \cap Y) = \bigcup_{k=1}^n U'_{i_k},$$

so $\{U'_{i_k}\}_{k=1}^n$ is a finite subcover of Y. Hence Y is compact. □

Example 9.7. In indiscrete topological space $(X, \{\emptyset, X\})$, every set is compact. Does it imply every set is closed?

Theorem 9.8. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let Y be a compact subspace of a Hausdorff space X. Claim: $X \setminus Y$ is open in X.
For $x_0 \in X \setminus Y$, it suffices to show that there is a neighborhood of x_0 disjoint from Y.
For each y in Y, choose two disjoint open sets U_y and V_y of X containing x_0 and y respectively.

Then $Y \subseteq \bigcup_{y \in Y} V_y \Rightarrow Y \subseteq \bigcup_{i \in J} V_{y_i}$ for some finite J.
Let $V = \bigcup_{i \in J} V_{y_i}$ and $U = \bigcap_{i \in J} U_{y_i}$.
Then U is the desired open neighborhood.
If $z \in U$, then $z \in U_{y_i}$ for all i, and hence $z \in U_{y_i}$ and $z \notin V$, so $z \notin Y$.

Thus $Y \subseteq V$ and $Y \cap U = \varnothing$.

The above statement we note down for later use.

□

Lemma 9.9. *If Y is a compact subspace of a Hausdorff space X, then for all $x_0 \in X \setminus Y$, there exist disjoint open sets U and V of X containing x_0 and Y respectively.*

Example 9.10. Each K is compact in $\mathbb{R}$, because any closed subspace of a compact space is compact.

Example 9.11. $[a, b)$ and $(a, b]$ are not compact in $(\mathbb{R}, \text{usual})$ because they are not closed in the Hausdorff space $(\mathbb{R}, \text{usual})$.

Remark 9.12. We need the Hausdorff condition for the previous theorem (theorem 9.8) because in $(\mathbb{R}, \text{cofinite})$ topological spaces, the only proper sets which are closed are finite sets, but every subset in the cofinite topology is compact.
Let $A \subset \mathbb{R}$, $A \subset \bigcup A_i$.
Then $A_i = \mathbb{R} \setminus \{a_{i_1}, \ldots, a_{i_k}\}$. Choose $a_j \in A_j$.
Then $A \subset A_{i_1} \cup \cdots \cup A_{i_r}$.

Remark 9.13. $(\mathbb{R}, \text{co-countable})$ is not compact.
$\mathbb{R} = \bigcup_{i \in I} ((\mathbb{R} \setminus \mathbb{N}) \cup \{i\})$ has no finite subcover.

Theorem 9.14. *Continuous image of a compact set is compact.*

Proof. Let $f : X \to Y$ be continuous and X be a compact subset space.
Let $f(X) = \bigcup_{i \in I} V_i$, $V_i \in \mathcal{T}_Y$ (a cover).
Then $X = \bigcup_{i \in I} f^{-1}(V_i)$ (since $f^{-1}(V_i)$ is open in X, and f is continuous).
$\Rightarrow X = \bigcup_{l=1}^{n} f^{-1}(V_{i_l})$
$\Rightarrow f(X) = \bigcup_{l=1}^{n} V_{i_l}$

□

Theorem 9.15. *Let $f : X \to Y$ (Hausdorff) be continuous.*
(a) If X is compact, then f is a closed map.
(b) If X is compact and f is a bijection, then f is a homeomorphism.

Proof of (a). Let $F \subset X$ be closed, then F is compact $\Rightarrow f(F)$ is compact in Y.
$\Rightarrow f(F)$ is closed as Y is a T_2 space (Hausdorff).

(b) If f is a bijection, then

$$(f^{-1})^{\#}(F) = f(F) \text{ is closed}$$

$$\Rightarrow f^{-1} \text{ is continuous} \Rightarrow f \text{ is a homeomorphism}$$

□

Theorem 9.16. *The product of finitely many compact topological spaces is compact.*

This is based on the following assertion:

Let X be a topological space and Y be compact.
Let $x_0 \in X$ and N be an open set in $X \times Y$ such that $\{x_0\} \times Y \subset N \subset X \times Y$.
We claim that there is a neighborhood W of x_0 in X such that $W \times Y \subset N$.
(Note: the set $W \times Y$ is called a tube about $\{x_0\} \times Y$.)
We can cover $\{x_0\} \times Y$ by basic open sets lying in N:

$$\textit{for every } y \in Y, \textit{ there exists } U_y \times V_y,\ (x_0, y) \in U_y \times V_y \subset N.$$

Since $\{x_0\} \times Y \cong Y$, and Y is compact,

$$\{x_0\} \times Y \subset \bigcup_{i=1}^{m} U_i \times V_i$$

(We assume that each $V_i \times V_i$ intersects $\{x_0\} \times Y$, otherwise those open sets are superfluous and hence can be discarded.)

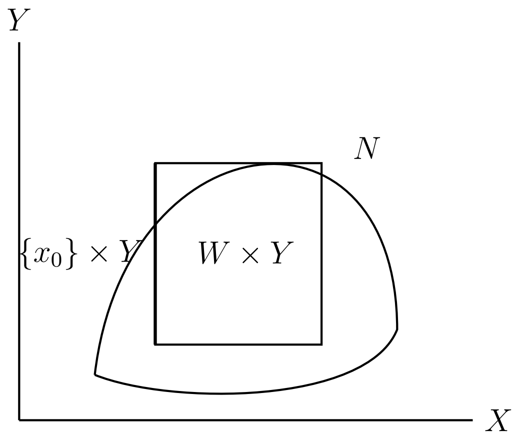
Let $W = \bigcap_{i=1}^{m} U_i$. Then $x_0 \in W$ (because $U_i \times V_i \ni (x_0, y_i) \neq \varnothing$).
We prove that

$$W \times Y \subset \bigcup_{i=1}^{m} U_i \times V_i$$

Let $(x, y) \in W \times Y$. Then $(x_0, y) \in \{x_0\} \times Y \subset \bigcup_{i=1}^{m} U_i \times V_i \subset N$.
Thus $(x_0, y) \in U_j \times V_j$ for some j with $j \in \{1, 2, \ldots, m\}$.
So, $y \in V_j$ and $x_0 \in U_j$, and j depends on y.
Thus, any y is in some V_j.
Therefore,

$$W \times Y \subset \bigcup_{i=1}^{m} U_i \times V_i \subset N.$$

If $\{x_0\} \times Y \subset N$, where N is open, then one can find a rectangle $W \times Y$ such that $\{x_0\} \times Y \subset W \times Y \subset N$.



Proof of the Theorem: Let X and Y be compact and

$$X \times Y = \bigcup_{i \in I} O_i, \quad O_i \text{ open in } X \times Y.$$

Then, for $x \in X$, $\{x\} \times Y \subset \bigcup_{i=1}^{m} O_i =: N_x$ (say).
Then, there exists a tube $W_x \times Y \subset N_x$.
Hence, each $W_x \times Y$ is covered by finitely many O_i's.
Also,

$$X = \bigcup_{\alpha \in X} W_\alpha \quad \text{(open cover)}$$

$$\Rightarrow X = \bigcup_{j=1}^{k} W_j$$

$$\Rightarrow \bigcup_{j=1}^{k} (W_j \times Y) = X \times Y$$

But $W_j \times Y \subset N_j = \bigcup_{i=1}^{m_j} O_i$

$$\Rightarrow X \times Y = \bigcup_{j=1}^{k} \bigcup_{i=1}^{m_j} O_i$$

Hence $X \times Y$ is compact.

□

Lemma 9.17 (Tube Lemma)**.** *Let X be a topological space, and Y be a compact space. If N is an open set in $X \times Y$ containing $\{x_0\} \times Y$ $(\subset N)$, then N contains a tube $W \times Y$ about $\{x_0\} \times Y$, where W is an open set of x_0 in X.*

Proof. For each $y \in Y$, since $(x_0, y) \in N$ and N is open in $X \times Y$, there exist open sets $U_y \subseteq X$ and $V_y \subseteq Y$ such that

$$(x_0, y) \in U_y \times V_y \subseteq N.$$

The family $\{V_y : y \in Y\}$ is an open cover of the compact space Y. Choose finitely many points $y_1, \dots, y_n \in Y$ such that

$$Y = \bigcup_{k=1}^{n} V_{y_k}.$$

Set

$$W = \bigcap_{k=1}^{n} U_{y_k}.$$

Then W is an open neighborhood of x_0. If $x \in W$ and $y \in Y$, choose k such that $y \in V_{y_k}$. Since $x \in U_{y_k}$, we have

$$(x, y) \in U_{y_k} \times V_{y_k} \subseteq N.$$

Hence $W \times Y \subseteq N$, as required. □

Proposition 9.18. *If Y is compact, then the projection*

$$p_X : X \times Y \to X, \qquad p_X(x, y) = x,$$

is a closed map.

Proof. Let $F \subseteq X \times Y$ be closed. We show that $X \setminus p_X(F)$ is open. Take $x_0 \in X \setminus p_X(F)$. Then

$$\{x_0\} \times Y \subseteq (X \times Y) \setminus F,$$

and the right-hand side is open. By the Tube Lemma, there exists an open neighborhood $W \ni x_0$ such that

$$W \times Y \subseteq (X \times Y) \setminus F.$$

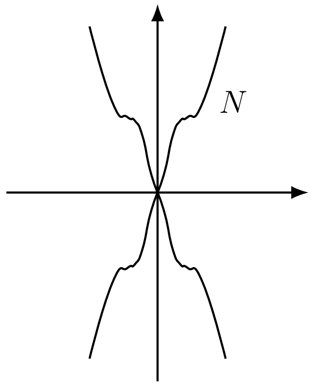
Thus no point of W lies in $p_X(F)$, so $W \subseteq X \setminus p_X(F)$. Hence $X \setminus p_X(F)$ is open and $p_X(F)$ is closed. □

Example 9.19. The lemma fails if Y is not compact.

Let $Y = y$-axis in $\mathbb{R}^2$

$$N' = \{(x, y) : |x| < f(|y|)\}$$

Then, N' is an open set containing $0 \times \mathbb{R}$, but it contains no tube around $0 \times \mathbb{R}$. We cannot look for a relative boundary of $0 \times \mathbb{R}$.



Next, we describe compactness through finite intersection property (FIP). This leads to a complete characterization of compact metric spaces too.

Definition 9.20. A collection $\mathcal{C} \subseteq \mathcal{P}(X)$ is said to satisfy the finite intersection property (FIP) if every finite subcollection of $\mathcal{C}$ has non-empty intersection.

Example 9.21. $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$; $\mathcal{C} = \{A_i\}$ has finite intersection property.

Theorem 9.22. *A topological space X is compact if and only if every collection $\mathcal{C}$ of closed sets in X having FIP has non-empty intersection.*

Proof. Let $\{F_\alpha\}_{\alpha \in A}$ be a family of closed subsets of X.

($\Rightarrow$) Assume X is compact and $\{F_\alpha\}$ has the finite intersection property. If $\bigcap_{\alpha \in A} F_\alpha = \varnothing$, then the open sets $U_\alpha := X \setminus F_\alpha$ form an open cover of X:

$$X = \bigcup_{\alpha \in A} U_\alpha.$$

By compactness, there exist $\alpha_1, \ldots, \alpha_n$ such that $X = \bigcup_{k=1}^n U_{\alpha_k}$, hence

$$\bigcap_{k=1}^n F_{\alpha_k} = X \setminus \bigcup_{k=1}^n U_{\alpha_k} = \varnothing,$$

contradicting the finite intersection property. Thus $\bigcap_{\alpha \in A} F_\alpha \neq \varnothing$.

(is implied by) Conversely, assume that every family of closed sets with the finite intersection property has nonempty intersection. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X. If it had no finite subcover, then for every finite subset $\{\alpha_1, \dots, \alpha_n\} \subset A$ we would have

$$X \neq \bigcup_{k=1}^{n} U_{\alpha_k},$$

so the closed sets $F_\alpha := X \setminus U_\alpha$ satisfy the finite intersection property:

$$\bigcap_{k=1}^{n} F_{\alpha_k} = X \setminus \bigcup_{k=1}^{n} U_{\alpha_k} \neq \varnothing.$$

By the hypothesis, $\bigcap_{\alpha \in A} F_\alpha \neq \varnothing$, which implies $\bigcup_{\alpha \in A} U_\alpha \neq X$, contradicting that $\{U_\alpha\}$ is a cover. Hence the cover admits a finite subcover, and X is compact. □

Example 9.23. Let $G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq \cdots$ be a sequence of closed sets in a compact space with $G_n \neq \emptyset$. Then

$$\bigcap G_n \neq \emptyset$$

Example 9.24. Show that every closed and bounded subset of $\mathbb{R}^2$ (and $\mathbb{R}^n$) is compact.

Example 9.25. $A = \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\}$ is closed but not bounded.

On $\mathbb{R}^2$, show that projection of closed set need not be closed. Same is true for bounded sets.

$\mathcal{B} = \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\}$ is bounded but not closed.

Theorem 9.26 (Extreme Value Theorem)**.** *Let X be a compact topological space.*

If $f : X \to \mathbb{R}$ is a continuous function, then f is bounded and attains its infimum and supremum.

That is,

$$\textit{there exists } c, d \in X \textit{ such that } f(c) \leq f(x) \leq f(d), \quad \textit{for every } x \in X.$$

Proof. Since f is continuous and X is compact, it implies that $f(X) = A$ is compact.

We show first that A has supremum and infimum.

If A has no largest element, then

Then $\{(-\infty, q_i) : a < q_i\}$ is an open cover of A. Since A is compact,

$$A \subset \bigcup_{i=1}^{n} (-\infty, q_i).$$

Let $q' = \min\{q_i : 1 \leq i \leq n\}$. Then $q' \notin A$, which is a contradiction. By similar argument, A has smallest element, i.e.,

$$m = \inf A = f(c_1), \quad M = \sup A = f(c_2),$$

where $c_1, c_2 \in X$.

Now, try to characterize compact sets in a metric space.
For $A \subset X$,

$$d(x, A) := \inf\{d(x, a) : a \in A\}$$

is uniformly continuous on X.

$$d(A) = \sup\{d(a_1, a_2) : a_1, a_2 \in A\}$$

is called *diameter* of A.

□

Definition 9.27. A number $\delta > 0$ is called a Lebesgue number for an open cover $\mathcal{A}$ of X if every subset $A \subset X$ with $d(A) < \delta$ implies $A \subset A_i$ for some $A_i \in \mathcal{A}$.

Theorem 9.28. *(Lebesgue Covering Lemma)*
If X is a compact metric space, then every open cover of X has a Lebesgue number.

Proof. Let $\mathcal{A}$ be an open cover of X. Then $X = \bigcup_{i \in I} A_i$, $\mathcal{A} = \{A_i : i \in I\}$.

If $X = A_i$ for some i, then every positive δ is a Lebesgue number for $\mathcal{A}$. So assume $X \neq A_i$ for all i.

Since X is compact,

$$X = \bigcup_{i=1}^{n} A_i.$$

Let $C_i = X \setminus A_i$, $i = 1, 2, \ldots, n$.

Define a function $f : X \to \mathbb{R}$ by average of $d(x, C_i)$.

i.e.

