DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA211(Minor): Real Analysis	End Semester Exam
Instructor: Rajesh Srivastava	November 29, 2022
Time duration: 03 hours	Maximum Marks: 50

N.B. Answer without proper justification will attract zero mark.

---

1. (a) Does there exist a set $A \subset \mathbb{R}$ such that outer measure of $\bar{A} \setminus A^\circ$ is infinite? **1**

   (b) Let $X = C[0,1]$. Is it necessary that $f_n \to f$ in $(X, \|\cdot\|_1)$ implies $f_n \to f$ in $(X, \|\cdot\|_\infty)$? **1**

   (c) Let $f, g : \mathbb{R} \to \bar{\mathbb{R}}$ be such that $\{x \in \mathbb{R} : f(x) = \pm\infty\}$ and $\{x \in \mathbb{R} : g(x) = \pm\infty\}$ are Lebesgue measurable. Does it imply that $\{x \in \mathbb{R} : (fg)(x) = \pm\infty\}$ is Lebesgue measurable? **1**

   (d) Does there exists an open and unbounded set A in $\mathbb{R}$ such that $m(A) = 2$? **1**

   (e) Let $C \subset [0,1]$ be the Cantor set, and define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in C; \\ \frac{1}{x} & \text{if } x \notin C. \end{cases}$$

   Whether f is a Lebesgue measurable function? **1**

2. Let $A \subset (1, \infty)$ be a closed set. Show that A^2 is a closed set. **2**

3. Let $1 \le p \le \infty$ and $d_i\,; i = 1, 2$ be two metric on a non-emply set X. Show that $d_p = (d_1^p + d_2^p)^{1/p}$ is a metric on X for $1 \le p < \infty$. Whether $d_\infty = \max\{d_1, d_2\}$ is a metric on X? **3**

4. For $x, y \in \mathbb{R}$, define $d(x, y) = \min\{1, \sqrt{|x-y|}\}$. Examine for $(\mathbb{R}, d)$ to be a complete metric space. **3**

5. For $x = (x_n) \in l^2$, write $\|x\| = \left(\sum_{n=1}^{\infty} a_n |x_n|^2\right)^{1/2}$. Find all possible sequences (a_n) such that $\|\cdot\|$ is a norm on l^2. **4**

6. Let $f_n, f : \mathbb{R} \to (0, \infty)$ be such that $f_n \to f$ uniformly on $\mathbb{R}$. Examine for $e^{f_n} \to e^f$ uniformly on $\mathbb{R}$. **4**

7. Let $f_n(t) = \sqrt{t^2 + n}$. Examine for the uniform convergence of f_n' on $\mathbb{R}$. **3**

8. Let $X = [-1, 1] \times [-1, 1]$. Define $\varphi : (X, \|\cdot\|_2) \to (X, \|\cdot\|_2)$ by $\varphi(x, y) = \left(x \sin\frac{y}{3}, y \cos\frac{x}{3}\right)$. Show that there exists a unique point $(x_o, y_o) \in X$ such that $\varphi(x_o, y_o) = (x_o, y_o)$. **3**

9. Let A be an arbitrary unbounded subset of in $\mathbb{R}$. Show that for any $\epsilon > 0$ there does not exist a compact set $K \subset A$ such that $m(A \setminus K) < \epsilon$. **3**

10. Let A be a Lebesguee measurable subset of $\mathbb{R}$. Show that $A + (0, 1]$ is Lebesguee measurable. **2**

11. Let $E \subset \mathbb{R}$ be such that $m^*(E) < \infty$. Let $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$ for each $A \subseteq \mathbb{R}$. Show that for each $\epsilon > 0$ there exists an open set O containing E such that $m^*(O \setminus E) < \epsilon$. Does the same conclusion holds true if $m^*(E) = \infty$? **4**

12. For Lebesgue measurable sets $A, B \subset [0, 1]$, define $f(x) = \inf\{|x - (a + b)| : a \in A, b \in B\}$. Show that f is continuous on $\mathbb{R}$. Further, if $m(A)+m(B) > 1$, then show that $f(1) = 0$. **4**

13. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sup\{|x - y| : y \in [0, 1]\}$. Show that f is Lebesgue measurable. **3**

14. Let E be a non-measurable subset of $\mathbb{R}$, and let $f = \chi_E$. Suppose g is a strictly monotone function on $\mathbb{R}$. Prove/disprove that $g \circ f$ is Lebesgue measurable. **2**

15. Let $f_n(t) = \sin t\, e^{-nt^2}$. Find the value of $\lim\limits_{n\to\infty} \int\limits_{\mathbb{R}} f_n dm$. **3**

16. $f : \mathbb{R} \to [0, \infty]$ be Lebesgue measurable and $\int\limits_{\mathbb{R}} f dm < \infty$. Show that

$$\int_{\mathbb{R}} \left(\sum_{n=1}^{\infty} f\left(2^n x + \frac{1}{n}\right)\right) ds < \infty.$$

3

END