MA1501H (CSE), EndSem 2025: Multivarable Calculus, Hint/Model solution

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1. (a) Whether the interior of $\{(x, \sin\frac{\pi}{x}) : x \neq 0 \text{ and } x \in \mathbb{R}\}$ is non-empty in $\mathbb{R}^2$? $\boxed{1}$

   Solution: The given set can be treated as graph of a continuous function, and that cannot keep any ball inside it in $\mathbb{R}^2$. Hence its interior is empty.

   (b) Whether the set $\{(x, \frac{1}{x}) : x \neq 0 \text{ and } x \in \mathbb{R}\}$ is closed in $\mathbb{R}^2$? $\boxed{1}$

   Solution: The given set is the graph of a rectangular hyperbola, which is closed by virtue of every convergent sequence in it has limit in itself.

   (c) Whether $(0, 0)$ is a saddle point of the function $f(x, y) = (x - y)(x - y^2)$? $\boxed{1}$

   Solution: Yes, $(0, 0)$ is a Saddle point of $f(x, y) = (x - y)(x - y^2)$. Do the calculation yourself.

2. Show that the content of the set $\{\frac{1}{n} : n \in \mathbb{N}\}$ in $[0, 1]$ is zero. $\boxed{2}$

   Solution: Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. For given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon \implies \frac{1}{n} < \epsilon$ for all $n \geq n_0$. Then $A \subset [0, \epsilon] \cup \cup_{k=1}^{N_0-1}(\frac{1}{k} - \epsilon, \frac{1}{k} + \epsilon)$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 - y^2} & \text{if } x^2 \neq y^2, \\ 0 & \text{otherwise.} \end{cases}$$

   Find all possible directions along which f has directional derivatives at $(0, 0)$. $\boxed{3}$

   Solution: Let $v = (v_1, v_2)$ and $\|v\| = 1$. Now

$$D_v f(0, 0) = \lim_{t \to 0} \frac{t^3 v_1^2 v_2}{t^3(v_1^2 - v_2^2)} = \lim_{t \to 0} \frac{v_1^2 v_2}{v_1^2 - v_2^2} = \begin{cases} \frac{v_1^2 v_2}{v_1^2 - v_2^2} & \text{if } v_1^2 \neq v_2^2 \\ 0 & \text{if } v_1^2 = v_2^2. \end{cases}$$

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} x^2 y^2 \dfrac{x - y}{x^2 + y^2} & \text{if } x^2 + y^2 \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

   Examine whether $f_{xy}(0, 0) = f_{yx}(0, 0)$. $\boxed{4}$

   Solution: Hint: Verify that all the partial derivative of order 2 are continuous at $(0, 0)$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}\sin\left(\frac{1}{x^2+y^2}\right) & \text{if } x^2+y^2 \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is continuous at $(0,0)$.

Solution: For $(x_0, y_0) \neq (0,0)$, we have $x_0^2 + y_0^2 > 0$, and the map

$$(x,y) \mapsto \frac{x^3}{x^2+y^2}\sin\left(\frac{1}{x^2+y^2}\right)$$

is a composition of continuous functions (polynomials, division by a nonzero quantity, and the sine function). Hence f is continuous at all $(x_0, y_0) \neq (0,0)$.

Thus, it remains to show continuity at $(0,0)$, i.e.

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0.$$

For $(x,y) \neq (0,0)$, using $|\sin t| \leq 1$ for all $t \in \mathbb{R}$, we get

$$|f(x,y)| = \left|\frac{x^3}{x^2+y^2}\sin\left(\frac{1}{x^2+y^2}\right)\right| \leq \frac{|x|^3}{x^2+y^2}.$$

Since $x^2 \leq x^2 + y^2$, we can write

$$\frac{|x|^3}{x^2+y^2} = |x| \cdot \frac{x^2}{x^2+y^2} \leq |x| \cdot 1 = |x|.$$

Moreover,

$$|x| \leq \sqrt{x^2+y^2} = \|(x,y)\|.$$

Therefore,

$$|f(x,y)| \leq |x| \leq \sqrt{x^2+y^2} = \|(x,y)\|.$$

Now, as $(x,y) \to (0,0)$ we have $\sqrt{x^2+y^2} \to 0$, hence by the squeeze theorem,

$$0 \leq |f(x,y)| \leq \|(x,y)\| \longrightarrow 0.$$

Thus

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0),$$

and so f is continuous at $(0,0)$.