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i).$$

Then $f(x) > 0$. For this, let $x \in X$, then $x \in A_i$ — A_i open $\Rightarrow$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset A_i$.

$$\Rightarrow d(x, C_i) \geq \varepsilon \Rightarrow f(x) \geq \frac{\varepsilon}{n} > 0.$$

Since f is continuous on compact space X, f has an infimum. Say $\delta > 0$.

We claim δ is the required Lebesgue number for $\mathcal{A}$.

Let $B \subset X$, $d(B) < \delta$.

Choose a point $x_0 \in B$.

Then $B \subset B(x_0, \delta)$.

Now,

$$\delta \leq f(x_0) \leq d(x_0, C_m)$$

where $d(x_0, C_m) = \min_i d(x_0, C_i)$.

Then $B(x_0, \delta) \subset A_m = X \setminus C_m$, i.e., $B \subset A_m$ for some m.

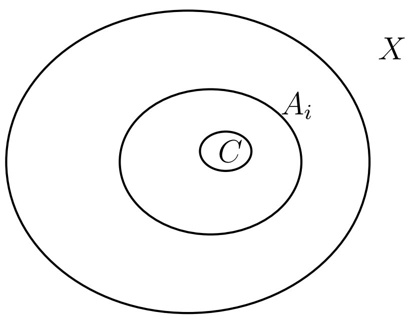
□

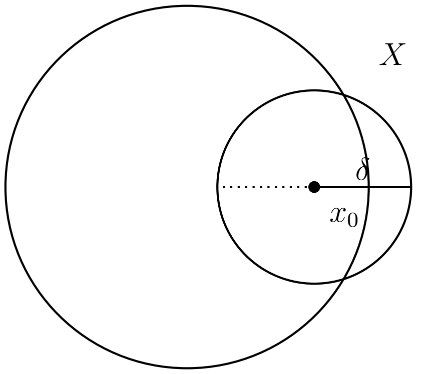
Definition 9.29. A function $f : (X, d_X) \to (Y, d_Y)$ is said to be *uniformly continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all pairs of points $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, it follows that

$$d_Y\Big(f(x_1), f(x_2)\Big) < \varepsilon.$$

Theorem 9.30. *Every continuous function on a compact metric space is uniformly continuous.*

Proof. Let X be a compact metric space and $f : (X, d_X) \to (Y, d_Y)$ be a continuous map.





For $\epsilon > 0$, let $Y = \bigcup_{y \in Y} B(y, \epsilon/2)$.
Let $\mathcal{A} = \{f^{-1}(B(Y, \epsilon/2)) : Y \subset Y\}$.
Then $\mathcal{A}$ is a cover of X. Hence, $\mathcal{A}$ has a Lebesgue number, say δ.
Let $\{x_1, x_2\} \subset X$ and $d_X(x_1, x_2) < \delta$.
Then $\{x_1, x_2\}$ has diameter $< \delta$.
So $\{f(x_1), f(x_2)\} \subset B(y, \epsilon/2)$ for some $y \in Y$.
$\Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon$.
Hence, f is uniformly continuous.

□

Definition 9.31. A point $x \in X$ (topological space) is said to be isolated if $\{x\}$ is open in X.

Theorem 9.32. *Let X be a non-empty compact Hausdorff space. If X has no isolated point, then X is uncountable.*

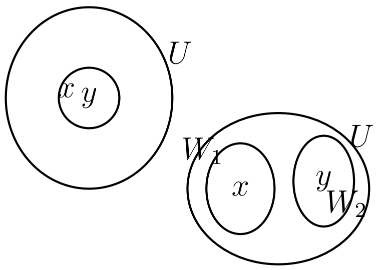
Proof. First, we show that for a given non-empty open set U and $x \in X$, there is an open set $V \subset U$ such that $x \notin \overline{V}$.

Choose $y \in U$ such that $y \neq x$. This is possible even if $x \in U$ since x is not an isolated point.

Since the space is T_2-space, there exists two disjoint open sets W_1 and W_2 such that $x \in W_1$ and $y \in W_2$.

Let $V = W_2 \cap U \subset U$.

Then $x \notin V$. If $x \in \overline{V}$, then $x \in V \cap W_1 = \varnothing$.



We complete the proof by showing that any function $f : \mathbb{N} \to X$ is not surjective. Let $x_n = f(n)$. Then for $x \in X = U$, there exists open set V' such that $x \notin \overline{V'}$. In general, for $V_{n_1} \neq \varnothing$ (open), choose $V_n \subset V_{n_1}$ such that $x_n \notin \overline{V_n}$.

$$\overline{V_1} \supset \overline{V_2} \supset \cdots \supset \overline{V_n} \supset \overline{V_{n+1}} \supset \cdots$$

Since X is compact, $\bigcap_n \overline{V_n} \neq \varnothing$.
Then $x \neq x_n$ for any n, since $x \in \bigcap \overline{V_n}$, but $x_n \notin \overline{V_n}$.
So, $x \neq f(n)$ for all $n \in \mathbb{N}$.
Hence f is not onto. Thus, X is uncountable.

□

Corollary 9.33. *Every closed interval in $\mathbb{R}$ is uncountable and hence (a, b) is uncountable.*

Example. Let $\mathbb{R}^K$ be endowed with K-topology, where K has basis

$$\mathcal{B} = \{C(a, S) : a \in \mathbb{R}, S \subset K, C(a, S) : c \in \mathbb{R}\},$$

where $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

Show that $[0, 1]$ is not compact in $\mathbb{R}_K$.

Definition 9.34. A topological space X is said to be limit point compact (LPC) if every infinite subset of X has a limit point.

Theorem 9.35. *Compactness $\Rightarrow$ limit point compactness, but converse need not be true.*

Proof. To claim every infinite set has a limit point, it is enough to show that if a set has no limit point, then it is finite.

Suppose $A \subset X$ has no limit point. Then A is closed. Further, for each $\alpha \in A$, there exists U_α such that

$$U_\alpha \cap A = \{\alpha\}$$

Then

$$X = (X \setminus A) \cup \bigcup_{\alpha \in A} U_\alpha \cup \varnothing$$

is an open cover of X.

$$\Rightarrow \; X = (X \setminus A) \cup \bigcup_{i=1}^{n} U_{\alpha_i}$$

$$\Rightarrow \; A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \Rightarrow A \text{ is finite.}$$

□

Example 9.36. Let $Y = \{y_1, y_2\}$, $\mathcal{T}_Y = \{\varnothing, Y\}$ and $X = \mathbb{N} \times Y$, where $\mathbb{N}$ has the usual topology. Then X is locally point compact but not compact, as

$$\{V_n = \mathbb{N} \setminus \{n\} \times Y\}$$

has no finite subcover.

(Since $\mathcal{T}_Y = \{\varnothing, Y\}$, every nonempty subset has a limit point.)

Theorem 9.37. *Let X be a metrizable space. Then the following statements are equivalent:*

(i) X is compact.

(ii) X is limit point compact.

(iii) X is sequentially compact.

Proof. $(i) \Rightarrow (ii)$ is already done.

$(ii) \Rightarrow (iii)$:

Let (x_n) be a sequence in X. Set $A = \{x_n : n \in \mathbb{N}\}$. If finite, then it trivially works.

If A is infinite, then A has a limit point, say x.

We find a subsequence (x_{n_k}) which converges to x.

Choose x_{m_1} so that $x_{m_1} \in B(x, 1)$. Since $B(x, 1)$ contains infinitely many points of A, we can choose $x_{m_2} > x_{m_1}$ such that $x_{m_2} \in B(x, \frac{1}{2})$.

Then $x_{n_i} \to x$.

$(iii) \Rightarrow (i)$:

First, we prove that if X is sequentially compact, then every cover has a Lebesgue number. We prove this via contradiction.

Suppose there is a cover $\mathcal{A}$ of X having no Lebesgue number. Then, for each $\delta = \frac{1}{n} > 0$, there exists a set C_n of diameter $d(C_n) < \frac{1}{n}$ but $C_n \nsubseteq A$ for any $A \in \mathcal{A}$.

Choose $x_n \in C_n$. Then, by hypothesis, a subsequence $x_{n_i} \to a$, and $a \in A$ for some $A \in \mathcal{A}$. But A is open, so $B(a, \epsilon) \subseteq A$ for some $\epsilon > 0$.

Note that $\frac{1}{n_i} < \epsilon_1$ for large i, and also for large n_i, $d(C_{n_i}, a) < \epsilon_1/2$.

$\Rightarrow C_{n_i} \subseteq B(a, \epsilon) \subseteq A$ (i.e. $\frac{1}{n_i} < \epsilon_1$), which is a contradiction.

Secondly, we prove that if X is sequentially compact, then for given $\epsilon > 0$, if x_i, $i = 1, 2, \ldots$ are in X,

$$X = \bigcup_{i=1}^{n} B(x_i, \epsilon). \qquad (*)$$

We prove this too via contradiction.

Assume that there exists $\epsilon > 0$ such that $(*)$ is false.

Let $x_1 \in X$, then $B(x_1, \epsilon) \neq X$.

Let $x_2 \in X \setminus B(x_1, \epsilon)$. In general, given $x_1, x_2, \ldots, x_n$, we can choose $x_{n+1} \in X$ such that

$$x_{n+1} \notin B(x_i, \epsilon) \text{ for every } i = 1, 2, \ldots, n.$$

$$\Rightarrow d(x_{n+1}, x_i) \geq \epsilon \quad \text{for every } i = 1, 2, \ldots, n$$

$$\Rightarrow (x_n) \text{ has no convergent subsequence.}$$

Finally, we show X is compact.

Let $\mathcal{U}$ be an open cover of X. Since X is sequentially compact, it has Lebesgue number say $\delta > 0$. Let $\epsilon = \delta/3$.

Also,

$$X = \bigcup_{i=1}^{l} B(x_i, \epsilon)$$

Each of these balls has diameter $2\delta/3 < \delta$, so it lies in some A_j.

$$\Rightarrow X = \bigcup B(x_i, \epsilon) = \bigcup_{j=1}^{l} A_i$$

□

Example 9.38. Let $X = [0, 1]^{\mathbb{N}} = [0, 1] \times [0, 1] \times \cdots$.

For $x = (x_n)$ and $y = (y_n)$ in X, let

$$d(x, y) = \sup_n |x_n - y_n|$$

Find an infinite subset of X having no limit point.

Example 9.39. Show that the subspace $[0, 1]$ is *not* compact in the *lower limit topology* on $\mathbb{R}$ (see theorem 2.16).

Definition 9.40. A space X is said to be countably compact if every countable cover has a finite subcover.

Example 9.41. If X is a T_1-space, show that X is countably compact if and only if X is limit point compact.

Example 9.42. Let $f : X \to X$ such that $d(f(x), f(y)) = d(x, y)$. If X is compact, then f is a homeomorphism.

Now, we discuss a most important result relating to compactness of sets in topology, known as Tychonoff's theorem.

Remark 9.43. (i) A topological space X is compact if every basis open cover has finite subcover.

(ii) A class $\mathcal{G}$ is called a closed basis for a topological space X if

$$\mathcal{G} = \{X \setminus C : C \in \mathcal{G}\}$$

is a basis for X. Similarly, closed subbases can be defined.

Theorem 9.44 (Alexander subbase theorem)**.** *Let X be a topological space and let $\mathcal{S}$ be a subbasis for the topology of X. Then X is compact if and only if every open cover of X by members of $\mathcal{S}$ admits a finite subcover.*

Equivalently, if

$$\mathcal{C} := \{ X \setminus S : S \in \mathcal{S} \}$$

is the corresponding closed *subbasis, then X is compact if and only if every family $\mathcal{F} \subseteq \mathcal{C}$ with the finite intersection property satisfies $\bigcap \mathcal{F} \neq \varnothing$.*

Proof. The open-cover and closed-intersection formulations are equivalent by taking complements, so we work with the closed-subbase formulation.

($\Rightarrow$) If X is compact and $\mathcal{F} \subseteq \mathcal{C}$ has the finite intersection property, then $\bigcap \mathcal{F} \neq \varnothing$ follows from theorem 9.22 (since every set in $\mathcal{C}$ is closed).

(is implied by) Assume that every $\mathcal{F} \subseteq \mathcal{C}$ with the finite intersection property has nonempty intersection. By theorem 9.22, it suffices to show that *every* family of closed subsets of X with the finite intersection property has nonempty intersection.

Let $\mathcal{A}$ be a family of closed subsets of X with the finite intersection property. By Zorn's lemma, extend $\mathcal{A}$ to a maximal family $\mathcal{M}$ of closed sets having the finite intersection property (with respect to inclusion among such families). Set $\mathcal{M}_0 := \mathcal{M} \cap \mathcal{C}$. Then $\mathcal{M}_0 \subseteq \mathcal{C}$ has the finite intersection property, hence by hypothesis there exists

$$x \in \bigcap \mathcal{M}_0.$$

We claim that $x \in \bigcap \mathcal{M}$. Suppose not. Then there exists $M \in \mathcal{M}$ with $x \notin M$, so $x \in X \setminus M$, which is open. Since $\mathcal{S}$ is a subbasis, there exist $S_1, \ldots, S_n \in \mathcal{S}$ such that

$$x \in S_1 \cap \cdots \cap S_n \subseteq X \setminus M.$$

Let $C_i := X \setminus S_i \in \mathcal{C}$. The above inclusion implies

$$M \subseteq C_1 \cup \cdots \cup C_n. \tag{9.1}$$

We now use maximality of $\mathcal{M}$ to show that at least one of the sets C_i belongs to $\mathcal{M}$. Indeed, if $C_i \notin \mathcal{M}$ for each i, then for every i the family $\mathcal{M} \cup \{C_i\}$ fails the finite intersection property, so we can choose finitely many sets $M_{i,1}, \ldots, M_{i,k_i} \in \mathcal{M}$ such that

$$C_i \cap M_{i,1} \cap \cdots \cap M_{i,k_i} = \varnothing.$$

Let

$$M^* := M \cap \bigcap_{i=1}^{n} \bigcap_{j=1}^{k_i} M_{i,j}.$$

This is an intersection of finitely many members of $\mathcal{M}$, hence $M^* \neq \varnothing$ by the finite intersection property of $\mathcal{M}$. On the other hand, by (9.1) we have $M^* \subseteq C_1 \cup \cdots \cup C_n$, while the displayed emptiness conditions imply $M^* \cap C_i = \varnothing$ for each i, a contradiction. Hence $C_i \in \mathcal{M}$ for some i.

But then $C_i \in \mathcal{M}_0$, so $x \in C_i$; this contradicts $x \in S_1 \cap \cdots \cap S_n$ (since $S_i = X \setminus C_i$). Therefore $x \in \bigcap \mathcal{M} \subseteq \bigcap \mathcal{A}$, and in particular $\bigcap \mathcal{A} \neq \varnothing$. This completes the proof. □

Theorem 9.45 (Heine–Borel theorem on $\mathbb{R}$). *Every closed and bounded subset of $\mathbb{R}$ (with the usual topology) is compact. In particular, every closed interval $[a, b] \subset \mathbb{R}$ is compact.*

Proof. It suffices to prove that $[a, b]$ is compact, since any closed and bounded $K \subset \mathbb{R}$ is contained in some $[a, b]$ and is closed in $[a, b]$, hence compact by theorem 9.6.

