MA1501H (CSE), 2025: Multivarable Calculus, Hint/Model solution

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1. (a) Whether the set $\{(x,y,z) \in \mathbb{R}^3 : |x| + 2|y| + 3|z|^2 < 1\}$ is bounded in $\mathbb{R}^3$? **1**

   Solution: Let $A = \{(x,y,z) \in \mathbb{R}^3 :\ |x| + 2|y| + 3|z|^2 < 1\}$. Then $|x| < 1,\ |y| < \frac{1}{2},\ |z|^2 < \frac{1}{3}$. Hence

   $$|x|^2 + |y|^2 + |z|^2 < 1 + \frac{1}{4} + \frac{1}{3} = \frac{19}{12} =: r.$$

   Thus $A \subset B_r(0)$, and therefore A is bounded.

   (b) Whether there exists an unbounded sequence (x_n) in $\mathbb{R}$ such that $((x_n, \sin x_n^2))$ has convergent subsequence? **1**

   Solution: The sequence $x_n = 1, \frac{1}{2}, 2, \frac{1}{3}, \ldots$ is unbounded, while the subsequence $x_{n_k} = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$ is convergent. Since $\sin t$ is continuous, $\left(x_{n_k}, \sin(x_{n_k}^2)\right)$ is a convergent subsequence.

   (c) Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(e^{-n^2}) = (n, \frac{1}{n})$ for each $n \in \mathbb{N}$? **1**

   Solution: No. A continuous function maps bounded sets to bounded sets, which would fail for such f since $f(e^{-n^2}) = \left(n, \frac{1}{n}\right)$ is unbounded in the first component.

2. Show that the set $\{x \in \mathbb{R}^m : 2 \le \|x\| < 3\}$ is neither open nor closed set in $\mathbb{R}^m$. **2**

   Solution: Let $A = \{x \in \mathbb{R}^m : 2 \le \|x\| < 3\}$. Note that the sequence $x_n = (3 - \frac{1}{n})e_1 \in A$, and converges to $3e_1 \notin A$. Hence A is not closed. Also, no ball with center $2e_1$ and any radius is complectly contained in A. Hence A is not open.

3. If (x_n) is sequence in $\mathbb{R}^m$ such that the series $\sum\limits_{n=1}^{\infty} n^3\|x_n\|^2 < \infty$. Show that the series $\sum\limits_{n=1}^{\infty} \|x_n\|^2$ is convergent. **2**

   Solution: Since $\sum n^3\|x_n\|^2 < \infty$, we have $\sum n^6\|x_n\|^4 < \infty$. Then

   $$\sum \|x_n\|^2 = \sum \frac{n^3\|x_n\|^2}{n^3} \le \Big(\sum \frac{1}{n^6}\Big)^{1/2} \Big(\sum n^6\|x_n\|^4\Big)^{1/2} < \infty$$

   by the Cauchy–Schwarz inequality.

4. Let function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

   $$f(x,y) = \begin{cases} \dfrac{\sin^2(x-y)}{|x|+|y|} & \text{if } |x|+|y| \ne 0, \\ 0 & \text{otherwise.} \end{cases}$$

   Check the continuity of f at $(0,0)$. **3**

Solution: For $(h,k) \neq (0,0)$,

$$|f(h,k)-f(0,0)| = \left|\frac{\sin^2(h-k)}{|h|+|k|}\right| \leq \frac{|h-k|^2}{|h|+|k|} \leq \frac{(|h|+|k|)^2}{|h|+|k|} = |h|+|k| \leq \sqrt{2}\sqrt{h^2+k^2},$$

where we used $|\sin t| \leq |t|$ and the Cauchy–Schwarz inequality. Hence f is continuous at $(0,0)$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f \circ g$ is differentiable for every function $g : \mathbb{R} \to \mathbb{R}^2$ with $g(0) = (0,0)$. Show that all the directional derivative of f exist $(0,0)$. **2**

   Solution: By assumption, for each such g the limit

$$\lim_{t\to 0} \frac{f(g(t)) - f(g(0))}{t}$$

   exists. Taking $g(t) = tv$ with $\|v\| = 1$ yields

$$\lim_{t\to 0} \frac{f(tv) - f(0,0)}{t} = D_v f(0,0),$$

   so every directional derivative $D_v f(0,0)$ exists.

6. Show that the function f defined by $f(x,y) = \dfrac{1}{1+x-y}$ is differentiable at $(0,0)$. **3**

   Solution: We have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h}-1}{h} = \lim_{h\to 0} \frac{1-(1+h)}{h(1+h)} = -1, \qquad f_y(0,0) = 1.$$

   Let

$$\begin{aligned}\epsilon(h,k) &= \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} \\ &= \frac{\frac{1}{1+h-k} - 1 + h - k}{\sqrt{h^2+k^2}} = \frac{(h-k)^2}{(1+h-k)\sqrt{h^2+k^2}}.\end{aligned}$$

   Using $|h|+|k| \leq \sqrt{2}\sqrt{h^2+k^2}$,

$$|e(h,k)| \leq \frac{|h-k|\,|h-k|}{(1+h-k)\sqrt{h^2+k^2}} \leq \frac{\sqrt{2}(|h|+|k|)}{1+h-k} \xrightarrow[(h,k)\to(0,0)]{} 0.$$

   Hence f is differentiable at $(0,0)$.

——End——