DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA224: Real Analysis — MidSem
Instructor: Rajesh Srivastava — March 1, 2018
Time duration: 02 hours — Maximum Marks: 30

N.B. Answer without proper justification will attract zero mark.

---

1. (a) Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(e^{-n^2}) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$? **[1]**

   (b) Let $f : [1, \infty) \to [1, \infty)$ be defined by $f(x) = x + \frac{1}{x}$. Whether f is a contraction map? **[1]**

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable. Find the necessary condition for $f(x, x^2 + 2x)$ has minimum at 0. **[2]**

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f_x(0, 0) = 0$. Does there exist some $\delta > 0$ such that $f(x, 0)$ is continuous on $(-\delta, \delta)$? **[2]**

4. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is satisfying $f(rx) = r^{\frac{3}{2}} f(x)$ for all $(x, r) \in \mathbb{R}^n \times (0, \infty)$. Whether f is differentiable at $\mathbf{0}$. **[3]**

5. Let Ω be an open subset of $\mathbb{R}^n$. Suppose $f : \Omega \to \mathbb{R}^n$ is a continuous function that satisfies $\|f(x_o)\| > 0$ for some $x_o \in \Omega$. Show that there exists an open ball B centered at x_o such that $\|f(x)\| > \frac{\|f(x_o)\|}{2}$ for all $x \in B$. **[2]**

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function that satisfies $f(x, y) > 0$ for all $(x, y) \in \mathbb{Q} \times \mathbb{R}$. Show that $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$. **[2]**

7. Show that there exists a linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that the strict inequality $\|A^2\| < \|A\|^2$ holds. **[3]**

8. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be such that $f^2 = f \circ f$ is a contraction map. Show that f has a unique fixed point. **[3]**

9. Show that equation $x^2 + yz - \cos(xz) = 0$ can be solved for x in some neighborhood of $(1, 1, 0)$. Whether it can be solved for y in a neighborhood of $(1, 1, 0)$? **[3]**

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable with $f(0, 0) = 0$. Find the condition under which $f(f(x, y), y) = 0$ can be solved for y in some neighborhood of $(0, 0)$. **[4]**

11. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $f'(0) \neq 0$. Show that the function $g(x, y) = (f(x), xf(x) - y)$ is locally invertible in some neighborhood of $(0, 0)$. Give an example of f (with justification) for which g is globally invertible. **[4]**

END