Consider the subspace $[a, b] \subset \mathbb{R}$. The collection

$$\mathcal{S} := \{\, [a, d) : a < d \leq b \,\} \;\cup\; \{\, (c, b] : a \leq c < b \,\}$$

is a subbasis for the subspace topology on $[a, b]$ (each set is the intersection of $[a, b]$ with an open ray in $\mathbb{R}$). The corresponding closed subbasis $\mathcal{C} = \{[d, b] : a \leq d \leq b\} \cup \{[a, c] : a \leq c \leq b\}$ consists of closed intervals.

Let $\mathcal{F} \subseteq \mathcal{C}$ have the finite intersection property. Define

$$A := \{\, c \in [a, b] : [a, c] \in \mathcal{F} \,\}, \qquad B := \{\, d \in [a, b] : [d, b] \in \mathcal{F} \,\}.$$

Let $c_0 := \inf A$ if $A \neq \varnothing$ (and $c_0 := b$ if $A = \varnothing$), and let $d_0 := \sup B$ if $B \neq \varnothing$ (and $d_0 := a$ if $B = \varnothing$). We claim that $d_0 \leq c_0$. Otherwise $c_0 < d_0$; choose $c \in A$ and $d \in B$ with $c < d$. Then $[a, c] \cap [d, b] = \varnothing$, contradicting the finite intersection property.

Choose any $x \in [d_0, c_0] \subseteq [a, b]$. If $[a, c] \in \mathcal{F}$, then $c \geq c_0 \geq x$, so $x \in [a, c]$. If $[d, b] \in \mathcal{F}$, then $d \leq d_0 \leq x$, so $x \in [d, b]$. Hence $x \in \bigcap \mathcal{F}$, so $\bigcap \mathcal{F} \neq \varnothing$. By the Alexander subbase theorem theorem 9.44, the interval $[a, b]$ is compact. □

Theorem 9.46 (Tychonoff's theorem). *Let $\{X_i\}_{i \in I}$ be a family of nonempty compact spaces. Then the product $\prod_{i \in I} X_i$, equipped with the product topology, is compact.*

Proof. Let $X := \prod_{i \in I} X_i$ and let $\pi_i : X \to X_i$ be the coordinate projections. A subbasis for the product topology is given by sets of the form $\pi_i^{-1}(U)$, where $U \subseteq X_i$ is open. Equivalently, a closed subbasis is given by sets of the form $\pi_i^{-1}(F)$, where $F \subseteq X_i$ is closed.

By the Alexander subbase theorem theorem 9.44, it is enough to show that any family $\mathcal{F}$ of closed subbasic sets $\pi_i^{-1}(F)$ with the finite intersection property has nonempty intersection.

Let $\mathcal{F}$ be such a family. For each $i \in I$, define

$$\mathcal{F}_i := \{ F \subseteq X_i \text{ closed} : \pi_i^{-1}(F) \in \mathcal{F} \}.$$

Then $\mathcal{F}_i$ has the finite intersection property in X_i: if $F_1, \ldots, F_n \in \mathcal{F}_i$, then

$$\pi_i^{-1}(F_1 \cap \cdots \cap F_n) = \pi_i^{-1}(F_1) \cap \cdots \cap \pi_i^{-1}(F_n) \neq \varnothing$$

by the finite intersection property of $\mathcal{F}$, hence $F_1 \cap \cdots \cap F_n \neq \varnothing$. Since each X_i is compact, theorem 9.22 yields $\bigcap \mathcal{F}_i \neq \varnothing$. Choose $x_i \in \bigcap \mathcal{F}_i$ (and if $\mathcal{F}_i = \varnothing$, choose $x_i \in X_i$ arbitrarily). Let $x := (x_i)_{i \in I} \in X$.

If $\pi_i^{-1}(F) \in \mathcal{F}$, then $F \in \mathcal{F}_i$ and hence $x_i \in F$, so $x \in \pi_i^{-1}(F)$. Therefore $x \in \bigcap \mathcal{F}$, proving $\bigcap \mathcal{F} \neq \varnothing$, as required. □

Theorem 9.47 (Generalized Heine-Borel Theorem)**.** *Every closed and bounded subset of* $\mathbb{R}^n$ *(with the usual topology) is compact.*

Proof. Let $K \subset \mathbb{R}^n$ be closed and bounded. Choose real numbers $a_i < b_i$ such that

$$K \subseteq R := \prod_{i=1}^{n} [a_i, b_i].$$

By theorem 9.45, each interval $[a_i, b_i]$ is compact; hence the product R, equipped with the product topology, is compact by theorem 9.46. Moreover, the product topology on R coincides with the subspace topology inherited from $\mathbb{R}^n$, since basic open sets in the product topology are intersections of R with open rectangles in $\mathbb{R}^n$.

Finally, K is closed in R (as K is closed in $\mathbb{R}^n$ and R has the subspace topology), so K is compact by theorem 9.6. □

Definition 9.48. A topological space X is said to be *locally compact* if every point $x \in X$ has an open neighborhood V such that $\overline{V}$ is compact.

Example. $\mathbb{R}^n$ and $\mathbb{C}^n$ are locally compact.

Example. Any infinite-dimensional Hilbert space is not locally compact.

Hint. In an infinite-dimensional Hilbert space, the closure of an open ball is never compact.

Proposition 9.49. *Let X be a Hausdorff space. Then X is locally compact if and only if for each $x \in X$, and each open set W containing x, there exists an open set V such that $\overline{V}$ is compact and $x \in V \subseteq \overline{V} \subseteq W$.*

Proof. The forward implication is immediate from the definition.

Conversely, assume that X is locally compact, and let $x \in X$ and W be an open neighborhood of x. Choose an open set V such that $x \in V$ and $\overline{V}$ is compact. Since X is Hausdorff, $\overline{V}$ is closed in X.

Set

$$Y := \overline{V} \setminus (W \cap \overline{V}).$$

Then Y is a closed subset of the compact space $\overline{V}$, hence compact, and $x \notin Y$. Because X is Hausdorff, there exist disjoint open sets separating x and Y; in particular, there is an open set D with

$$x \in D \subseteq \overline{D} \subseteq W \cap V.$$

Since $\overline{D} \subseteq \overline{V}$ and $\overline{V}$ is compact, it follows that $\overline{D}$ is compact. This proves the claim.

Compactification

When a space X is not compact, it is often useful to embed it into a compact space that contains it densely or at least as a distinguished subspace. Such a process is called a *compactification*. A familiar example is obtained by adjoining the points $-\infty$ and $+\infty$ to $\mathbb{R}$, thereby producing the compact ordered space $[-\infty, +\infty]$.

The simplest compactification procedure is to adjoin a single point. Geometrically, this is what happens when one identifies $\mathbb{R}^n$ with the sphere S^n minus one point. For a general space X, the underlying set of the one-point compactification is

$$X^* = X \cup \{\infty\},$$

and the open sets are either the open subsets of X itself or the sets $U \subseteq X^*$ that contain ∞ and satisfy the condition that $X^* \setminus U$ is compact and closed in X. This construction always yields a compact space; it is Hausdorff precisely under the familiar local compactness and Hausdorff hypotheses.

□

Example 9.50. Let $X = (-1, 1)$. Then, $X \subset X^* = [-1, 1]$, which is compact. But also $(-1, 1) \subset S^1$, and S^1 is compact, which is a one-point compactification of $(-1, 1)$. But also $[-1, 1]$ is not homeomorphic to S^1. Hence, we get different compactifications.

The following theorem records the basic properties of this construction and clarifies exactly when the resulting space is Hausdorff.

Theorem 9.51 (Alexandroff). *The one-point compactification X^* of a topological space X is compact, and X is a subspace of X^*.*

The space X^* is Hausdorff if and only if X is locally compact and Hausdorff.

Proof. $(X^*, \mathcal{T}^*)$ is a topological space.

(i) If $\infty \in U_1 \cap U_2$, $U_1, U_2 \in \mathcal{T}^*$, then

$$X^* \setminus (U_1 \cap U_2) = (X^* \setminus U_1) \cup (X^* \setminus U_2)$$

is closed and compact. Hence, $U_1 \cap U_2 \in \mathcal{T}^*$.

(ii) If $\infty \in \bigcup_{i\in I} U_i$, then $\infty \in U_{i_0}$ for some i_0. Therefore,

$$X^* \setminus \bigcup_{i\in I} U_i = \bigcap_{i\in I} (X^* \setminus U_i) \subset X^* \setminus U_{i_0}$$

is closed and compact. Therefore, $X^* \setminus \bigcup_{i\in I} U_i$ is closed and compact. Hence, $\bigcup_{i\in I} U_i$ is open in X^*.

(iii) Let $X^* = \bigcup_{i\in I} U_i$ (open cover). Then $\infty \in U_{i_0}$ for some i_0. Therefore, $X^* \setminus U_{i_0}$ is compact. Hence, $X^* \setminus U_{i_0} \subset \bigcup_{i\neq i_0} U_i$.

Therefore, X^* is compact in X^* and X is a subspace of X^*.

Suppose X^* is a Hausdorff space. Then X is locally compact and Hausdorff.

$(C \subset X \subset X^*$ implies $X \subset C \cap U^* \subset C \cap X^*)$

If X is locally compact and Hausdorff, we need to show X^* is Hausdorff. For this, it is enough to find disjoint open sets around $x \in X$ and $\infty \in X^*$ in X^*.

Since X is locally compact, for $x \in X$, there exists an open set U such that $x \in U \subset \overline{U}$, where $\overline{U}$ is compact.

Let $V = X^* \setminus \overline{U}$, then $\infty \in V$ (open).

□

Remark 9.52. If X is compact, then ∞ is an isolated point of X^*, i.e., ∞ is both open and closed in X^*.

Conversely, if ∞ is an isolated point of X^*, then X is closed in X^* and hence compact.

Stone-Čech Compactification:

As a motivating prototype, observe that if K is a compact metric space and $f\colon (0,1) \to K$ is uniformly continuous, then f extends continuously (indeed uniformly continuously) to the compactification $[0,1]$.

Let $x \in [0,1]$, then if $x_n \in (0,1)$ st $x_n \to x$, then $f(x_n)$ is a sequence in compact space K, hence by passing to subsequence, we can assume that $f(x_n) \to \tilde{f}(x)$.

Let $\tilde{f}(x) := \lim f(x_n)$ when $x_n \to x$.

Suppose $x_n, y_n \in [0,1]$ and $|x_n - y_n| < \varepsilon$ (which is to be chosen later). Then, for large n, $|x_n - y_n| < \varepsilon$, hence by uniform continuity of f on $(0,1)$, it follows that for $\varepsilon > 0$, there exists $\delta > 0$ such that

$|x_{n_k} - y_{n_k}| < \delta \;\; \& \;\; |f(x_{n_k}) - f(y_{n_k})| < \varepsilon$
i.e. $|\hat{f}(a) - \hat{f}(y)| < \varepsilon$.

Hence $\hat{f}$ is uniformly continuous on the compactification $[0,1]$.

The above example can be generalized to topological sets, known as Stone-Čech compactification.

Note that in the above example we assume uniform continuity instead of continuity to avoid complication.

Exercises

Exercise 9.53. Prove that a closed subset of a compact space is compact.

Exercise 9.54. Prove that a compact subset of a Hausdorff space is closed.

Exercise 9.55. Show that if X is compact and $f : X \to \mathbb{R}$ is continuous, then f attains its maximum and minimum values.

Exercise 9.56. Let X be locally compact Hausdorff. Construct explicitly the one-point compactification $X^* = X \cup \{\infty\}$ and show that X^* is compact Hausdorff.

Exercise 9.57. Show that a Hausdorff space X is locally compact if and only if every $x \in X$ has a neighborhood whose closure is compact.

Exercise 9.58 (Tychonoff for countable products)**.** Let $\{X_n\}_{n\in\mathbb{N}}$ be compact spaces. Prove that $\prod_{n\in\mathbb{N}} X_n$ is compact.

Chapter 10

Urysohn's lemma and Tietze's extension theorem

Normality is the separation hypothesis under which closed sets can be separated by continuous functions. Urysohn's lemma provides such separating functions, and Tietze's theorem upgrades this to extend continuous real-valued functions from closed subsets. These results are fundamental in analysis and in metrization theory.

10.1 Urysohn's lemma

Theorem 10.1 (Urysohn's lemma)**.** *Let X be a normal space and let $F, G \subset X$ be disjoint closed sets. Then there exists a continuous function $u : X \to [0,1]$ such that $u|_F \equiv 0$ and $u|_G \equiv 1$.*

Proof. Since X is normal, for any disjoint closed sets $A, B \subset X$ there exist disjoint open sets separating them. Let $\mathbb{D} = \left\{ \frac{k}{2^n} : n \in \mathbb{N},\ 0 \le k \le 2^n \right\}$ be the dyadic rationals in $[0,1]$.

We construct open sets $U_r \subset X$ for $r \in \mathbb{D}$ satisfying

(U1) $F \subset U_0$ and $U_1 = X \setminus G$.

(U2) If $r < s$, then $\overline{U_r} \subset U_s$ for all $r, s \in \mathbb{D}$.

Start with U_0 open with $F \subset U_0 \subset \overline{U_0} \subset X \setminus G$ (normality with F and G). Set $U_1 = X \setminus G$. Assume $U_{k/2^n}$ are constructed for a fixed n satisfying (U2) for adjacent dyadic rationals. For each odd k (so $k/2^{n+1}$ lies strictly between $(k-1)/2^{n+1}$ and $(k+1)/2^{n+1}$), apply normality to the disjoint closed sets $\overline{U_{(k-1)/2^{n+1}}}$ and $X \setminus U_{(k+1)/2^{n+1}}$

to obtain an open set $U_{k/2^{n+1}}$ with

$$\overline{U_{(k-1)/2^{n+1}}} \subset U_{k/2^{n+1}} \subset \overline{U_{k/2^{n+1}}} \subset U_{(k+1)/2^{n+1}}.$$

Iterating yields the family $\{U_r\}_{r\in\mathbb{D}}$ satisfying (U1)–(U2).

Define $u : X \to [0,1]$ by

$$u(x) = \inf\{\, r \in \mathbb{D} : x \in U_r \,\},$$

with the convention $\inf \emptyset = 1$. Then $u(x) = 0$ on F and $u(x) = 1$ on G by (U1). To see continuity, fix $a \in [0,1]$. Using (U2), one checks that

$$\{x : u(x) < a\} = \bigcup_{\substack{r\in\mathbb{D} \\ r<a}} U_r \quad \text{and} \quad \{x : u(x) > a\} = \bigcup_{\substack{r\in\mathbb{D} \\ r>a}} (X \setminus \overline{U_r}),$$

both open; hence u is continuous. □

10.2 Tietze's extension theorem

Theorem 10.2 (Tietze extension theorem)**.** *Let X be a normal space and let $A \subset X$ be closed. Every continuous function $f : A \to [-1,1]$ admits a continuous extension $\tilde{f} : X \to [-1,1]$ with $\tilde{f}|_A = f$.*

Proof. We sketch a standard constructive proof based on Urysohn's lemma. Let $f : A \to [-1,1]$ be continuous. Using Urysohn's lemma repeatedly, one constructs a sequence of continuous functions $g_n : X \to [-2^{-n}, 2^{-n}]$ such that the partial sums $s_N = \sum_{n=0}^{N} g_n$ satisfy

$$|f(x) - s_N(x)| \le 2^{-(N+1)} \qquad (x \in A).$$

Then $\{s_N\}$ converges uniformly on X to a continuous function $\tilde{f}$ (uniform limit of continuous functions), and the estimate forces $\tilde{f}|_A = f$. Moreover $|\tilde{f}| \le 1$ on X by construction. □

Remark 10.3. The full details amount to approximating f by step functions on A and extending each step using Urysohn functions. Uniform convergence then gives a continuous extension.

