DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA543: Functional analysis — Mid semester exam
Instructor: Rajesh Srivastava — September 24, 2013
Time duration: **Two hours** — **Maximum Marks: 30**

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1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function which vanishes at most at one point in $\mathbb{R}^n$ and satisfies $f(\alpha x) = |\alpha| f(x)$, for each $(\alpha, x) \in \mathbb{R} \times \mathbb{R}^n$. Show that f is a norm on $\mathbb{R}^n$. $\boxed{\mathbf{3}}$

2. Let $B[0,1]$ be the space of all bounded functions on $[0,1]$. Show that the closed unit ball $D = \{f \in B[0,1] : \|f\|_\infty \leq 1\}$ is **not** compact in $(B[0,1], \|\cdot\|_\infty)$. $\boxed{\mathbf{3}}$

3. Prove that $(c_{oo}(\mathbb{N}), \|\cdot\|_\infty)$ is a dense subspace of $(c_o(\mathbb{N}), \|\cdot\|_\infty)$. Deduce that the space $(c_o(\mathbb{N}), \|\cdot\|_\infty)$ is separable. $\boxed{\mathbf{2+2}}$

4. Let $1 < p < q < \infty$. Prove that $l^p(\mathbb{N})$ is a **proper** dense subspace of $l^q(\mathbb{N})$. $\boxed{\mathbf{3}}$

5. Let $X = \{f \in C^1[0,1] : f(0) = 0\}$. For $f \in X$, define $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$. Prove that $\|f\|_1 \leq 2\|f'\|_\infty$. $\boxed{\mathbf{3}}$

6. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Let X be the space of all analytic function on D which are continuous on the closure $\bar{D}$. Define $\|f\|_\infty = \sup\left\{|f(e^{i\theta})| : 0 \leq \theta \leq 2\pi\right\}$. Suppose $f, f_n \in X$ and $\lim\limits_{n\to\infty} \|f_n - f\|_\infty = 0$. Prove that $\lim\limits_{n\to\infty} \|f_n' - f'\|_\infty = 0$. $\boxed{\mathbf{3}}$

7. Show that the norms $\|\cdot\|_\infty$ and $\|.\|_2$ are **not** equivalent on $C[0,1]$. $\boxed{\mathbf{3}}$

8. Denote $X = \left\{f : f \in BV[0,1] \text{ and } \lim\limits_{x\to 0^+} f(x) = 0 = f(0)\right\}$. For $f \in X$, define $\|f\| = V_0^1(f)$. Prove that $\|f\| = 0$ if and only if $f \equiv 0$. Further, prove that every Cauchy sequence in $(X, \|\cdot\|)$ converges to a point in X.
(**Hint:** Use $f_n \xrightarrow{\textbf{p.w.}} f \Rightarrow V(f_n, P) \to V(f, P)$.) $\boxed{\mathbf{2+3}}$

9. Let $1 < p < q < \infty$ and $\Omega \subset \mathbb{R}$ is a measurable set with $m(\Omega) < \infty$. For $f \in L^q(\Omega)$, prove that
$$\|f\|_p \leq (m(\Omega))^{\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_q.$$
$\boxed{\mathbf{3}}$

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