Another Method: We use the sequential criterion for continuity. To show that f is continuous at $(0,0)$, it suffices to prove that for every sequence

$$(x_n, y_n) \to (0,0) \quad \text{in } \mathbb{R}^2,$$

we have

$$f(x_n, y_n) \to f(0,0) = 0.$$

Let (x_n, y_n) be an arbitrary sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (0,0)$. If $(x_n, y_n) = (0,0)$ for infinitely many n, then $f(x_n, y_n) = 0$ for those n, so they do not affect the limit. Thus, without loss of generality, we may assume that

$$x_n^2 + y_n^2 \neq 0 \quad \text{for all sufficiently large } n.$$

For each such n, define

$$r_n := \sqrt{x_n^2 + y_n^2}, \qquad \theta_n \in \mathbb{R}$$

such that

$$x_n = r_n \cos\theta_n, \qquad y_n = r_n \sin\theta_n.$$

Since $(x_n, y_n) \to (0,0)$, we have $r_n = \sqrt{x_n^2 + y_n^2} \to 0$.
For $x_n^2 + y_n^2 \neq 0$ we can write

$$f(x_n, y_n) = \frac{x_n^3}{x_n^2 + y_n^2} \sin\left(\frac{1}{x_n^2 + y_n^2}\right).$$

Using $x_n = r_n \cos\theta_n$ and $x_n^2 + y_n^2 = r_n^2$, this becomes

$$f(x_n, y_n) = \frac{(r_n \cos\theta_n)^3}{r_n^2} \sin\left(\frac{1}{r_n^2}\right) = r_n \cos^3\theta_n \sin\left(\frac{1}{r_n^2}\right).$$

Now estimate the absolute value:

$$\big|f(x_n, y_n)\big| = \left|r_n \cos^3\theta_n \sin\left(\frac{1}{r_n^2}\right)\right| \leq r_n \,|\cos^3\theta_n|\, \left|\sin\left(\frac{1}{r_n^2}\right)\right|.$$

Since $|\cos^3\theta_n| \leq 1$ and $\left|\sin\left(\frac{1}{r_n^2}\right)\right| \leq 1$, we obtain

$$\big|f(x_n, y_n)\big| \leq r_n.$$

But $r_n \to 0$ as $n \to \infty$, hence by the squeeze theorem,

$$f(x_n, y_n) \to 0.$$

Since $(x_n, y_n) \to (0,0)$ was arbitrary, the sequential criterion implies that

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0,$$

so f is continuous at $(0,0)$.

6. Let $A = \int\limits_0^1 e^{-x^2} dx$. Show that $\int_0^1 \int_0^x e^{-t^2} dt dx = A + \frac{1}{2}\left(\frac{1}{e} - 1\right)$. **4**

Solution: Let

$$A = \int_0^1 e^{-x^2}\, dx$$

and set

$$I := \int_0^1 \int_0^x e^{-t^2}\, dt\, dx.$$

We compute I by changing the order of integration.

The region of integration is

$$D = \{(t, x) \in \mathbb{R}^2 : 0 \le t \le x \le 1\},$$

a triangle under the line $x = 1$, above $x = t$, and to the right of $t = 0$.