Exercises

Exercise 10.4. Let X be normal and let $F, G \subseteq X$ be disjoint closed sets. Use Urysohn's lemma to show that there exists a continuous $f : X \to [0,1]$ with $f(F) = \{0\}$ and $f(G) = \{1\}$.

Exercise 10.5. Show that a T_1 space X is normal if and only if for every closed $F \subseteq X$ and every open set $U \supseteq F$, there exists an open V with $F \subseteq V \subseteq \overline{V} \subseteq U$.

Exercise 10.6. Assuming Tietze's extension theorem, prove that if X is normal then every continuous $f : F \to [-1,1]$ defined on a closed $F \subseteq X$ extends to a continuous $F : X \to [-1,1]$.

Exercise 10.7. Show that normality is inherited by closed subspaces: if X is normal and $F \subseteq X$ is closed, then F is normal.

Exercise 10.8. Let X be normal. Prove that any two disjoint closed sets can be separated by disjoint open neighborhoods.

Chapter 11

Urysohn's metrization theorem

Metrization theorems characterize when a purely topological space arises from a metric. Urysohn's theorem states that a second countable regular space is metrizable. We present a standard approach: use Urysohn functions to embed the space into a countable product of intervals, and then pull back a metric.

11.1 Statement and proof

Theorem 11.1 (Urysohn metrization theorem)**.** *Let X be a T_1 regular, second countable space. Then X is metrizable; that is, there exists a metric d on X whose induced topology equals the given topology.*

Proof. Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable base for X. Regularity implies: for any $x \in B_n$ there exists an open set U with

$$x \in U \subset \overline{U} \subset B_n.$$

Using second countability, we may enumerate a countable family $\{U_k\}_{k\in\mathbb{N}}$ of open sets such that for every pair $x \in O$ with O open, there exists k with $x \in U_k \subset \overline{U_k} \subset O$. (One obtains such a family by choosing, for each B_n and each $x \in B_n$, one U as above and then selecting countably many representatives using the countable base.)

For each k, apply Urysohn's lemma to the disjoint closed sets $\overline{U_k}$ and $X \setminus B_{n(k)}$ (where $B_{n(k)}$ is a basic open set containing $\overline{U_k}$) to obtain a continuous function $f_k : X \to [0,1]$ such that $f_k = 1$ on $\overline{U_k}$ and $f_k = 0$ on $X \setminus B_{n(k)}$.

Define an evaluation map

$$F : X \longrightarrow [0,1]^{\mathbb{N}}, \qquad F(x) = (f_k(x))_{k\in\mathbb{N}},$$

where $[0,1]^{\mathbb{N}}$ carries the product topology. By construction each coordinate f_k is continuous, hence F is continuous.

Claim 1: F is injective. If $x \neq y$, use the T_0 consequence of regularity to find an open set O containing x but not y. Choose k with $x \in U_k \subset \overline{U_k} \subset O$. Then $f_k(x) = 1$ while $y \notin O \subset X \setminus B_{n(k)}$ forces $f_k(y) = 0$, so $F(x) \neq F(y)$.

Claim 2: F is a topological embedding. Let $O \subset X$ be open and $x \in O$. Choose k with $x \in U_k \subset \overline{U_k} \subset O$. Then

$$x \in f_k^{-1}((1/2, 1]) \subset O,$$

so the family $\{f_k^{-1}((1/2, 1])\}$ refines the topology. This implies that the topology on X is the initial topology induced by $\{f_k\}$, hence coincides with the subspace topology inherited from $F(X) \subset [0,1]^{\mathbb{N}}$. Therefore F is a homeomorphism of X onto $F(X)$.

Finally, endow $[0,1]^{\mathbb{N}}$ with the metric

$$D(u,v) = \sum_{k=1}^{\infty} 2^{-k} \, |u_k - v_k|.$$

This metric induces the product topology on $[0,1]^{\mathbb{N}}$. Then $d(x,y) = D(F(x), F(y))$ defines a metric on X inducing the given topology. □

Remark 11.2. The proof shows more: X embeds homeomorphically into the Hilbert cube $[0,1]^{\mathbb{N}}$.

Exercises

Exercise 11.3. Let X be second countable and regular. Show that for every $x \in X$ and every open neighborhood $O \ni x$, there exists a continuous $f : X \to [0,1]$ such that $f(x) = 1$ and $f \equiv 0$ on $X \setminus O$.

Exercise 11.4. Show that the metric

$$D(u,v) = \sum_{k=1}^{\infty} 2^{-k} |u_k - v_k|$$

induces the product topology on $[0,1]^{\mathbb{N}}$.

Exercise 11.5. Let X be T_1, regular, and second countable. Using the embedding into $[0,1]^{\mathbb{N}}$ from Urysohn's metrization theorem, prove that X is Hausdorff.

Exercise 11.6. Prove that every second countable metrizable space admits a compatible complete metric if and only if it is completely metrizable (state any needed definitions precisely).

Notation and Glossary

Notation

X, Y, Z	topological spaces (unless otherwise specified).
$\mathcal{T}$	a topology on X; the pair $(X, \mathcal{T})$ denotes a topological space.
$\mathcal{P}(X)$	the power set of X.
$\overline{A}$	closure of $A \subseteq X$.
$\mathrm{Int}(A)$	interior of $A \subseteq X$.
∂A	boundary of $A \subseteq X$.
A'	derived set (set of limit points) of $A \subseteq X$.
$\partial(A)$	boundary operator used in parts of the notes, typically $\partial(A) = \overline{A} \cap \overline{(X \setminus A)}$.
$\mathcal{T}_Y$	subspace topology on $Y \subseteq X$.
$\prod_{\lambda \in I} X_\lambda$	product space with index set I, equipped with the product topology unless stated otherwise.
π_λ	the λ-th coordinate projection in a product.
$q : X \to Y$	a quotient map; Y carries the quotient topology induced by q.
T_0, T_1, T_2	separation axioms (with T_2 meaning Hausdorff).
First/second countable	countability axioms: existence of a countable neighborhood base at each point / existence of a countable base for the topology.

Glossary of key terms

Topology A collection $\mathcal{T} \subseteq \mathcal{P}(X)$ containing $\varnothing$ and X, closed under arbitrary unions and finite intersections.

Basis / subbasis A basis $\mathcal{B}$ generates $\mathcal{T}$ by unions; a subbasis $\mathcal{S}$ generates $\mathcal{T}$ by finite intersections and unions.

Continuous map
$f : X \to Y$ is continuous if $f^{-1}(U)$ is open in X whenever U is open in Y.

Homeomorphism
A bijection $f : X \to Y$ that is continuous with continuous inverse; equivalently, a continuous bijection that is open (or closed).

Connected X is connected if it is not the union of two disjoint nonempty open sets.

Path connected Any two points can be joined by a continuous map $[0, 1] \to X$.

Component A maximal connected subset of X; components partition X.

Compact Every open cover admits a finite subcover.

Locally compact Each point has a neighborhood with compact closure (in Hausdorff spaces, this is equivalent to the usual definition).

One-point compactification
For locally compact Hausdorff X, the space $X^* = X \cup \{\infty\}$ obtained by adjoining one point so that complements of compact sets form neighborhoods of ∞.

Normal Disjoint closed sets can be separated by disjoint open neighborhoods.

Urysohn function
In a normal space, a continuous $f : X \to [0, 1]$ separating two disjoint closed sets.

Metrizable The topology arises from a metric.

Problem Sets

Problem Set 1: Topological spaces and bases

1. List all possible topologies on the set $\{a, b\}$ and also on the set $\{a, b, c\}$.

2. Let X be a set with exactly 4 elements. Does there exist a topology τ on X such that there are precisely 14 open sets in the topological space (X, τ)? Justify.

3. Examine whether τ is a topology on X, where

   (a) $X = \mathbb{R}$ and $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.

   (b) $X = \mathbb{R}$ and $\tau = \{[a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.

   (c) $X = \mathbb{R}$ and $\tau = \{(a, \infty) : a \in \mathbb{Q}\} \cup \{\emptyset, \mathbb{R}\}$.

   (d) $X = \mathbb{R}$ and $\tau = \{[-a, a) : a \in \mathbb{R},\ a > 0\} \cup \{\emptyset, \mathbb{R}\}$.

   (e) $X = \mathbb{N}$ and $\tau = \{\{n, n+1, n+2, ...\} : n \in \mathbb{N}\} \cup \{\emptyset\}$.

   (f) $X = \mathbb{R}$ and $\tau = \{G \subset \mathbb{R} : \mathbb{Q} \not\subset G\} \cup \{\mathbb{R}\}$.

   (g) $X = \mathbb{R}$ and $\tau = \{G \subset \mathbb{R} : G \subset \mathbb{Q} \text{ or } \mathbb{Q} \subset G\}$.

   (h) $X = [0, 1]$ and $\tau = \{G \subset [0, 1] : GG \subset G\}$.
   (For each $A \subset [0, 1]$, define $AA = \{xy : x, y \in A\}$.)

   (i) X is an infinite set and $\tau = \{G \subset X : X \setminus G \text{ is infinite or } \emptyset\}$.

   (j) X is a nonempty set, $x_0 \in X$ and $\tau = \{G \subset X : X \setminus G \text{ is finite or } x_0 \in X \setminus G\}$.

   (k) X is an uncountable set and $\tau = \{G \subset X : G \text{ is countable or } X \setminus G \text{ is countable}\}$.

4. Let $\tau = \{G_k : k \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}^2\}$, where for each $k \in \mathbb{R}$, $G_k = \{(x, y) \in \mathbb{R}^2 : x > y + k\}$. Prove that τ is a topology on $\mathbb{R}^2$.
   Is τ a topology on $\mathbb{R}^2$ if '$k \in \mathbb{R}$' is replaced by (a) '$k \in \mathbb{N}$'? (b) '$k \in \mathbb{Q}$'? Justify.

5. Let X, Y be nonempty sets and let $f : X \to Y$.

   (a) If $\mathfrak{J}'$ is a topology on Y, then show that $\mathfrak{J} = \{f^{-1}(H) : H \in \mathfrak{J}'\}$ is a topology on X.

   (b) If τ is a topology on X and if f is one-one and onto, then show that $\tau' = \{f(G) : G \in \tau\}$ is a topology on Y.
   Show also that, in general, both conditions that f be one-to-one and onto are necessary.

6. Let $\mathcal{X}$ be the class of all (proper) prime ideals of a commutative ring R with unity and for each $E \subset R$, let $\mathcal{V}(E) = \{P \in \mathcal{X} : E \subset P\}$. Prove that $\tau = \{\mathcal{X} \setminus \mathcal{V}(E) : E \subset R\}$ is a topology on $\mathcal{X}$.
   (τ is called the Zariski topology on $\mathcal{X}$ and the topological space $(\mathcal{X}, \tau)$ is called the prime spectrum of R, denoted by Spec(R).)

7. Let τ_f and τ_c denote respectively the cofinite and the cocountable topologies on a nonempty set X. Find a necessary and sufficient condition on X such that $\tau_f = \tau_c$.

8. Let (X, τ) be a topological space in which every singleton subset of X is open. Prove that τ is the discrete topology on X.

9. Let τ be a topology on an infinite set X such that every infinite subset of X is open in (X, τ). Show that τ is the discrete topology on X.

10. Let τ be a topology on an infinite set X such that every infinite subset of X is closed in (X, τ). Prove that τ is the discrete topology on X.

11. Let τ and τ' be topologies on $\mathbb{R}$ such that every countable subset of $\mathbb{R}$ is open in $(\mathbb{R}, \tau)$ and every uncountable subset of $\mathbb{R}$ is open in $(\mathbb{R}, \tau')$. Determine (with justification) which of the following is true.

    (a) τ is strictly finer than τ'.

    (b) τ' is strictly finer than τ.

    (c) τ and τ' are not comparable.

    (d) $\tau = \tau'$.

12. Prove that on a finite set with exactly n elements there are at most $2^{(2^n - 2)}$ distinct topologies.

(The problem of determining the exact number of topologies on a finite set is still open. However, in some particular cases the results are known. For example, if $k(n)$ denotes the number of distinct topologies on a set with exactly n elements, then it is already known that $k(4) = 355$, $k(5) = 6942$, $k(6) = 209527$, $k(7) = 9535241$.)

13. Prove that uncountably many distinct topologies can be defined on any infinite set.

14. Let X be a nonempty set and let $\lambda > 0$. Consider the metric d on X defined by
$$d(x, y) = \begin{cases} \lambda & \text{if } x \neq y, \\ 0 & \text{if } x = y \ (x, y \in X). \end{cases}$$
Find the topology on X induced by d.

15. Let d be a metric on a nonempty set X and let $\lambda > 0$. Consider the metrics d_1, d_2 and d_3 on X defined by $d_1(x, y) = \lambda d(x, y)$, $d_2(x, y) = \min\{1, d(x, y)\}$ and $d_3(x, y) = \frac{d(x,y)}{1+d(x,y)}$ for all $x, y \in X$. Prove that d, d_1, d_2 and d_3 induce the same topology on X.

16. Let d be a metric on a nonempty set X and let τ_d be the topology on X induced by d. Prove that every finite subset of X is closed in (X, τ_d). Without using this fact, show also that every finite subset of $\mathbb{R}$ is closed in $\mathbb{R}$ with the usual topology.

17. Let X be a topological space. If A is an open subset of X and B is a closed subset of X, then prove that $A \setminus B$ is an open subset of X and $B \setminus A$ is a closed subset of X.

18. Let $\mathcal{X}$ be the class of all (proper) prime ideals of a commutative ring R with unity. Consider the Zariski topology $\tau = \{\mathcal{X} \setminus \mathcal{V}(E) : E \subset R\}$ on $\mathcal{X}$, where $\mathcal{V}(E) = \{P \in \mathcal{X} : E \subset P\}$ for each $E \subset R$. If $P \in \mathcal{X}$, then show that $\{P\}$ is a closed set in the topological space $(\mathcal{X}, \tau)$ if and only if P is a maximal ideal of R.

19. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a class of closed subsets of a topological space X. Assume that for each $x \in X$, there exists an open subset G_x of X containing x such that $\{\alpha \in \Lambda : G_x \cap A_\alpha \neq \emptyset\}$ is a finite set. Show that $\bigcup_{\alpha \in \Lambda} A_\alpha$ is closed in X.

20. Let G be a nonempty open subset of $\mathbb{R}$ (with the usual topology) such that $x - y \in G$ for all $x, y \in G$. Show that $G = \mathbb{R}$.

(Thus $\mathbb{R}$ is the only open subgroup of the additive group $\mathbb{R}$ with the usual topology.)

21. Let F be a nonempty closed subset of $\mathbb{R}$ (with the usual topology) such that $x - y \in F$ for all $x, y \in F$. Show that either $F = \mathbb{R}$ or $F = \alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$.
(This provides the class of all closed subgroups of the additive group $\mathbb{R}$ with the usual topology.)

22. Prove that every subgroup of the group $(\mathbb{R}, +)$ is either cyclic (and hence closed in $(\mathbb{R}, \tau_u)$) or dense in $(\mathbb{R}, \tau_u)$.

