DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA211(Minor): Real Analysis   Quiz - I
Instructor: Rajesh Srivastava   September 7, 2022
Time duration: 1.5 hours   Maximum Marks: 10

N.B. Answer without proper justification will attract zero mark.

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1. (a) Does there exist a continuous function f from $\{(x,y) : |x| \leq 1 \text{ and } 1 \leq y \leq 2\}$ onto $\mathbb{R}^2$? `1`

   (b) Is it possible that for every open set $O \subset \mathbb{R}^2$ and for every function $f : O \to \mathbb{R}$ with $f_x(X) = f_y(X) = 0$ for all $X \in O$, implies that f is constant on O ? `1`

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = \begin{cases} \frac{x^2 y}{x^2 - y^2} & \text{if } x^2 \neq y^2, \\ 0 & \text{otherwise.} \end{cases}$

   Determine all possible directions along which f has directional derivative at $(0,0)$. `2`

3. Prove /disprove that $A = \left\{ \left(x, \frac{1}{x}\right) : x \neq 0, \ x \in \mathbb{R} \right\}$ is closed. What is the closure of A if it is not closed? `2`

4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be satisfying $\|f(X) - f(Y)\| \leq \frac{1}{2}\|X - Y\|$ for every $X, Y \in \mathbb{R}^n$. Show that the function $g : \mathbb{R}^n \to \mathbb{R}^n$ given by $g(X) = X - f(X)$ is injective. `2`

5. Let $\mathbb{D} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$. Define a function $f : \mathbb{D} \to \mathbb{R}$ by $f(x_1, x_2) = \sin(x_1^2 + x_2)$. Show, by using appropriate MVT, that $|f(X) - f(Y)| \leq \sqrt{5}\|X - Y\|$ for every $X, Y \in \mathbb{D}$. `2`

END