DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA541: Real Analysis — Mid Semester Exam
Instructor: Rajesh Srivastava — September 20, 2017
Time duration: 02 hours — Maximum Marks: 30

N.B. Answer without proper justification will attract zero mark.

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1. (a) What is the cardinality of the set $\{f : \mathbb{R} \to \mathbb{R},\ f \text{ is nowhere continuous}\}$? $\boxed{1}$

   (b) Whether the function $f(x) = \frac{\sin x}{x}$ is uniformly continuous on $(0, 1)$? $\boxed{1}$

   (c) If f and g are non-constant uniformly continuous functions on $\mathbb{R}$. Is it necessary that f and g are bounded for fg to be uniformly continuous? $\boxed{1}$

2. For a monotone increasing function $f : [a, b] \to \mathbb{R}$, define $g(x) = \sup\{f(y) : y < x\}$. If f has limit at c, then show that $f(c) = g(c)$. $\boxed{4}$

3. Let f be a continuous function on $\mathbb{R}$ such that $\inf f(x) = f(x_o)$ and $\sup f(x) = f(y_o)$. Show that for any x_1 and x_2 in $\mathbb{R}$, there exists $c \in \mathbb{R}$ such that $f(c) = \frac{f(x_1)+f(x_2)}{2}$. $\boxed{2}$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that f takes integral values on the set of rationals. Prove that f is constant. $\boxed{2}$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for any sequence $x_n \in \mathbb{R}$, the sequences $f(x_{2n}), f(x_{2n-1})$ and $f(x_{5n})$ are convergent. Show that f is continuous except on a countable set. $\boxed{4}$

6. For non-empty sets A and B in $\mathbb{R}$, let $f : A \cup B \to \mathbb{R}$ be such that $f|_A$ and $f|_B$ are uniformly continuous. Can f uniformly continuous on $A \cup B$ if $A \cap B = \emptyset$? $\boxed{2}$

7. Let $f : (0, \infty) \to [0, \infty)$ be a continuous function satisfying $f(x + y) \le \frac{f(x)+f(y)}{3}$. Show that f is bounded and hence deduce that f is uniformly continuous. $\boxed{4}$

8. For $x \in (0, 1)$, show that the series $\sum_{n=0}^{\infty} (-1)^n x^n$ is point-wise convergent. Whether the given series is uniformly convergent on $(0, 1)$? $\boxed{3}$

9. Does there exist a sequence of differential functions f_n on $(0, \infty)$ such that f_n' is uniformly convergent on $(0, \infty)$ but f_n is nowhere point-wise convergent? $\boxed{2}$

10. Show that the polynomial $p_n(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n}$ has exactly one real root if n is odd. For n even, show that there exists $\delta > 0$ such that $\frac{1}{2}x^n \le p_n(x) \le \frac{3}{2}x^n$, whenever $|x| > \delta$ and hence deduce that p_n has no real root. $\boxed{4}$

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