23. Consider the topology $\tau = \{G \subset \mathbb{R} : \mathbb{R} \setminus G \text{ is finite or } 2 \notin G\}$ on $\mathbb{R}$. If τ_u is the usual topology on $\mathbb{R}$, examine whether $\tau \cup \tau_u$ is a topology on $\mathbb{R}$.

24. State TRUE or FALSE with justification for each of the following statements.
    (a) There exists a topological space with exactly 100 (distinct) open sets.
    (b) If $\{\tau_n\}_{n=1}^{\infty}$ is an ascending chain of topologies on a nonempty set X, then $\bigcup\limits_{n=1}^{\infty} \tau_n$ must be a topology on X.

25. Let τ_u be the usual topology on $\mathbb{R}$. Show that $\tau_u|_{\mathbb{N}}$ is the discrete topology on $\mathbb{N}$ but that $\tau_u|_{\mathbb{Q}}$ is not the discrete topology on $\mathbb{Q}$.

26. Consider $\mathbb{R}$ with the usual topology. Find the relative topology on $\{\frac{1}{n} : n \in \mathbb{N}\}$.

27. Is it possible to define a topology τ on $\mathbb{R}$ such that τ is not the discrete topology on $\mathbb{R}$ and the relative topology on $[0, 1]$ induced by τ is the discrete topology on $[0, 1]$?

28. Let τ_l denote the lower limit topology on $\mathbb{R}$. Examine whether the set $\{x \in \mathbb{Q} : x^2 > 5\}$ is closed in the topological space $(\mathbb{Q}, \tau_l|_{\mathbb{Q}})$.

29. Let (X, τ) be a topological space and let $A(\neq \emptyset) \subset X$. Prove that $\tau' = \{G \cup (H \cap A) : G, H \in \tau\}$ is a topology on X such that A is open in the topological space (X, τ') and $\tau|_A = \tau'|_A$.

30. Let (X, τ) be a topological space such that for each nonempty finite subset A of X (or, for each subset A of X containing exactly two elements), $\tau|_A$ is the indiscrete topology on A. Is it necessary that τ is the indiscrete topology on X?

31. Let F be a closed subspace of a topological space X and let G be an open set in F. If H is an open set in X containing G, then prove that $G \cup (H \setminus F)$ is open in X.

32. Let Y and Z be subspaces of a topological space X and let $A \subset Y \cap Z$. If A is open in both Y and Z, then show that A is open in $Y \cup Z$.

33. Let Y be a nonempty subset of a topological space X and let Z be a nonempty open subset of $X \setminus Y$. If $A \subset Z$ such that A is open in $Y \cup Z$, then show that A is open in X.

34. Let X be a topological space and $A \subset X$. Show that A is closed in X if and only if for every $x \in \overline{A}$, there exists an open set G in X containing x such that $G \cap A$ is closed in G.

35. Let X be a topological space and $A \subset X$. If for each $a \in A$, there exists an open set G in X containing a such that $G \cap A$ is closed in G, then show that there exist an open set V in X and a closed set F in X such that $A = V \cap F$.

36. Consider the subspace $(X, \tau_u|_X)$ of the topological space $(\mathbb{R}, \tau_u)$, where $X = \{m + n\pi : m, n \in \mathbb{Z}\}$ and τ_u is the usual topology on $\mathbb{R}$. Examine whether $\{0\}$ is an open set in $(X, \tau_u|_X)$.

37. Let (X, d) be a metric space and let $Y(\neq \emptyset) \subset X$. If ρ is the metric on Y induced by d (*i.e.* $\rho = d|_{Y \times Y}$), τ_d is the topology on X induced by d and τ_ρ is the topology on Y induced by ρ, then show that $\tau_\rho = \tau_d|_Y$.

38. Consider $\mathbb{R}$ with the usual topology and let $x \in \mathbb{R}$. Find the closure of the set $\{x + r : r \in \mathbb{Q}\}$.

39. Consider $\mathbb{R}$ with the lower limit topology. Does there exist a subset A of $\mathbb{R}$ such that
$\overline{A} = [0, 1) \cup (1, 2)$? Justify.

40. Let X be a topological space and let $A, B \subset X$. Prove that $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$, and also that the inclusion can be strict.

41. Let τ, τ' be topologies on a nonempty set X. Prove that $\tau \subset \tau'$ if and only if $\overline{A}^{\tau} \supset \overline{A}^{\tau'}$ for all $A \subset X$.
(Hence $\tau = \tau'$ if and only if $\overline{A}^{\tau} = \overline{A}^{\tau'}$ for all $A \subset X$. However, even if $\tau \subsetneq \tau'$, it can happen that $\overline{A}^{\tau} = \overline{A}^{\tau'}$ for some $A \subset X$.)

42. If G is an open set in a topological space X, then show that $\overline{G} = \overline{X \setminus \overline{X \setminus \overline{G}}}$.

43. Let A be a subset of a topological space X. Show that $\overline{A} \setminus A$ is closed in X if and only if there exist a closed set F in X and an open set G in X such that $A = F \cap G$.

44. Let G be an open set in a topological space X and let $A \subset X$. Prove that $G \cap A = \emptyset$ if and only if $G \cap \overline{A} = \emptyset$.
Hence, or otherwise, show that if U and V are disjoint open sets in X, then $(\overline{U})^0 \cap (\overline{V})^0 = \emptyset$.

45. Prove that a subset G of a topological space X is open in X if and only if $\overline{G \cap \overline{A}} = \overline{G \cap A}$ for all $A \subset X$.

46. Let Y be a closed subspace of a topological space X. Let $A \subset X$ and let H be an open set in Y such that $A \cap Y \subset H$. Prove that $A \cap \overline{Y \setminus H}^X = \emptyset$.

47. Let A, B be subsets of a topological space. Show that $(A \setminus B)^0 \subset A^0 \setminus B^0$ and also show that the equality need not occur in the above inclusion.

48. Prove that a subset F of a topological space X is closed in X if and only if $(F \cup A^0)^0 = (F \cup A)^0$ for all $A \subset X$.

49. Let Y be a subspace of a topological space X and let $A \subset Y$. Show that $\mathrm{Int}_Y(A) \supset Y \cap \mathrm{Int}_X(A)$ and $\mathrm{Int}_X(A) = \mathrm{Int}_Y(A) \cap \mathrm{Int}_X(Y)$, although in general, $\mathrm{Int}_Y(A) \neq Y \cap \mathrm{Int}_X(A)$.

50. For every subset A of a topological space X, show that $(X \setminus A)^0 = X \setminus \overline{A}$.
(Other similar 'commutative' relations are true for closure, interior and complement. For example, $X \setminus A^0 = \overline{X \setminus A}$.)

51. Let A, B be subsets of a topological space. Prove that $\overline{A} \cap B^0 = \overline{A \cap B} \cap B^0$.

52. Let G be an open subset of a topological space. Prove that $G \subset (\overline{G})^0$, and that the inclusion can be strict. Prove, however, that $\overline{G} = \overline{(\overline{G})^0}$.

53. If F is a closed set in a topological space, then show that $(\overline{F^0})^0 = F^0$.

54. If A is a subset of a topological space, then show that $\overline{A^0} = \overline{(\overline{A^0})^0}$.

55. Let A be an open set in a topological space X and let $B \subset X$. Show that $(\overline{A \cap B})^0 = (\overline{A})^0 \cap (\overline{B})^0$.

56. If $\tau = \{\{n, n+1, ...\} : n \in \mathbb{N}\} \cup \{\emptyset\}$, then in the topological space $(\mathbb{N}, \tau)$, find $\{4, 13, 28, 37\}'$ and also find all the subsets A of $\mathbb{N}$ for which $A' = \mathbb{N}$.

57. If $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, then in the topological space $(\mathbb{R}, \tau)$, determine $[4, 10)'$ and $\mathbb{Z}'$.

58. If $\tau = \{G \subset \mathbb{R} : 2 \in G\} \cup \{\emptyset\}$, then determine (with justification) $(2, 4)'$ and $[2, 4]'$ in the topological space $(\mathbb{R}, \tau)$.

59. Let A be a subset of a topological space X such that each subset of A is closed in X. Show that A has no limit point in X.

60. If $\tau = \{G \subset \mathbb{R} : \mathbb{R} \backslash G$ is finite or $0 \in \mathbb{R} \backslash G\}$, then determine (with justification) $\mathbb{Q}^0$, $\overline{\mathbb{R} \setminus \mathbb{Q}}$, $[0, 1]'$ and $\partial \mathbb{Q}$ in the topological space $(\mathbb{R}, \tau)$.

61. Let A be a subset of a topological space. Prove that $\partial A^0 \subset \partial A$ and $\partial \overline{A} \subset \partial A$ and that both these inclusions can be strict.

62. Let A be a subset of a topological space. Show that $(A \cap \partial A)^0 = \emptyset$. Is it necessary that $(\partial A)^0 = \emptyset$?

63. Consider $\mathbb{R}$ with the usual topology. Does there exist a set $A \subset \mathbb{R}$ such that $\partial A = [0, 1]$? Justify.

64. Let A, B be subsets of a topological space X. Prove that $\partial(A \cup B) \subset \partial A \cup \partial B$, and that this inclusion can be strict. Show, however, that if $\overline{A} \cap \overline{B} = \emptyset$, then $\partial(A \cup B) = \partial A \cup \partial B$.

65. Let A, B be subsets of a topological space such that $\partial A \cap \partial B = \emptyset$. Prove that $(A \cup B)^0 = A^0 \cup B^0$.

66. State TRUE or FALSE with justification for each of the following statements.

    (a) Every uncountable subset of $\mathbb{R}$ (with the usual topology) has a limit point in $\mathbb{R}$.

    (b) If every singleton subset of a topological space X is dense in X, then X must be an indiscrete space.

    (c) If no proper subset of a topological space X is dense in X, then X must be a discrete space.

    (d) A nonempty open set in a topological space X cannot be nowhere dense in X.

67. Let A be a nonempty subset of a topological space X. Show that A is dense in the subspace $\overline{A}^X$.

68. Let Y be a subspace of a topological space X. If $D \subset Y$ is dense in Y, then prove that D is dense in the subspace $\overline{Y}^X$.
(Hence, if Y is a dense subspace of a topological space X and if $D \subset Y$ is dense in Y, then D is dense in X.)

69. Let D be a dense set in a topological space X and let $Y(\neq \emptyset) \subset X$. Show that $D \cap Y$ need not be dense in Y, but if moreover Y is open in X, then $\overline{D \cap Y}^X = \overline{Y}^X$ and hence deduce that $D \cap Y$ is dense in Y.

70. Let D_1 and D_2 be dense sets in a topological space X. Show that $D_1 \cap D_2$ need not be dense in X, but if moreover D_1 or D_2 is open in X, then $D_1 \cap D_2$ is dense in X.

71. Let F be a closed subset of a topological space X. Prove that F is nowhere dense in X if and only if $X \setminus F$ is dense in X. Is this result true for an arbitrary subset F of X? Justify.

72. Prove that a subset A of a topological space X is nowhere dense in X if and only if $A \subset \overline{X \setminus \overline{A}}$.

73. Let A be a subset of a topological space X. Prove that the following statements are equivalent.

    (a) A is nowhere dense in X.

    (b) $(X \setminus A)^0$ is dense in X.

    (c) $A \subset \partial\overline{A}$.

74. Let Y be a nonempty open set in a topological space X and let $A \subset Y$. If A is nowhere dense in the subspace Y, then show that A is nowhere dense in X.

75. Prove that for each $1 \le m < n$, $A = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_{m+1} = x_{m+2} = \cdots = x_n = 0\}$ is nowhere dense in $\mathbb{R}^n$ with the usual topology.

76. Show that for a set A in a topological space X, ∂A need not be nowhere dense in X, but that ∂A is nowhere dense in X if A is either open or closed in X.

77. Prove that in a topological space X, every closed nowhere dense set is the boundary of some open set in X.

78. Prove that every finite union of nowhere dense sets in a topological space X is nowhere dense in X.

Give examples to show that an infinite union of nowhere dense sets in a topological space X can/need not be nowhere dense in X.

79. Let X be a topological space and let $x \in X$. If x has a finite local basis in X, then show that x has a local basis in X consisting of precisely one member.

80. Let $\mathcal{B}$ be a class of subsets of a nonempty set X. Show that $\mathcal{B}$ is a basis for the discrete topology on X iff $\mathcal{B} \supset \{\{x\} : x \in X\}$.

81. Show that $\{(a, b) : a, b \in \mathbb{Q}, a < b\}$ is a basis for the usual topology on $\mathbb{R}$.

82. Show that $\{[a, b) : a, b \in \mathbb{Q}, a < b\}$ is a basis for a topology τ on $\mathbb{R}$ which is strictly coarser than the lower limit topology on $\mathbb{R}$ but strictly finer than the usual topology on $\mathbb{R}$.
Determine the closures of $(0, \sqrt{2})$ and $(\sqrt{2}, 3)$ in the topological space $(\mathbb{R}, \tau)$.

83. Examine whether each of the following classes is a basis for some topology on $\mathbb{R}$.

    (a) $\{[a, b] : a \in \mathbb{Q}, b \in \mathbb{R} \setminus \mathbb{Q}, a < b\}$

    (b) $\{[a, b] : a, b \in \mathbb{R}, a < b\}$

    (c) $\{[a, b] : a, b \in \mathbb{Q}, a < b\}$

84. Let τ_u and τ_c denote respectively the usual topology and the cocountable topology on $\mathbb{R}$. Examine whether $\tau_u \cup \tau_c$ is a basis for some topology on $\mathbb{R}$.

85. Let X be any nonempty set. Prove that $\mathcal{S} = \{X \setminus \{x\} : x \in X\}$ is a subbasis for the cofinite topology τ_f on X. Is $\mathcal{S}$ a basis for τ_f on X? Is $\mathcal{S}$ a basis for some topology on X? Justify.

86. For each $(a, b, c) \in \mathbb{R}^3$ with $a^2 + b^2 \neq 0$, let $E(a, b, c) = \{(x, y) \in \mathbb{R}^2 : ax + by + c = 0\}$ and let τ be the topology on $\mathbb{R}^2$ generated by the class $\{E(a, b, c) : a, b, c \in \mathbb{R}, a^2 + b^2 \neq 0\}$. Examine whether $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is an open set in the topological space $(\mathbb{R}^2, \tau)$.

87. Let τ be the topology on $\mathbb{R}$ having $\mathcal{S} = \{\mathbb{R} \setminus \{x\} : x \in \mathbb{R}\} \cup \{\{x\} : x (\neq 2) \in \mathbb{R}\}$ as a subbasis. If A is an infinite subset of $\mathbb{R}$, then show that $A' \neq \emptyset$ in the topological space $(\mathbb{R}, \tau)$.

88. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$ and let $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$. Show that $\mathcal{B}$ is a basis for some topology on $\mathbb{R}$. (This topology is

called the K-topology on $\mathbb{R}$.)
Prove that the K-topology on $\mathbb{R}$ is strictly finer than the usual topology on $\mathbb{R}$ but that it is not comparable with the lower limit topology on $\mathbb{R}$.

89. Let $\mathcal{B}$ be a basis for a topology τ on a nonempty set X. Prove that a set $D \subset X$ is dense in X if and only if $D \cap B \neq \emptyset$ for every $B(\neq \emptyset) \in \mathcal{B}$.
Does this remain true if $\mathcal{B}$ is merely a subbasis for τ on X? Justify.

