DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA543: Functional Analysis — Mid Semester Exam
Instructor: Rajesh Srivastava — November 23, 2019
Time duration: Two hours — Maximum Marks: 30

N.B. Answer without proper justification will attract zero mark.

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1. (a) Suppose T is a continuous injective operator on a Hilbert space H. Does it imply that $\dim R(T^*) \geq 1$? **1**
   (b) Let X be a Banach space and $f_n \to 0$ in the week star topology of X^*. Is it necessary that f_n is bounded in X^*? **1**
   (c) Let M be proper dense subspace of normed linear space X. Does it imply that every continuous linear functional on M has a unique Hahn Banach extension to X? **1**
   (d) Let $T : l^2 \to l^2$ be defined by $T(x_1, x_2, x_3, \ldots) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots)$. Whether T is a self-adjoint operator? **1**

2. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of a Hilbert space H and $y = (y_1, y_2, \ldots) \in l^1$. Define a linear map $T : H \to H$ by $T(x) = \sum_{n=1}^{\infty} \langle x, e_n \rangle y_n e_n$. Show that T is continuous. Find the norm of T. **3**

3. Let M be a proper dense subspace of a real inner product space X. If $T : X \to X$ is a bounded self-adjoint operator that satisfying $\langle Tx, x \rangle = 0$ and $\langle Tx, y \rangle \in \mathbb{R}$ for all $x, y \in M$. Show that $T = 0$. **3**

4. Let $M = \{(x, y) \in \mathbb{R}^2 : \; 2x - 3y = 0\}$. Define a linear functional f on M by $f(x, y) = x$. Find all possible Hahn Banach extensions of f to $(\mathbb{R}^2, \| \cdot \|_2)$. **3**

5. Let X be an inner product space. For any three points $x, y, z \in X$. Show that $\|x - y\|^2 + \|x - z\|^2 = 2\left(\|x - \frac{1}{2}(y + z)\|^2 + \|\frac{1}{2}(y - z)\|^2\right)$. **4**

6. Let $T : L^2[0, 1] \to L^2[0, 1]$ be a linear map defined by $T(f)(x) = \int_0^x e^{ixy} f(y) dy$. Show that $\|T\| \leq 1$. Find the adjoint of T. **5**

7. Let C be an open subset of a normed linear space X and $0 \in C$. For $x \in X$, define $p(x) = \inf\{t > 0 : t^{-1}x \in C\}$. Show that there exists $M > 0$ such that $p(x) \leq M\|x\|$ for all $x \in X$. **2**

8. Let c_o be the space of all sequences on $\mathbb{C}$ that converges to 0. Show that the dual of $(c_o, \| \cdot \|_\infty)$ is isomorphic to $(l^1, \| \cdot \|_1)$ **4**

9. Let H be complex Hilbert space and $T \in B(H, H)$. Show that T is a normal operator if and only if $\|T^*x\| = \|Tx\|$. **4**

10. Let X and Y be two Banach spaces and $T_n, T \in B(X,Y)$. If $f \circ T_n(x) \to f \circ T(x)$ for all $f \in Y^*$ and for all $x \in X$, then show that $\sup \|T_n\| < \infty$. **4**

11. Let M be a proper closed subspace of a Hilbert space H. Define $\pi : H \to H/M$ by $\pi(x)\| = x + M$. Show that $\|\pi\| = 1$. Further, if there exists $0 \neq x \in H$ such that $\|\pi(x) = \|x\|$, then there exists a unique $0 \neq z \in M$ such that $\|z\|^2 = 2\text{Re}\langle x, z\rangle$. **6**

12. Let $E = \{e_\alpha : \alpha \in I\}$ be an orthonormal basis of a Hilbert space H. For each $x \in H$, show that the set $\left\{e_\alpha \in E : |\langle x, e_\alpha\rangle|^2 > \left(\frac{2}{n} - e^{-n}\right)\|x\|\right\}$ is a finite set for each fixed $n \in \mathbb{N}$. **3**

END