If we fix t instead, then for $0 \le t \le 1$, x runs from $x = t$ to $x = 1$. Thus

$$I = \int_0^1 \int_t^1 e^{-t^2}\, dx\, dt.$$

Since e^{-t^2} does not depend on x, we get

$$\int_t^1 e^{-t^2}\, dx = (1 - t)e^{-t^2},$$

and therefore

$$I = \int_0^1 (1 - t)e^{-t^2}\, dt = \int_0^1 e^{-t^2}\, dt - \int_0^1 te^{-t^2}\, dt.$$

The first term is A:

$$\int_0^1 e^{-t^2}\, dt = A.$$

To evaluate the second term, use the substitution $u = t^2$, $du = 2t\, dt$, so $t\, dt = \frac{1}{2}\, du$ and when t goes from 0 to 1, u goes from 0 to 1. Hence

$$\int_0^1 te^{-t^2}\, dt = \frac{1}{2}\int_0^1 e^{-u}\, du = \frac{1}{2}\left[-e^{-u}\right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right).$$

Thus

$$I = A - \frac{1}{2}\left(1 - \frac{1}{e}\right) = A + \frac{1}{2}\left(\frac{1}{e} - 1\right).$$

Therefore,

$$\int_0^1 \int_0^x e^{-t^2}\, dt\, dx = A + \frac{1}{2}\left(\frac{1}{e} - 1\right),$$

as required.

7. Let $f(x, y, z) = xyz$ and $S = \{(x, y, z) : x^2 + y^2 + z^2 = 6\}$. Use Lagrange multiplier method to find maximum and minimum values of f on S. **5**

Solution: We want to extremize

$$f(x, y, z) = xyz$$

subject to the constraint

$$g(x, y, z) = x^2 + y^2 + z^2 - 6 = 0.$$

By the method of Lagrange multipliers, at an extremum we must have

$$\nabla f = \lambda \nabla g.$$

Compute

$$\nabla f = (yz,\, xz,\, xy), \qquad \nabla g = (2x,\, 2y,\, 2z).$$

Thus,

$$yz = 2\lambda x, \qquad xz = 2\lambda y, \qquad xy = 2\lambda z, \qquad x^2 + y^2 + z^2 = 6. \tag{$*$}$$

Case 1: $xyz \neq 0$ (so $x, y, z \neq 0$).

From $(*)$ we have

$$yz = 2\lambda x, \quad xz = 2\lambda y, \quad xy = 2\lambda z.$$

Multiplying by x, y, z respectively, we get

$$xyz = 2\lambda x^2, \qquad xyz = 2\lambda y^2, \qquad xyz = 2\lambda z^2.$$

Since $xyz \neq 0$, we can divide to obtain

$$x^2 = y^2 = z^2.$$

Hence there exists $a > 0$ such that $x^2 = y^2 = z^2 = a^2$. Using the constraint,

$$x^2 + y^2 + z^2 = 3a^2 = 6 \quad \Rightarrow \quad a^2 = 2 \quad \Rightarrow \quad a = \sqrt{2}.$$

Therefore

$$(x, y, z) = (\pm\sqrt{2},\, \pm\sqrt{2},\, \pm\sqrt{2}),$$

with all 8 choices of signs.

At such a point,

$$f(x, y, z) = xyz = (\pm\sqrt{2})(\pm\sqrt{2})(\pm\sqrt{2}) = \pm(\sqrt{2})^3 = \pm 2\sqrt{2}.$$

If the number of minus signs is even, then $f = 2\sqrt{2}$; if the number of minus signs is odd, then $f = -2\sqrt{2}$.

Case 2: $xyz = 0$.

Then at least one of x, y, z is zero, so $f(x, y, z) = 0$. On the sphere $x^2 + y^2 + z^2 = 6$, such points exist (for example $(\pm\sqrt{6}, 0, 0)$, etc.). These are not global maxima or minima because the value 0 lies strictly between $-2\sqrt{2}$ and $2\sqrt{2}$.

Since S is compact and f is continuous, the extreme values on S are attained and must be among the above critical values.

Conclusion.

