DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA541: Real Analysis — End Semester Exam
Instructor: Rajesh Srivastava — November 26, 2017
Time duration: Three hours — Maximum Marks: 45

N.B. Answer without proper justification will attract zero mark.

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1. (a) Does there exist an unbounded set A in $(\mathbb{R}, u)$ such that $\text{diam}(A^\circ) = 1$? **[1]**
   (b) Is it possible for every metric d on $\mathbb{R}$, each closed and bounded set in $((\mathbb{R}, d)$ is compact? **[1]**
   (c) Let $1 < p < q < \infty$. Whether $l^p \cap l^q$ is a dense subspace of l^q? **[1]**
   (d) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be satisfying $\|f(x)\| \leq \|x\|_2^2$. Does f differentiable at $\mathbf{0}$? **[1]**
   (e) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous function. Does it imply that f sends every bounded set to a bounded set? **[1]**

2. Using Baire category theorem, show that the open interval $(0, 1)$ is uncountable. **[2]**

3. Show that $X = \{f \in C[0, 1] : \int_0^1 f(t)dt = 0\}$ is a closed subspace of $(C[0, 1], \|.\|_\infty)$. Whether the space X is a finite dimensional? **[1+2]**

4. Let $X \neq \emptyset$ and (X, d) be metric space. Show that (X, d) is complete if and only if $\left(X, \frac{d}{1+d}\right)$ is complete. Does every bounded set in $\left(X, \frac{d}{1+d}\right)$ is bounded in (X, d)? **[2+1]**

5. Let (X, d) be a metric space. For $A \subseteq X$, write $\delta(A) = \sup\limits_{x,y \in A} d(x, y)$. Show that any subset $A \subseteq X$ satisfies $\delta(A) = \delta(\bar{A})$. **[2+1]**

6. For $x \in [0, 1]$, define $f_n(x) = \frac{x}{1+nx}$. Show that $A = \{f_n : n \in \mathbb{N}\}$ is an equicontinuous set in $(C[0, 1], \|.\|_\infty)$. Whether the set A is compact in $(C[0, 1], \|.\|_\infty)$? **[2+1]**

7. Let $X = \{f : \mathbb{R} \to \mathbb{R}, f \text{ is bounded}\}$ and $Y = \{f : \mathbb{R} \to \mathbb{R}, \text{f is continuous and} \lim\limits_{|x| \to \infty} f(x) = 0\}$. Show that $(X, \|.\|_\infty)$ is complete. Further, show that $(Y, \|.\|_\infty)$ is a closed subspace of $(X, \|.\|_\infty)$. **[2+2]**

8. Let K and F be two non-empty subsets of a metric space (X, d). If K is compact and F closed, then show that $\text{dist}(K, F) > 0$, whenever $K \cap F = \emptyset$. **[4]**

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} 1 & \text{if } y^2 < x < 2y^2, \\ 0 & \text{otherwise.} \end{cases}$
   Find all those $v \in S^1$ such that $D_v f(0, 0)$ exists. Does f continuous at $(0, 0)$? **[2+1]**

10. Show that equation $x^2 + ye^x - \sin(xy) = 0$ can be solved for y in some neighborhood of $(0, 0)$ but cannot be solved for x in any neighborhood of $(0, 0)$. **[3]**

11. For $(x, y) \in \mathbb{R}^2$, let $A(x, y) = (3x, 4y)$. Show that $\sup_{x^2+y^2=1} \|A(x, y)\|_2 = 4$. **1+2**

12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable. Prove that the necessary condition for f has minimum on the curve $y = x + x^2$ at $(0, 0)$ is $f_x(0, 0) + f_y(0, 0) = 0$. **3**

13. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x - e^{-y}, y - e^x)$. Show that f is locally invertible at $(0, 0)$. Find $(f^{-1})'(0, 0)$. **2+1**

14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1-map satisfying $f(0, 0) = 0$ and $f_x(0, 0) = 1$. For $(x, y) \in \mathbb{R}^2$, let $g(x, y) = (f(x, y), y)$. Show that g is injective in some neighborhood of $(0, 0)$. Does f injective in any neighborhood of $(0, 0)$? **1+2**

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