90. Let τ be the topology on $\mathbb{R}$ having $\{[a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b] : a, b \in \mathbb{R}, a < b\}$ as a subbasis. Examine whether $\mathbb{Q}$ is a dense set in the topological space $(\mathbb{R}, \tau)$.

91. Let $\{\tau_\alpha\}_{\alpha\in\Lambda}$ be a family of topologies on a nonempty set X. Show that there is a unique smallest topology on X containing all τ_α $(\alpha \in \Lambda)$ and a unique largest topology on X contained in all τ_α $(\alpha \in \Lambda)$.

92. Consider the topologies $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$ on the set
$X = \{a, b, c\}$. Find the smallest topology on X containing both τ_1 and τ_2 and the largest topology on X contained in both τ_1 and τ_2.

93. State TRUE or FALSE with justification for each of the following statements.

    (a) If τ is the topology on $\mathbb{R}$ having $\{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$ as a basis, where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$, and if τ_u is the usual topology on $\mathbb{R}$, then the relative topology on $A = \{x \in \mathbb{R} : x > 0\}$ induced by τ is strictly finer than the relative topology on A induced by τ_u.

    (b) If a topology τ on $\mathbb{R}$ is strictly finer than both the lower limit topology and the cocountable topology on $\mathbb{R}$, then τ must be the discrete topology on $\mathbb{R}$.

    (c) If $\mathcal{B}$ and $\mathcal{B}'$ are bases for topologies τ and τ' respectively on a nonempty set X, then $\{B \cap B' : B \in \mathcal{B},\ B' \in \mathcal{B}'\}$ must be a basis for some topology on X.

    (d) If τ and τ' are topologies on a nonempty set X, then $\tau \cup \tau'$ must be a basis for some topology on X.

94. It is known that if $\mathbb{R}$ is equipped with either the indiscrete topology or the cofinite topology, then the sequence $(\frac{1}{n})_{n=1}^{\infty}$ converges to 1. Mention (with

justification) another topology on $\mathbb{R}$ with respect to which the sequence $(\frac{1}{n})_{n=1}^{\infty}$ converges to 1.

95. Consider $X = (0, 1]$ with the relative topology τ induced by the usual topology on $\mathbb{R}$. Show that the sequence $(\frac{1}{n})_{n=1}^{\infty}$ in X does not converge in the topological space (X, τ).

96. If $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, then in the topological space $(\mathbb{R}, \tau)$, prove that the sequence

    (a) $(n)_{n=1}^{\infty}$ converges to every point of $\mathbb{R}$.

    (b) $(-n)_{n=1}^{\infty}$ does not converge.

    (c) $((-1)^n)_{n=1}^{\infty}$ converges to x if and only if $x \in (-\infty, -1]$.

97. If $\tau = \{G \subset \mathbb{R} : 0 \in G\} \cup \{\emptyset\}$, then determine (with justification) all the convergent sequences and all their limits in the topological space $(\mathbb{R}, \tau)$.

98. State TRUE or FALSE with justification for each of the following statements.

    (a) The sequence $\left(\frac{(-1)^n}{n}\right)_{n=1}^{\infty}$ converges in $\mathbb{R}$ with the lower limit topology.

    (b) There exists a topology τ on $\mathbb{R}$ such that the sequence $(1, 2, 3, ...)$ converges to the unique limit 0 in $(\mathbb{R}, \tau)$.

    (c) If (X, τ) is a topological space such that every sequence in X converges to every point of X in (X, τ), then τ must be the indiscrete topology on X.

Problem Set 2: Continuous maps and homeomorphisms

1. Let X, Y be topological spaces and let $x_0 \in X$. If $f : X \to Y$ is continuous at x_0 and G is an open set in Y containing $f(x_0)$, then is it necessary that $f^{-1}(G)$ is an open set in X? Justify.

2. Let A be a nonempty subset of a topological space X and let $f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus A. \end{cases}$
   Prove that $f : X \to (\mathbb{R}, \tau_u)$ is continuous if and only if $\partial A = \emptyset$, where τ_u is the usual topology on $\mathbb{R}$.

3. Let τ_f and τ_c denote respectively the cofinite topology and the cocountable topology on $\mathbb{R}$. If $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$
then determine all the points of $\mathbb{R}$ at which $g : (\mathbb{R}, \tau_c) \to (\mathbb{R}, \tau_f)$ is continuous.

4. Let τ_l denote the lower limit topology on $\mathbb{R}$ and let $f(x) = -x$ for all $x \in \mathbb{R}$. Show that $f : (\mathbb{R}, \tau_l) \to (\mathbb{R}, \tau_l)$ is discontinuous at each point of $\mathbb{R}$.

5. Let $f : X \to \mathbb{R}$ be continuous at a point $x_0 \in X$, where X is a topological space and $\mathbb{R}$ is equipped with the usual topology. If $f(x_0) > 0$, then show that there exists an open set G in X containing x_0 such that $f(x) > 0$ for all $x \in G$.

6. Let (X, τ) and (Y, τ') be topological spaces such that every map from (X, τ) to (Y, τ') is continuous. Prove that τ is the discrete topology on X or τ' is the indiscrete topology on Y.

7. Let X, Y be nonempty sets and let $x_0 \in X$, $y_0 \in Y$. If $\tau = \{G \subset X : x_0 \in G\} \cup \{\emptyset\}$ and $\tau' = \{H \subset Y : y_0 \in H\} \cup \{\emptyset\}$, then show that a map $f : (X, \tau) \to (Y, \tau')$ is continuous if and only if $f(x_0) = y_0$ or f is constant.

8. Let τ_f denote the cofinite topology on $\mathbb{R}$. Examine whether $\varphi : (\mathbb{R}, \tau_f) \to (\mathbb{R}, \tau_f)$ is continuous, if φ is defined as

   (a) $\varphi(x) = x^3 - 4x - 1$ for all $x \in \mathbb{R}$.

   (b) $\varphi(x) = \sin x$ for all $x \in \mathbb{R}$.

   (c) $\varphi(x) = x + \sin x$ for all $x \in \mathbb{R}$.

   (d) $\varphi(x) = e^x$ for all $x \in \mathbb{R}$.

9. Show that $\tau = \{G \subset \mathbb{N} : n \in G \text{ and } m|n \Rightarrow m \in G\}$ is a topology on $\mathbb{N}$ which is different from the discrete topology on $\mathbb{N}$.
Show also that a map $f : (\mathbb{N}, \tau) \to (\mathbb{N}, \tau)$ is continuous if and only if for all $m, n \in \mathbb{N}$, $m|n \Rightarrow f(m)|f(n)$.

10. Let $\{\tau_\alpha\}_{\alpha \in \Lambda}$ be a family of topologies on a nonempty set X, let (Y, τ) be a topological space and let $f : X \to Y$. Prove that $f : \left(X, \bigcap\limits_{\alpha \in \Lambda} \tau_\alpha\right) \to (Y, \tau)$ is continuous if and only if for each $\alpha \in \Lambda$, $f : (X, \tau_\alpha) \to (Y, \tau)$ is continuous.

11. Let $\{\tau_\alpha\}_{\alpha \in \Lambda}$ be a family of topologies on a nonempty set Y, let (X, τ) be a topological space and let $f : X \to Y$. If τ' is the unique smallest topology on

Y containing $\bigcup\limits_{\alpha \in \Lambda} \tau_\alpha$, then show that $f : (X, \tau) \to (Y, \tau')$ is continuous if and only if for each $\alpha \in \Lambda$, $f : (X, \tau) \to (Y, \tau_\alpha)$ is continuous.

12. Let τ_f and τ_u denote respectively the cofinite topology and the usual topology on $\mathbb{R}$. If $\varphi : (\mathbb{R}, \tau_f) \to (\mathbb{R}, \tau_u)$ is continuous, then prove that φ is a constant map.

13. Let X, Y be topological spaces and let $f : X \to Y$. Prove that f is continuous iff
$\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for all $B \subset Y$.
Also, show that the continuity of f need not give the equality in the above inclusion for some $B \subset Y$.

14. Let X, Y be topological spaces and let $f : X \to Y$. Prove that f is continuous iff
$\partial(f^{-1}(B)) \subset f^{-1}(\partial B)$ for all $B \subset Y$.
Also, show that the continuity of f need not give the equality in the above inclusion for some $B \subset Y$.

15. Let X, Y be topological spaces and let $f : X \to Y$. Prove that f is continuous iff
$f^{-1}(B^0) \subset (f^{-1}(B))^0$ for all $B \subset Y$.
Also, show that the continuity of f need not give the equality in the above inclusion for some $B \subset Y$.

16. Let X, Y be topological spaces and let $f : X \to Y$. Prove that f is continuous if and only if $f(A') \subset \overline{f(A)}$ for all $A \subset X$.
Also, show that the continuity of f need not give the equality in the above inclusion for some $A \subset X$.

17. Let $f : X \to Y$ be continuous, where X, Y are topological spaces. If $x \in X$ is a limit point of a subset A of X, then is it necessary that $f(x)$ is a limit point of $f(A)$? Justify.

18. Let X, Y be topological spaces and let $f : X \to Y$ be continuous and onto. If A is a dense set in X, then prove that $f(A)$ is dense in Y and that the onto condition on f is, in general, necessary for this.
Show also that $f(A)$ can be dense in Y even when $A \subset X$ is not dense in X and $f : X \to Y$ is neither continuous nor onto.

19. Consider the topology $\tau = \{G \subset \mathbb{R} : \mathbb{R} \setminus G \text{ is finite or } 0 \in \mathbb{R} \setminus G\}$ on $\mathbb{R}$. Let τ_u be the usual topology on $\mathbb{R}$ and let $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_u)$ be continuous. Show that there exists a countable subset A of $\mathbb{R}$ such that $f(x) = f(0)$ for all $x \in \mathbb{R} \setminus A$.

20. Consider the topology τ on $\mathbb{R}$ having $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \cap \mathbb{Q} : a, b \in \mathbb{R}, a < b\}$ as a basis. Also, let τ_u be the usual topology on $\mathbb{R}$.

    (a) Examine whether $\mathbb{R} \setminus \mathbb{Q}$ is closed in the topological space $(\mathbb{R}, \tau)$.

    (b) If $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_u)$ is a continuous map such that $f(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$, then show that $f(x) = 0$ for all $x \in \mathbb{R}$.

21. Let X, Y be topological spaces. Prove that a map $f : X \to Y$ is an open map iff
    $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$ for all $B \subset Y$.

22. Let X, Y be topological spaces. Show that $f : X \to Y$ is a closed map if and only if for each $y \in Y$ and for each open set G in X with $f^{-1}(\{y\}) \subset G$, there exists an open set H in Y containing y such that $f^{-1}(H) \subset G$.

23. Let X, Y, Z be topological spaces and let $f : X \to Y$ and $g : Y \to Z$ be continuous maps such that $g \circ f : X \to Z$ is a homeomorphism. If g is one-one, then show that both f and g are homeomorphisms.

24. Examine whether the following pairs of topological spaces are homeomorphic.

    (a) $(\mathbb{R}, \tau_u)$ and $(\mathbb{R}, \tau_c)$, where τ_u and τ_c are respectively the usual topology and the cocountable topology on $\mathbb{R}$.

    (b) $(\mathbb{R}, \tau_f)$ and $(\mathbb{R}, \tau_c)$, where τ_f and τ_c are respectively the cofinite topology and the cocountable topology on $\mathbb{R}$.

    (c) $(\mathbb{R}, \tau)$ and $(\mathbb{R}, \tau')$, where $\tau = \{G \subset \mathbb{R} : 0 \in G\} \cup \{\emptyset\}$ and $\tau' = \{G \subset \mathbb{R} : 0 \notin G\} \cup \{\mathbb{R}\}$.

25. Consider $\mathbb{R}^2$ with the usual topology. Show that the subspaces $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, $\{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$ and $\{(x, y) \in \mathbb{R}^2 : 2x^2 + 3y^2 < 1\}$ of $\mathbb{R}^2$ are homeomorphic.

26. State TRUE or FALSE with justification for each of the following statements.

(a) If X is an indiscrete topological space, then there cannot exist any non-constant continuous map $f : X \to (\mathbb{R}, \tau_u)$.

(b) If τ_u is the usual topology on $\mathbb{R}$, then every one-one, onto and continuous map
$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_u)$ is a homeomorphism.

(c) Every continuous map $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_u)$ is open or closed.

(d) If τ is the cofinite topology on $\mathbb{R}$, then every bijection $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ is a homeomorphism.

(e) If X, Y are topological spaces such that X is homeomorphic to a subspace of Y and Y is homeomorphic to a subspace of X, then X and Y must be homeomorphic.

27. Let X, Y be topological spaces and let $f : X \to Y$. Show that the map φ of X to the subspace $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $X \times Y$, defined by $\varphi(x) = (x, f(x))$ for all $x \in X$, is a homeomorphism if and only if f is continuous.
Deduce that every topological space X is homeomorphic to the diagonal $\triangle = \{(x, x) : x \in X\}$ of $X \times X$.

28. Let τ_l be the lower limit topology on $\mathbb{R}$. Examine whether the set $\{(x, y) \in \mathbb{R}^2 : x + y < 0\}$ is closed in the product space $(\mathbb{R}, \tau_l) \times (\mathbb{R}, \tau_l)$.

29. If $X = (\mathbb{R}, \tau_c)$, where τ_c is the cocountable topology on $\mathbb{R}$, then determine the interior of the set $(\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q}) \times \cdots$ in the countably infinite product $X^{\mathbb{N}}$ with the product topology.

30. If τ_c is the cocountable topology on $\mathbb{R}$, then show that the product topology of $(\mathbb{R}, \tau_c)$ and $(\mathbb{R}, \tau_c)$ is strictly finer than the cocountable topology on $\mathbb{R} \times \mathbb{R}$.

31. Let X, Y be topological spaces and let $A \subset X$, $B \subset Y$. Prove that
$\partial(A \times B) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$.

32. Let A, B, C, D be topological spaces and let $f : A \to B$ and $g : C \to D$ be continuous. If $\varphi(a, c) = (f(a), g(c))$ for all $(a, c) \in A \times C$, then show that $\varphi : A \times C \to B \times D$ is continuous.

33. Let Λ be an infinite set and let for each $\alpha \in \Lambda$, X_α be a discrete space. Prove that $\prod\limits_{\alpha \in \Lambda} X_\alpha$ is a discrete space in the box topology, but if for each $\alpha \in \Lambda$, X_α

has more than one point, then $\prod_{\alpha\in\Lambda} X_\alpha$ is not a discrete space in the product topology.

34. Consider the usual topology on $\mathbb{R}$ and let $A = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \{n \in \mathbb{N} : x_n \neq 0\} \text{ is finite}\}$. Show that A is dense in $\mathbb{R}^{\mathbb{N}}$ with the product topology.

35. Determine the closure of $\{(x_n) \in \mathbb{R}^{\mathbb{N}} : \{n \in \mathbb{N} : x_n \neq 0\} \text{ is finite}\}$ in $\mathbb{R}^{\mathbb{N}}$ with the box topology.

Problem Set 3: Quotient spaces and compact spaces

1. State TRUE or FALSE with justification for each of the following statements.

   (a) If $A = \{x \in \mathbb{R} : x^8 - x^7 \leq 200\}$ and $B = \{x^2 - 2x : x \in (0, \infty)\}$, then $A \cap B$ is a compact set in $\mathbb{R}$ with the usual topology.