- The *maximum* value of f on S is

  $$f_{\max} = 2\sqrt{2},$$

  attained at the four points with an even number of minus signs:

  $$(\sqrt{2}, \sqrt{2}, \sqrt{2}),\ (\sqrt{2}, -\sqrt{2}, -\sqrt{2}),\ (-\sqrt{2}, \sqrt{2}, -\sqrt{2}),\ (-\sqrt{2}, -\sqrt{2}, \sqrt{2}).$$

- The *minimum* value of f on S is

$$f_{\min} = -2\sqrt{2},$$

attained at the four points with an odd number of minus signs:

$$(\sqrt{2}, \sqrt{2}, -\sqrt{2}),\ (\sqrt{2}, -\sqrt{2}, \sqrt{2}),\ (-\sqrt{2}, \sqrt{2}, \sqrt{2}),\ (-\sqrt{2}, -\sqrt{2}, -\sqrt{2}).$$

8. Evaluate the double integral $\iint_D \sqrt{x+y}(y-2x)^2 dydx$ over the domain D bounded by $x = 0$, $y = 0$, and $x + y = 1$. **5**

$$I = \iint_D \sqrt{x+y}\,(y-2x)^2\,dy\,dx,$$

where D is the triangular region bounded by $x = 0$, $y = 0$ and $x + y = 1$.

Solution: We use the change of variables

$$u = x + y, \qquad v = y - 2x.$$

Step 1: Express (x, y) in terms of (u, v).

We have

$$\begin{cases} u = x + y, \\ v = y - 2x. \end{cases}$$

From $u = x + y$ we get $y = u - x$. Substitute into $v = y - 2x$:

$$v = (u - x) - 2x = u - 3x \ \Rightarrow\ x = \frac{u - v}{3}.$$

Then

$$y = u - x = u - \frac{u - v}{3} = \frac{3u - (u - v)}{3} = \frac{2u + v}{3}.$$

Thus

$$x = \frac{u - v}{3}, \qquad y = \frac{2u + v}{3}.$$

Step 2: Jacobian.

Compute

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3} \cdot \frac{1}{3} - \left(-\frac{1}{3} \cdot \frac{2}{3}\right) = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}.$$

Hence

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv = \frac{1}{3}\,du\,dv.$$

Also

$$x + y = u, \qquad y - 2x = v,$$

so the integrand becomes

$$\sqrt{x+y}\,(y-2x)^2 = \sqrt{u}\,v^2.$$

Step 3: Image of the region D.

The region D is

$$D = \{(x,y) : x \geq 0,\ y \geq 0,\ x+y \leq 1\}.$$

In terms of (u,v):

- $x+y \leq 1 \iff u \leq 1$, and $x,y \geq 0 \implies u = x+y \geq 0$, so $0 \leq u \leq 1$.
- $x \geq 0 \iff \dfrac{u-v}{3} \geq 0 \iff u-v \geq 0 \iff v \leq u$.
- $y \geq 0 \iff \dfrac{2u+v}{3} \geq 0 \iff 2u+v \geq 0 \iff v \geq -2u$.

Thus the image region R in the (u,v)-plane is

$$R = \{(u,v) : 0 \leq u \leq 1,\ -2u \leq v \leq u\}.$$

Step 4: Transform and evaluate the integral.

We get

$$I = \iint_D \sqrt{x+y}\,(y-2x)^2\,dy\,dx = \iint_R \sqrt{u}\,v^2 \cdot \frac{1}{3}\,dv\,du.$$

So

$$I = \int_{u=0}^{1} \int_{v=-2u}^{u} \frac{1}{3}\sqrt{u}\,v^2\,dv\,du.$$

First compute the inner integral:

$$\int_{-2u}^{u} v^2\,dv = \left[\frac{v^3}{3}\right]_{-2u}^{u} = \frac{u^3}{3} - \frac{(-2u)^3}{3} = \frac{u^3}{3} - \frac{-8u^3}{3} = \frac{9u^3}{3} = 3u^3.$$

Therefore

$$I = \int_0^1 \frac{1}{3}\sqrt{u}\,(3u^3)\,du = \int_0^1 u^3 u^{1/2}\,du = \int_0^1 u^{7/2}\,du.$$

