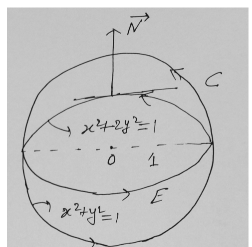
# MA15010H: Multi-variable Calculus

(Practice problem set 7 Hint/ Model solutions: Line and surface integrals)

September - November, 2025

1. Let $\vec{N}$ be the unit outward normal vector on the ellipse $x^2 + 2y^2 = 1$. Evaluate the line integral $\int\limits_C \vec{N}.d\vec{R}$ along the unit circle $C = \{(x, y) : \ x^2 + y^2 = 1\}$.

   **Solution:** The ellipse $x^2 + 2y^2 = 1$ can be represented by $E(t) = \left(\cos t, \frac{\sin t}{\sqrt{2}}\right)$ with $0 \le t < 2\pi$. This implies that normal vector to $E$ will be $(y'(t), -x'(t)) = \left(\frac{\cos t}{\sqrt{2}}, \cos t\right)$. Hence the unit normal vector $\vec{N}(t) = \sqrt{\frac{2}{3}}\left(\frac{\cos t}{\sqrt{2}}, \cos t\right)$. Let $C$ be represented by $R(t) = (\cos t, \sin t),\ 0 \le t < 2\pi$. Please refer to Figure 1.

   
   
   Figure 1

   Thus,

   $$\int\limits_C \vec{N}.d\vec{R} = \sqrt{\frac{2}{3}}\left(\frac{\cos t}{\sqrt{2}}, \cos t\right).(-\sin t, \cos t)dt.$$

2. Use second fundamental theorem of calculus for the line integral to show that $\int_C ydx + (x + z)dy + ydz$ is independent of any path $C$ joining the points $(2, 1, 4)$ and $(8, 3, -1)$.

   **Solution:** Let $F(x, y, z) = (y, x + z, y)$. Consider $f(x, y, z) = xy + yz + c$. Then $\nabla f(x, y, z) = F(x, y, z)$. Hence, by second FTC for line integral

   $$\int_C \nabla f \cdot dR = f(2, 1, 4) - f(8, 3, -1).$$

   That is, the given line integral is path independent. **Note that** one can $\nabla f = F$ for $f$ by doing indefinite integral.

3. Consider the curve $C$ which is the intersection of the surfaces $x^2 + y^2 = 1$ and $z = x^2$. Assume that $C$ is oriented counterclockwise as seen from the positive $z$-axis. Evaluate $\int\limits_C zdx - xydy - xdz$.

   **Solution:**

Let $F(x,y,z) = (z, -xy, -x)$. The cylinder $x^2 + y^2 = 1$ can be parameterized as

$$\{(\cos\theta, \sin\theta, z) : \ 0 \le \theta < 2\pi, z \in \mathbb{R}\}.$$

Since $C$ also lies in $z = x^2$, this implies $C = \{(\cos\theta, \sin\theta, \cos^2\theta) : \ 0 \le \theta < 2\pi\}$. Please see the Figure 2.

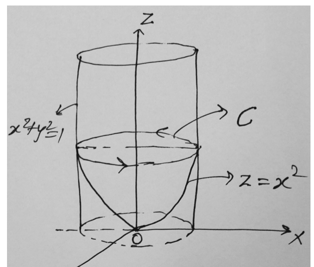


Figure 2

The required line integral is

$$\int\limits_C f \cdot dR = \int_0^{2\pi} f(R(\theta)) \cdot R'(\theta) d\theta = 0.$$

4. Let $f(x,y,z) = (x^2, xy, 1)$. Show that that there is no $\phi$ such that $\nabla\phi = f$.
**Solution:** If there exists $\phi$ such that $\nabla\phi = f$, then $0 = \text{curl } \nabla\phi = \text{curl } f$ should be satisfied. But that is not the case here.
5. Let $C$ be a curve represented by two parametric representations such that $C = \{R_1(s) : \ s \in [a,b]\} = \{R_2(t) : \ t \in [c,d]\}$, where $R_1 : [a,b] \to \mathbb{R}^3$ and $R_2 : [c,d] \to \mathbb{R}^3$ be two distinct differentiable one-one maps.
(a) Show that there exists a function $h : [c,d] \to [a,b]$ such that $R_2(t) = R_1(h(t))$.
(b) If $R_1$ and $R_2$ trace out $C$ in the same direction, then $\int\limits_C f \cdot dR_1 = \int\limits_C f \cdot dR_2$.
(c) If $R_1$ and $R_2$ trace out $C$ in the opposite direction, then $\int\limits_C f \cdot dR_1 = -\int\limits_C f \cdot dR_2$.
**Solution:** (a) Consider $h(t) = R_1^{-1}(R_2(t))$.
(b & c) By chain rule $R_2'(t) = R_1'(t)h'(t)$. Therefore, the required line integral