   (b) If every proper closed subset of a topological space X is compact, then X must be compact.

   (c) If every subset of a Hausdorff space X is compact, then X must be a discrete space.

   (d) A topological space in which every compact subset is closed must be Hausdorff.

   (e) If X is a topological space such that every continuous map $f : X \to \mathbb{R}$ (with the usual metric on $\mathbb{R}$) is bounded, then X must be compact.

   (f) Every topological space which is a continuous image of a topological space with the Bolzano-Weierstrass property must have the Bolzano-Weierstrass property.

   (g) If τ_c denotes the cocountable topology on $\mathbb{R}$, then the topological space $(\mathbb{R}, \tau_c)$ is locally compact.

   (h) The one-point compactification of $\mathbb{N}$ (with the usual topology) is homeomorphic to $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ (with the usual topology).

2. Let $P : X \to Y$ be continuous. Let $f : Y \to X$ be a continuous function such that $p \circ f = I$. Show that p is a quotient map.

3. Let A be subset of topological space X, and let $r : X \to A$ be such that $r(a) = a$ for each $a \in A$. Show that (retraction) r is a quotient map.

4. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $\pi_1(x, y) = x$. Let $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} :$ either $x \geq 0$ or $y = 0($ or both$)\}$. Let q be the restriction of p to A. Show that q is a quotient map that is neither open nor closed.

5. Let $p : X \to Y$ be an open map and A is open in X. Show that $q :\to P(A)$, where $q = p|A$, is an open map.

6. Define equivalence relations on $X = \mathbb{R}^2$ by

   (a) $x_o, y_o) \sim (x_1, y_1)$ if $x_o + y_o^2 = x_1 + y_1^2$,

   (b) $x_o, y_o) \sim (x_1, y_1)$ if $x_o^2 + y_o^2 = x_1^2 + y_1^2$.

   Let X^* be the corresponding quotient space. Find the surfaces in each case which is homeomorphic to corresponding X^*.

7. Let Y be the quotient space obtained from $K-$topology $\mathbb{R}_K$ by collapsing the set K to a point. Let $p : \mathbb{R}_K \to Y$ be a quotient map.

   (a) Show that Y satisfies T_1 axiom but not Hausdorff.

   (b) Show that $p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is not a quotient map.

8. Show that $[0, 1]$ is not limit point compact in the lower limit topological space $\mathbb{R}_l$.

9. If X is compact Hausdorff space under two topologies τ and τ', then show that either $\tau = \tau'$ or not comparable.

10. Let X be any nonempty set and let $x_0 \in X$. If $\tau = \{G \subset X : x_0 \notin G\} \cup \{X\}$, then find all the compact subsets of the topological space (X, τ).

11. Consider the topology τ on $\mathbb{R}$ having $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$ as a basis, where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Examine whether $[0, 1]$ is a compact set in the topological space $(\mathbb{R}, \tau)$.

12. Let τ_c be the cocountable topology on a nonempty set X. Find all the (a) compact (b) sequentially compact subsets of the topological space (X, τ_c).

13. If $a, b \in \mathbb{R}$ such that $a < b$, then show that $[a, b]$ is not a compact subset of $\mathbb{R}$ with the lower limit topology.

14. Let (x_n) be a sequence in a topological space X and let $x_n \to x \in X$. Show that
$\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is a compact subset of X.

15. Prove that the union of finitely many compact sets in a topological space is compact.
Also, show that the union of infinitely many compact sets in a topological space need not be compact.

16. Let A and B be compact subsets of a topological space X. Show that $A \cap B$ need not be compact but if X is Hausdorff, then $A \cap B$ must be compact.

17. Let G be topological group.
    (a) If A and B are compact in G, then show that $A \cdot B$ is compact.
    (b) Let H be subgroup of G and $p :\to G/H$ be the quotient map. If H is compact, show that p is closed.
    (c) Let H be a compact subgroup of G. Show that if G/H is compact, then G is compact.

18. Let A and B be compact subspaces of the topological spaces X and Y respectively. If W is an open set in $X \times Y$ containing $A \times B$, then prove that there exist open sets G and H in X and Y respectively such that $A \times B \subset G \times H \subset W$.

19. For each $n \in \mathbb{N}$, let $X_n = [0, 1]$ with the usual topology, and $X = \prod_{n=1}^{\infty} X_n$.
    (a) Show that X is not limit point compact with respect to uniform topology.
    (b) Show that $\prod_{n=1}^{\infty} X_n$ with the box topology is not compact.

20. Let X, Y be topological spaces with Y compact. Prove that the projection map $p_X : X \times Y \to X$ is a closed map.

21. Let X, Y be topological spaces with Y compact. If $f : X \to Y$ is such that $\{(x, f(x)) : x \in X\}$ (*i.e.* the graph of f) is closed in the product space $X \times Y$, then prove that f is continuous.

22. Show that the closure of a compact subset of a topological space need not be compact.
Show, however, that the closure of a compact subset of a regular space is compact.

23. Let X be a topological space. If for each $x \in X$, there exists an open set G in X containing x such that $\overline{G}$ is a compact Hausdorff space, then show that X is Hausdorff.

24. Show that a compact topological space need not be separable but that every compact metrizable space must be separable.

25. If X is a Lindelöf space and Y is a compact space, then show that the product space $X \times Y$ is a Lindelöf space.

26. Let A_n be sequence of closed nowhere dense sets in compact Hausdorff space X. Show that $(\cup A_n)^\circ = \emptyset$.

27. Let X and Y be topological spaces such that Y is compact. Let $p : X \to Y$ be continuous, closed and onto such that $p^{-1}(\{y\})$ is a compact set in X for each $y \in Y$. Show that X is compact.

28. Let X and Y be topological spaces such that X is second countable. Let $f : X \to Y$ be continuous, closed and onto such that $f^{-1}(\{y\})$ is a compact set in X for each $y \in Y$. Show that Y is second countable.

29. Let f be a continuous map from a compact space X onto a Hausdorff space Y and let g be a map from Y to a topological space Z. If $g \circ f$ is continuous, then prove that g is continuous.

30. Let X be a metrizable topological space. Prove that the following statements are equivalent.

    (a) X is compact.

    (b) X is bounded with respect to every metric on X that induces the topology of X.

    (c) Every continuous map $f : X \to \mathbb{R}$ (with the usual metric on $\mathbb{R}$) is bounded.

31. Let (X, d) be a compact metric space and let $f : X \to X$ be such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Prove that f is onto and so f is a homeomorphism.
    Show also that the compactness of (X, d) is, in general, necessary.

32. Let X be a compact Hausdorff space and let $f : X \to X$ be continuous. Show that there exists a nonempty compact set K in X such that $f(K) = K$.

33. Show that a T_1-space X has the Bolzano-Weierstrass property if and only if every countable open cover of X has a finite subcover.

34. Let X be a topological space with the Bolzano-Weierstrass property and let Y be a first countable space. Show that the projection map $p_Y : X \times Y \to Y$ is closed.

35. Let Y be a nonempty dense subset of a Hausdorff space X. If the subspace Y is locally compact, then show that Y is open in X.

36. Let X be a compact Hausdorff space and let $\mathbb{R}$ be equipped with the usual topology. Consider the real vector space $C(X, \mathbb{R})$ of all continuous maps from X to $\mathbb{R}$, where the vector space operations are defined pointwise. Prove that $C(X, \mathbb{R})$ is finite dimensional if and only if X is finite.

37. Let K be a nonempty compact subset and let C be a nonempty closed subset of a locally compact Hausdorff space X such that $K \cap C = \emptyset$. Prove that there exists a continuous map $f : X \to [0, 1]$ (with the usual topology) such that $f(K) = \{0\}$ and $f(C) = \{1\}$.

38. Let f_n be real-valued sequence of monotone increasing continuous function on a compact topological space X. If f_n converges point wise to f, than show that f_n converges to f uniformly.

39. Show that a compact Hausdorff space is metrizable if and only if it is second countable.

Problem Set 4: Connected spaces

1. State TRUE or FALSE with justification for each of the following statements.

   (a) It is possible that $\mathbb{R}^2$ can be written as countable union of connected paths.

   (b) The cardinality of set of all the polynomials on $\mathbb{R}$ such that complement of their zero set are connected is countable.

   (c) There exists a non-empty open and connected set $A \subset \mathbb{R}^n$ such that every real valued function on A is continuous.

(d) If a metric space X is path connected, then there exists a continuous function $f : [0, 1] \to X$ which is onto.

(e) There exists a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ such that the graph G_f is connected in $\mathbb{R}^2$ but $\text{int}(\overline{G_f})$ is non-empty in $\mathbb{R}^2$.

(f) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^3 \in \mathbb{R} \setminus \mathbb{Q}\}$ is a disconnected subset of $\mathbb{R}^2$ (with the usual topology).

(g) If a topological space X has only finitely many components, then each component of X must be open in X.

(h) If τ_l denotes the lower limit topology on $\mathbb{R}$, then the topological space $(\mathbb{R}, \tau_l)$ is totally disconnected.

(i) $\mathbb{R}$ with the cofinite topology is a locally connected space.

2. Show that a topological space X is connected if and only if for every $x, y \in X$ with $x \neq y$, there exists a connected subspace of X containing both x and y.

3. Prove that a topological space X is connected if and only if every nonempty proper subset of X has a nonempty boundary.

4. Let C be a connected subspace of a topological space X and let $A \subset X$ such that $C \cap A \neq \emptyset$ and $C \cap (X \setminus A) \neq \emptyset$. Prove that $C \cap \partial A \neq \emptyset$.

5. Consider the topology τ on $\mathbb{R}$ having $\{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \cap \mathbb{Q} : a, b \in \mathbb{R}, a < b\}$ as a basis. Show that the topological space $(\mathbb{R}, \tau)$ is connected.

6. Show that $\tau = \{G \subset \mathbb{R} : \mathbb{R} \setminus G \text{ is a compact set in } (\mathbb{R}, \tau_u)\} \cup \{\emptyset\}$ is a topology on $\mathbb{R}$, where τ_u is the usual topology on $\mathbb{R}$. Examine whether the topological space $(\mathbb{R}, \tau)$ is (a) compact (b) connected.

7. Let A and B be nonempty closed (or, open) subsets of a topological space such that both $A \cup B$ and $A \cap B$ are connected. Show that A and B are also connected.

8. If A and B are connected subsets of a topological space such that $A \cap \overline{B} \neq \emptyset$, then prove that $A \cup B$ is connected.

9. If X, Y are connected spaces and if $A \subsetneq X$, $B \subsetneq Y$, then show that $(X \times Y) \setminus (A \times B)$ is connected.

10. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected subspaces of a topological space X such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Show that $\bigcup\limits_{n=1}^{\infty} A_n$ is connected.

11. If $\{A\} \cup \{A_\alpha : \alpha \in \Lambda\}$ is a class of connected subsets of a topological space such that $A \cap A_\alpha \neq \emptyset$ for all $\alpha \in \Lambda$, then show that $A \cup \left(\bigcup\limits_{\alpha \in \Lambda} A_\alpha \right)$ is connected.

12. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a class of connected subsets of a topological space such that $A_\alpha \cap A_\beta \neq \emptyset$ for all $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$. Prove that $\bigcup\limits_{\alpha \in \Lambda} A_\alpha$ is connected.

13. Let $\{A_n\}_{n=1}^{\infty}$ be a decreasing sequence of nonempty compact connected sets in a Hausdorff space. Prove that $\bigcap\limits_{n=1}^{\infty} A_n$ is nonempty, compact and connected.

14. Let X be a connected space and let there exist a non-constant continuous map from X to $\mathbb{R}$ (with the usual topology). Show that X is uncountable.

15. Prove that every connected T_3-space containing at least two points must be uncountable.

16. Let X be a connected space and let τ_l be the lower limit topology on $\mathbb{R}$. Prove that every continuous map from X to $(\mathbb{R}, \tau_l)$ is a constant map.
(Hence, in particular, every continuous map from $(\mathbb{R}, \tau_u)$ to $(\mathbb{R}, \tau_l)$ is a constant map, where τ_u denotes the usual topology on $\mathbb{R}$.)

17. For each $m, n \in \mathbb{N}$, let $B_{m,n} = \{mk + n : k \in \mathbb{Z}\} \cap \mathbb{N}$. Show that
    (a) $\{B_{m,n} : m, n \in \mathbb{N}, \text{ g.c.d.}(m, n) = 1\}$ is a basis for some topology τ on $\mathbb{N}$.
    (b) for each prime $p \in \mathbb{N}$, $\{pk : k \in \mathbb{N}\}$ is a closed set in $(\mathbb{N}, \tau)$ and hence deduce that there exist infinitely many prime numbers.
    (c) $(\mathbb{N}, \tau)$ is connected, Hausdorff but not compact.
    (d) if P is the set of all primes, then $P^0 = \emptyset$.

18. Consider $\mathbb{R}^2$ with the usual topology. If A is a countable subset of $\mathbb{R}^2$, then show that $\mathbb{R}^2 \setminus A$ is path connected.

19. Prove that every open connected subspace of $\mathbb{R}^2$ (with the usual topology) is path connected.

Problem Set 5: Countability and separation axioms

1. State TRUE or FALSE with justification for each of the following statements.

   (a) If A and B are nonempty subsets of topological spaces X and Y respectively such that $A \times B$ is closed in the product space $X \times Y$, then A must be closed in X and B must be closed in Y.

   (b) If τ_l denotes the lower limit topology on $\mathbb{R}$, then $\{(x, y) \in \mathbb{R}^2 : x^2+y^2 \geq 1\}$ is an open set in the product space $(\mathbb{R}, \tau_l) \times (\mathbb{R}, \tau_l)$.

   (c) There cannot exist topologies τ, τ' on an infinite set X such that the product topology for (X, τ) and (X, τ') coincides with the cofinite topology on $X \times X$.

   (d) If τ_u and τ_l denote respectively the usual topology and the lower limit topology on $\mathbb{R}$, then the product space $(\mathbb{R}, \tau_u) \times (\mathbb{R}, \tau_l)$ is not metrizable.

2. Prove that every closed (respectively, open) subset of a metrizable space is a G_δ (respectively, an F_σ) set.
   Also, show that the metrizability condition is, in general, necessary.

3. Let (X, τ) be a metrizable topological space and let τ_u be the usual topology on $\mathbb{R}$. Show that τ is the weakest topology on X with respect to which every continuous map from (X, τ) to $(\mathbb{R}, \tau_u)$ remains continuous.

4. Let X be a first countable space and let $A \subset X$. Prove that

   (a) A is closed in X if and only if for every sequence $(a_n) \subset A$ and for every $x \in X$, $a_n \to x \Rightarrow x \in A$.

   (b) A is open in X if and only if for every sequence $(x_n) \subset X$ and for every $a \in A$, $x_n \to a \Rightarrow x_n \in A$ eventually.

5. If $\tau = \{G \subset \mathbb{R} : 0 \in G\} \cup \{\emptyset\}$, then show that the topological space $(\mathbb{R}, \tau)$ is first countable but not second countable.

6. Let X be a first countable space and let $G \subset X$. If for every nonempty countable set A in X, $G \cap A$ is open in the subspace A, then show that G is open in X.

