DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA543: Functional Analysis — Endsem Exam
Instructor: Rajesh Srivastava — November 25, 2013
Time duration: **Three hours** — **Maximum Marks: 45**

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1. Let K be a compact subset of an infinite dimensional normed linear space X. Prove or disprove that interior of K is empty. $\boxed{3}$

2. Let M be a closed subspace of a normed linear space X such that M and X/M both are complete. Show that X is also complete. $\boxed{4}$

3. Suppose $\{x_1, x_2, \ldots\}$ is a sequence in a Banach space X such that each $x \in X$ has a unique representation $x = \sum\limits_{n=1}^{\infty} a_n x_n$, where $a_n \in \mathbb{C}$. Show that X is separable. $\boxed{5}$

4. For $f \in C^1[0,1]$, define its norm by $\|f\| = \max\{\|f\|_\infty, \|f'\|_\infty\}$. Show that the linear map $T : (C^1[0,1], \|.\|) \to (C[0,1], \|.\|_\infty)$ given by $T(f) = f'$ is continuous and $\|T\| = 1$. $\boxed{4}$

5. Let $X = C[0,1]$ and $M = \{f \in X : f(0) = 0\}$. Give an example of a linear functional T on $(X, \|.\|_\infty)$ such that $T(M) = \{0\}$ and $\|T\| = 1$. $\boxed{3}$

6. Let $M = \{(x_1, x_2, \ldots) \in l^1 : x_1 + x_2 = 0\}$ and define $f_o : M \to \mathbb{C}$ by $f_o(x_1, x_2, \ldots) = 2x_1$. Find a norm preserving extension of f_o to l^1. $\boxed{4}$

7. Let $T_n : l^1 \to l^1$ be a sequence of linear transformations such that for eachlinebreak[4] $x = (x_1, \ldots, x_n, x_{n+1}, \ldots) \in l^1$, $T_n(x) = (x_{n+1}, x_{n+2}, \ldots)$. Show that $\|T_n(x)\| \to 0$ but $\|T_n\| = 1$. $\boxed{4}$

8. Let X^* be the dual space of a normed linear space X. Let $x, x_n \in X$ such that $x_n \xrightarrow{\text{weakly}} x$. If $f, f_n \in X^*$ satisfy $\|f_n - f\| \to 0$, then prove that $f_n(x_n) \to f(x)$. $\boxed{3}$

9. Let f_n be a sequence of linear functionals on $(c_o, \|.\|_\infty)$ such that for each $x = (x_1, x_2 \ldots,) \in c_o$, $f_n(x) = \frac{1}{n+1} \sum\limits_{j=1}^{n} x_j$. Prove that $\lim f_n(x) = 0$, but $\lim \|f_n\| = 1$. $\boxed{3}$

10. Let $T : l^2 \to l^2$ be a linear map such that $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. Find the adjoint T^* of T. $\boxed{4}$

11. Let $M = \{(y_1, y_2, \ldots) \in l^2 : 2y_1 - y_2 = 0\}$ and $x_o = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$. Find $y_o \in M$ such that $\inf\limits_{y \in M} \|x_o - y\|_2 = \|x_o - y_o\|_2$. $\boxed{4}$

12. Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H and $(\alpha_1, \alpha_2, \ldots) \in l^\infty$. Define a linear map $T : H \to H$ by $T(x) = \sum\limits_{n=1}^{\infty} \langle x, e_n \rangle \alpha_n e_n$. Prove that T is continuous and $\|T\| = \sup |\alpha_n|$. $\boxed{2+2}$

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