DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA224: Real Analysis	Mid Semester Exam
Instructor: Rajesh Srivastava	February 28, 2014
Time duration: Two hours	Maximum Marks: 30

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1. Let $f(x,y) = \dfrac{x^m y^n}{(x^2+y^2)^p}$; if $x^2+y^2 \neq 0$ and $f(0,0)=0$. Find condition on (m,n,p) for which f is continuous at $(0,0)$. Further, find condition for the function f to be bounded on $\mathbb{R}^2$. **[2]**

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be map $f(x,y) = \begin{cases} \sqrt{x^2+y^2}\ \sin\left(\dfrac{y^2}{x-y}\right) & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$
Show that f is continuous at $(0,0)$. Does f differentiable at $(0,0)$? **[2]**

3. Let $u = (1,0,0)$, $v = \frac{1}{\sqrt{2}}(1,1,0)$ and $w = \frac{1}{\sqrt{3}}(1,1,1)$. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable at $\mathbf{0}$. If $D_u(\mathbf{0}) = 1$, $D_v(\mathbf{0}) = 2$ and $D_w(\mathbf{0}) = -1$. Then find $f'(\mathbf{0})$. **[3]**

4. Let Ω be an open subset of $\mathbb{R}^n$. Let $\varphi : \Omega \to \mathbb{R}^n$ be continuous at $x_0 \in \Omega$. Suppose $\psi : \Omega \to \mathbb{R}$ is map satisfying $\psi(x+h) - \psi(x) = \varphi(x+h)\cdot h$, for all $x \in \Omega$. Then show that ψ is continuously differentiable at x_0. **[3]**

5. Let $\varphi,\ \psi : \mathbb{R} \to \mathbb{R}$ be twice differentiable. Suppose $f(x,y) = \varphi(x^2+y^2) + \psi(x^2-y^2)$. Show that $f_{xy}(x,y) = f_{yx}(x,y)$. **[2]**

6. Let $f : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}$ be differentiable maps. Suppose $F_y \neq 0$ and $F(x, f(x)) = 0$. Show that $f'(x) = -\dfrac{F_x(x,y)}{F_y(x,y)}$, where $y = f(x)$. **[2]**

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a map given by $f(x) = x^3 + x + \cos x$. Show that f is one-one and onto. Find the points where f^{-1} is differentiable. **[4]**

8. Let $A \in GL(\mathbb{R}^n)$ and $\alpha \geq 2$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\|f(x)\| \leq k\|x\|^\alpha$, for some $k > 0$. Show that the map $g = f + A$ is continuously differentiable at $\mathbf{0}$ and g is invertible in the neighborhood of $\mathbf{0}$. **[4]**

9. Show that the system of equations $xy - 1 = 0$ and $y^2 + z^2 - 1 = 0$ can be solved for x and y in terms of z near $(2, \frac{1}{2}, \frac{\sqrt{3}}{2})$ as $x = \varphi(z)$ and $y = \psi(z)$. Further, find the values of $\varphi'(\frac{\sqrt{3}}{2})$ and $\psi'(\frac{\sqrt{3}}{2})$. **[4]**

10. Let $z = y + x\sin z$. Show that there exists unique function φ such that $z = \varphi(x,y)$ on an open set U containing $(0,0)$. Further, prove that φ has Taylor's series expansion in a neighborhood of $(0,0)$ as $\varphi(x,y) = y + 2xy + \cdots$. **[4]**

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