7. Let $X = \mathbb{Z}_+ \times \mathbb{Z}_+$, where $\mathbb{Z}_+ = \{0, 1, 2, ...\}$, and let
   $\tau = \mathcal{P}(X \setminus \{(0, 0)\}) \cup \{G \subset X : (0, 0) \in G, \{m \in \mathbb{Z}_+ : \{n \in \mathbb{Z}_+ : (m, n) \notin$

$G\}$ is infinite$\}$ is finite$\}$. Prove that τ is a topology on X which is different from the discrete topology on X.
Determine all the convergent sequences in the topological space (X, τ) and hence conclude that (X, τ) is not first countable.
(The topological space (X, τ) is called the Arens-Fort space. It shows that a topological space $(Z, \mathfrak{J})$ need not be first countable even if Z is a countable set.)

8. Prove that a topological space is second countable if and only if it has a countable subbasis.

9. Let A be an uncountable subset of a second countable space. Show that uncountably many points of A are limit points of A.

10. Let X be a separable space. Prove that every class of pairwise disjoint open sets in X is countable.
Hence deduce that the set of all isolated points of X is countable.

11. Prove that every topological space can be considered as a subspace of a separable space.
(From this it follows immediately that a subspace of a separable space need not be separable.)

12. If Y is a nonempty open subset of a separable space (X, τ), then show that the subspace $(Y, \tau|_Y)$ is separable.

13. Let Y be a dense subspace of a first countable separable space X. Show that Y is separable.

14. Let τ_l denote the lower limit topology on $\mathbb{R}$. Prove that every subspace of the topological space $(\mathbb{R}, \tau_l)$ is separable.

15. Prove or disprove: $\mathbb{R}$ with the cocountable topology is a Lindelöf space.

16. Prove that every metrizable Lindelöf space is second countable.

17. Let X be any set with at least two elements. Show that there exists a topology τ on X such that (X, τ) is a T_0-space but not a T_1-space.

18. Prove that a topological space X is a T_0-space if and only if for all $x, y \in X$ with $x \neq y$, $\overline{\{x\}} \neq \overline{\{y\}}$.

19. For a topological space (X, τ), prove that the following statements are equivalent.

    (a) (X, τ) is a T_1-space.

    (b) For each $x \in X$, $\{x\} = \bigcap\{G \in \tau : x \in G\}$.

    (c) τ is finer than the cofinite topology on X.

20. Let (X, τ) be a T_1-space and $A \subset X$. Show that $A = \bigcap\{G \in \tau : A \subset G\}$.

21. Show that for every convergent sequence in a topological space X to have a unique limit in X, it is necessary but not sufficient that X is a T_1-space.

22. Let X be a T_1-space and let $A \subset X$, $x \in X$. Show that

    (a) $x \in A'$ if and only if every open set in X containing x contains infinitely many points of A.

    (b) A' is closed in X.

23. Let X be a first countable T_1-space and let $A \subset X$, $x \in X$. Prove that $x \in A'$ if and only if there exists a sequence of distinct points in A converging to x in X.
    Also, show that both T_1 and first countability conditions are, in general, necessary.

24. Show that in a first countable T_1-space, every singleton set is a G_δ set.
    Show that both first countability and T_1 conditions are, in general, necessary.
    Also, give an example of a topological space which is not first countable but in which every singleton set is a G_δ set.

25. Prove that a topological space (X, τ) is a Hausdorff space if and only if for each $x \in X$,
    $\{x\} = \bigcap\{\overline{G} : G \in \tau, x \in G\}$.

26. If X is a Hausdorff space, then show that

    (a) for each $x \in X$, $\bigcap\{F \subset X : F \text{ is closed in } X, x \in F\} = \{x\}$.

    (b) for each $x \in X$, $\bigcap\{G \subset X : G \text{ is open in } X, x \in G\} = \{x\}$.

    Also, show that a topological space X satisfying (a) and (b) need not be Hausdorff.

27. Show that a topological space X is a Hausdorff space if and only if $\{(x, x) : x \in X\}$ is a closed subset of the product space $X \times X$.

28. Show that a topological space X is a discrete space if and only if $\{(x, x) : x \in X\}$ is an open subset of the product space $X \times X$.

29. Let X be a Hausdorff space and let $f : X \to X$ be a continuous map such that $f \circ f = f$. Prove that $f(X)$ is a closed subset of X.

30. Let X, Y be topological spaces and let $f : X \to Y$ and $g : Y \to X$ be continuous such that $g(f(x)) = x$ for all $x \in X$. If Y is a Hausdorff space, then show that X is a Hausdorff space and $f(X)$ is closed in Y.

31. Let D be a dense subset of a topological space X and let Y be a Hausdorff space. If $f : X \to Y$ and $g : X \to Y$ are continuous such that $f(x) = g(x)$ for all $x \in D$, then show that $f(x) = g(x)$ for all $x \in X$.

32. Let X_0 be a dense subspace of a topological space X and let Y be a Hausdorff space. Prove that every continuous map $f_0 : X_0 \to Y$ can have at most one continuous extension $f : X \to Y$.
    Show also that a continuous map $f_0 : X_0 \to Y$ need not have any continuous extension $f : X \to Y$.

33. Let X and Y be Hausdorff spaces and let $f : X \to Y$ be onto. Show that $f : X \to Y$ is a homeomorphism if and only if $\overline{A} = f^{-1}(\overline{f(A)})$ for all $A \subset X$.

34. Let $x_1, ..., x_n$ be distinct points of a Hausdorff space X. Prove that there exist pairwise disjoint open sets $G_1, ..., G_n$ in X such that $x_i \in G_i$ for $i = 1, ..., n$.

35. Prove that every infinite Hausdorff space contains an infinite set A such that each point of A is an isolated point of A.
    (Hence every infinite Hausdorff space contains an infinite discrete subspace).

36. Prove that a first countable space X is a Hausdorff space if and only if every convergent sequence in X has a unique limit in X.

37. Let X, Y be topological spaces with Y Hausdorff. If $f : X \to Y$ is continuous, then prove that $G_f = \{(x, f(x)) : x \in X\}$ (the graph of f) is closed in the product space $X \times Y$.
    Show that the Hausdorff condition on Y is, in general, necessary.

38. Prove that a topological space Y is Hausdorff if and only if for every topological space X and for any continuous maps $f : X \to Y$ and $g : X \to Y$, the set $\{x \in X : f(x) = g(x)\}$ is closed in X. Hence show that the set of all fixed points of a continuous map from a Hausdorff space Y to itself is closed in Y.

39. State TRUE or FALSE with justification for each of the following statements.

    (a) If τ and τ' are topologies on a nonempty set X such that both (X, τ) and (X, τ') are T_1-spaces, then $(X, \tau \cap \tau')$ must be a T_1-space.

    (b) There exists a Hausdorff space with exactly 100 (distinct) open sets.

    (c) If $x_1, x_2, \ldots$ are distinct points in a Hausdorff space X, then there must exist pairwise disjoint open sets $G_1, G_2, \ldots$ in X such that $x_n \in G_n$ for all $n \in \mathbb{N}$.

    (d) Every second countable Hausdorff space is metrizable.

    (e) If X and Y are topological spaces such that the product space $X \times Y$ is Hausdorff, then both X and Y must be Hausdorff.

    (f) If $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, then the topological space $(\mathbb{R}, \tau)$ is normal but not regular.

40. Prove that every regular T_0-space is a T_3-space but that a normal T_0-space need not be a T_4-space.

41. Let X be a T_3-space and let $x, y \in X$ with $x \neq y$. Prove that there exist open sets G and H in X containing x and y respectively such that $\overline{G} \cap \overline{H} = \emptyset$.

42. Let A and B be disjoint closed subsets of a normal space X. Prove that there exist open subsets G and H of X containing A and B respectively such that $\overline{G} \cap \overline{H} = \emptyset$.

43. Let X be a T_4-space and let Y be a topological space. If $f : X \to Y$ is continuous, closed and onto, then show that Y is a T_4-space.

44. Let X be a normal space. Show that X is regular if and only if X is completely regular.

45. Let (X, τ) be a completely regular space. Show that the weak topology on X induced by the family of all continuous maps $f : (X, \tau) \to (\mathbb{R}, \tau_u)$ is the same as τ.

46. Let K be a compact set and C be a closed set in a regular space X such that $K \cap C = \emptyset$. Prove that there exist disjoint open sets G and H in X such that $K \subset G$ and $C \subset H$.

47. Show that a compact Hausdorff space is metrizable if and only if it is second countable.

Problem Set 6: Nets, filters, and general convergence

1. Give an example of a topological space X and a subset $A \subset X$ such that there exists a point $x \in \overline{A}$ which is *not* the limit of any sequence from A.

2. Let $(x_i)_{i \in I}$ be a net in a topological space X and let $x \in X$. Prove that the following are equivalent:

   (a) $x_i \to x$;

   (b) for every neighborhood U of x there exists $i_0 \in I$ such that $i \geq i_0$ implies $x_i \in U$;

   (c) for every open set U with $x \in U$, the set $\{i \in I : x_i \in U\}$ is cofinal in I.

3. Prove that $x \in \overline{A}$ if and only if there exists a net (a_i) in A converging to x.

4. Prove that a function $f : X \to Y$ between topological spaces is continuous if and only if it preserves limits of nets: whenever $x_i \to x$ in X, we have $f(x_i) \to f(x)$ in Y.

5. Let $\mathcal{F}$ be a filter on X and $x \in X$. Prove that x is a limit point of $\mathcal{F}$ if and only if every neighborhood of x belongs to $\mathcal{F}$ (equivalently, $\mathcal{N}(x) \subset \mathcal{F}$).

6. Show that if (X, τ) is first countable, then every net has a subnet which is a sequence. Conclude that in first countable spaces, closure can be tested using sequences.

7. [$\star$] Let (x_i) be a net in X. Prove that x is a cluster point of (x_i) (every neighborhood of x meets $\{x_i : i \geq i_0\}$ for all i_0) if and only if some subnet of (x_i) converges to x.

8. Let X be compact. Prove that every net in X has a convergent subnet. (This is the net version of sequential compactness, and it holds without countability assumptions.)

9. Let $(X_j)_{j \in J}$ be a family of spaces and $X = \prod_{j \in J} X_j$ with the product topology. Prove that a net $x_i = (x_i^{(j)})$ converges to $x = (x^{(j)})$ in X if and only if for every $j \in J$ we have $x_i^{(j)} \to x^{(j)}$ in X_j.

10. [$\star$] Let $\mathcal{U}$ be an ultrafilter on X. Show that X is compact if and only if every ultrafilter on X converges.

Problem Set 7: Compactness, products, and compactifications

1. Prove that a continuous image of a compact space is compact. Give an example showing that the preimage of a compact set under a continuous map need not be compact.

2. Let X be Hausdorff. Prove that every compact subset of X is closed.

3. Show that if X is compact and Y is Hausdorff, then every continuous bijection $f : X \to Y$ is a homeomorphism.

4. Let X be a space. Prove that X is compact if and only if every family of closed sets with the finite intersection property has nonempty intersection.

5. Let X be compact and $A \subset X$. Show that A is compact if and only if it is closed in X.

6. Let (X, d) be a metric space. Prove that compactness is equivalent to completeness plus total boundedness.

7. Prove that a product of finitely many compact spaces is compact. (*Hint:* use the finite-intersection characterization.)

8. [$\star$] (Tychonoff in a special case.) Let $X = \{0, 1\}^J$ with the product topology. Prove that X is compact. (*Hint:* identify X with the set of ultrafilters on $\mathcal{P}(J)$, or use the finite intersection property directly.)

9. Let X be locally compact Hausdorff. Construct the one-point compactification $X^* = X \cup \{\infty\}$ and prove:

   (a) X^* is compact and Hausdorff;

   (b) X is open and dense in X^*;

   (c) X^* is Hausdorff if and only if X is locally compact Hausdorff.

10. [$\star$] Let X be non-compact locally compact Hausdorff. Show that X^* is connected if and only if X is connected and not compact.

Problem Set 8: Baire category and applications

1. A set $A \subset X$ is *nowhere dense* if $\text{int}(\overline{A}) = \emptyset$, and *meagre* if it is a countable union of nowhere dense sets. Show that:
   (a) every nowhere dense set has empty interior;
   (b) the closure of a nowhere dense set is nowhere dense;
   (c) a countable union of meagre sets is meagre.
2. Prove the Baire Category Theorem for complete metric spaces: if (X, d) is complete and (U_n) is a sequence of open dense sets, then $\bigcap_{n=1}^{\infty} U_n$ is dense.
3. Conclude that $\mathbb{R}$ is not a countable union of nowhere dense sets (in particular, not countable).
4. Let X be a Baire space. Prove that every nonempty open subset of X is not meagre in X.
5. [$\star$] Let $C([0,1])$ be the space of real-valued continuous functions on $[0,1]$ with the sup norm. Show that the set of nowhere differentiable functions is dense (and in fact comeagre) in $C([0,1])$. (*You may quote a standard theorem, for example, the Banach–Mazur theorem, if known.*)
6. Use Baire category to show that if (X, d) is complete and $f_n : X \to \mathbb{R}$ are continuous with $f_n(x) \to f(x)$ pointwise, then the set of points where f is continuous is a G_δ set. (*Hint:* oscillation of f.)
7. Let X be compact Hausdorff and $C(X)$ the Banach space of continuous real functions with the sup norm. Show that if (f_n) is Cauchy in $\|\cdot\|_\infty$, then (f_n) converges uniformly to some $f \in C(X)$.
8. [$\star$] Show that every locally compact Hausdorff space is a Baire space.

Problem Set 9: Urysohn–Tietze, metrization, and separation

1. Let X be normal and $A, B \subset X$ be disjoint closed sets. Prove Urysohn's Lemma: there exists $f : X \to [0,1]$ continuous with $f|_A = 0$ and $f|_B = 1$.

2. Use Urysohn's Lemma to show that every normal T_1 space is completely regular.

3. State and prove the Tietze Extension Theorem for normal spaces: if X is normal, $A \subset X$ closed, and $f : A \to [-1, 1]$ continuous, then f extends to a continuous $F : X \to [-1, 1]$.

4. Let X be a space and $\mathcal{F} \subset C(X, [0, 1])$ a family of continuous functions separating points from closed sets. Show that the evaluation map

$$e : X \to [0,1]^{\mathcal{F}}, \qquad e(x) = (f(x))_{f \in \mathcal{F}}$$

is an embedding.

5. Prove that every completely regular T_1 space embeds in a cube $[0, 1]^I$ for some index set I.

6. (Urysohn metrization.) Prove that every regular second countable space is metrizable. (*You may assume the existence of a countable base and construct a metric via Urysohn functions.*)

7. Show that a compact Hausdorff space is metrizable if and only if it is second countable.

8. [$\star$] (Nagata–Smirnov, special case.) Prove that every metrizable space admits a σ-locally finite base.

9. Let X be Hausdorff and second countable. Prove that X is separable. Is the converse true? Give an example.

10. [$\star$] Give an example of a normal space X for which $X \times X$ is not normal. (*Hint:* the Sorgenfrey line.)

Bibliography

[1] J. R. Munkres: *Topology*, Pearson India, 2015.

[2] C. W. Patty, *Foundations of Topology*, Second Edition, Jones & Bartlett Student Edition, 2010.

[3] G. F. Simmons: *Introduction to Topology and Modern Analysis*, McGraw-Hill India, 2017.

[4] S. Willard: *General Topology*, Dover, 2004.

[5] B. Mendelson: *Introduction to Topology*, Third Edition, Dover, 2003.

[6] J. L. Kelley: *General Topology*, Springer, 1982.