Now

$$\int_0^1 u^{7/2}\,du = \left[\frac{u^{9/2}}{9/2}\right]_0^1 = \frac{2}{9}.$$

Hence

$$\boxed{I = \frac{2}{9}}.$$

9. Let C be the intersection of the cylinder $x^2 + y^2 = 1$ with plane $x + 2y + 3z = 3$ which is parameterised by R. Let F be a vector field with $curlF = i + 2j - \alpha k$ for some $\alpha \in \mathbb{R}$. If $\oint\limits_C F \cdot dR = -\frac{3\pi}{2}$, then find α. **5**

Solution: By Stoke's Theorem,

$$\oint\limits_C F \cdot dR \;=\; \iint_S (\nabla \times F) \cdot \mathbf{n}\, d\sigma.$$

Here $\mathbf{n} = \frac{i+2j+3k}{\sqrt{14}}$, and so $\nabla \times F \cdot \mathbf{n} = \frac{5-3\alpha}{\sqrt{14}}$. Thus,

$$\iint_S \frac{5-3\alpha}{\sqrt{14}} d\sigma = \iint_{x^2+y^2 \le 1} \frac{5-3\alpha}{\sqrt{14}} \sqrt{1 + f_x^2 + f_y^2}\; dxdy,$$

where $f(x, y) = 1 - \frac{1}{3}x - \frac{2}{3}y$. So, $\sqrt{1 + f_x^2 + f_y^2} = \frac{\sqrt{14}}{3}$. This implies

$$\oint\limits_C F \cdot dR = \iint_S \frac{5-3\alpha}{\sqrt{14}} d\sigma = \iint_{x^2+y^2 \le 1} \frac{5-3\alpha}{3} dxdy = \frac{5-3\alpha}{3}\pi.$$

Since $\oint\limits_C F \cdot dR = -\frac{3\pi}{2}$, we have

$$\boxed{\alpha = \frac{19}{6}}.$$

10. Evaluate the line integral $\oint\limits_C 2xyzdx + x^2zdy + x^2ydz$, where C is parametrised by $R(t) = \cos t\, i + \frac{t}{2\pi}\, j + \sin t\, k$, $0 < t < \frac{\pi}{2}$. **3**

Solution: Here $x = \cos t$, $y = \frac{t}{2\pi}$, $z = \sin t$. Observe that $\nabla\varphi = (2xyz, x^2z, x^2y)$ such that $\varphi(x, y, z) = x^2yz$.

$$I = \oint_C 2xyzdx + x^2zdy + x^2ydz = \oint_C \nabla\varphi.dr = \varphi(R(\pi/2)) - \varphi(R(0)).$$

But $R(0) = (1, 0, 0), R(\pi/2) = (0, 1/4, 1) \implies I = 0$.

11. Find the area of the surface $z = 2xy$ inside the cylinder $x^2 + y^2 = 2$. **4**

Solution: $z = f(x, y) = 2xy$ and cylinder $x^2 + y^2 = 2$. $f_x = 2y, f_y = 2x$.

$$\text{Area} \;=\; \iint_{x^2+y^2 \le 1} \sqrt{1 + 4(y^2 + x^2)}dxdy.$$

Let $x = r\cos\theta, y = r\sin\theta$. Then

$$\begin{aligned}\text{Area} \;&=\; \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{1 + 4r^2}\; rdrd\theta \\ &= 2\pi \int_0^{\sqrt{2}} \sqrt{1 + 4r^2}\; rdr = \frac{13\pi}{3}.\end{aligned}$$

12. Evaluate the line integral $\oint\limits_C xydx + 2x^2dy$, where C is the line joining $(-2, 0)$ and $(2, 0)$ and the upper half of the circle $x^2 + y^2 = 4$. **4**

Solution: $I = \oint xydx + 2x^2dy$.

Let $M = xy$ and $N = 2x^2$. Since C is closed and piecewise smooth, then by Green's theorem

$$I = \iint_D (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy = \iint_D 3xdxdy$$

Let $x = r\cos\theta, y = r\sin\theta$, then

$$I = 3\int_0^\pi \int_{r=0}^2 r^2dr\cos\theta d\theta = 3\left(\int_{r=0}^2 r^2dr\right)\left(\int_0^\pi \cos\theta d\theta\right) = 0.$$

—–End—–