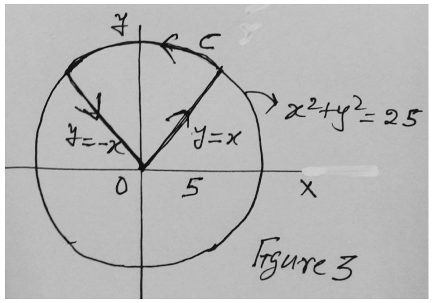
$$\int_C f.dR_2 = \int_c^d f(R_2(t)) \cdot R_2'(t)dt = \int_c^d f(R_1(h(t)))) \cdot R_1'(h(t))h'(t)dt.$$

Let $u = h(t)$. Then

$$\int_C f.dR_2 = \int_{h(c)}^{h(d)} f(R_1(u)) \cdot R_1'(u)du = \pm\int_a^b f(R_1(u)) \cdot R_1'(u)du = \pm\int_C f \cdot dR_1.$$

6. Evaluate the line integral $\oint\limits_C (x^2\sin^2 x - y^3)dx + (y^2\cos^2 y - y)dy$, where $C$ is the closed curve consisting $x + y = 0$, $x^2 + y^2 = 25$ and $y = x$ and lying in the first and fourth quadrants.

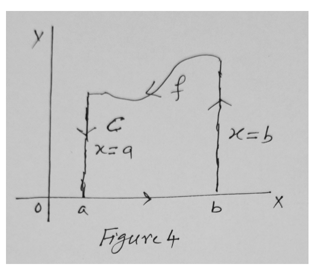
**Solution:** Let $D$ be the domain enclosed by $C$ as shown in Figure 3.




By Green's Theorem,

$$\oint_C (x^2 \sin^2 x - y^3)dx + (y^2 \cos^2 y - y)dy = \iint_D 3y^2 dxdy = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \int_0^5 3r^2 \sin^2 \theta dr d\theta.$$

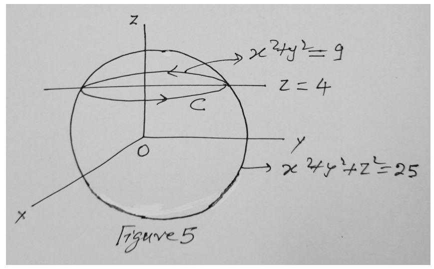
7. Let $f : [a, b] \to \mathbb{R}$ be a non-negative continuously differentiable function. Suppose $C$ is the boundary of the region bounded above by the graph of $f$, below by the $x$-axis and on the sides by the lines $x = a$ and $x = b$. Show that $\int\limits_a^b f(x)dx = -\oint\limits_C ydx$.
**Solution:** Let $D$ be the domain enclosed by $C$ as shown in Figure 4.




It follows from Green's theorem that

$$\int_a^b f(x)dx = \text{Area}(D) = \iint_D 1dxdy = -\int_C (ydx + 0dy).$$

8. Let $F(x, y, z) = (y, -x, 2z^2 + x^2)$ and $S$ be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies below the plane $z = 4$. Evaluate $\iint\limits_S \text{curl}\, F \cdot \hat{n} d\sigma$, where $\hat{n}$ is the unit outward normal of $S$.
**Solution:** Let $C$ be the boundary of the surface $S$ as shown in Figure 5.




Then $C = \{(3\sin\theta, 3\cos\theta, 4) : 0 \le \theta < 2\pi\}$. Note that C is oriented clockwise when viewed from above. By Stoke's Theorem

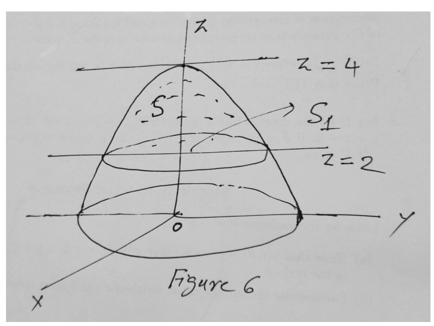
$$\iint_S \operatorname{curl} F \cdot \hat{n} d\sigma = \oint_C F \cdot dR = \int_0^{2\pi} F(R(\theta)) \cdot R'(\theta) d\theta = 18\pi.$$

9. Let $C$ be the boundary of the cone $z = x^2 + y^2$ and $0 \le z \le 1$. Use Stoke's theorem to evaluate the line integral $\int_C \vec{F}.d\vec{R}$ where $\vec{F} = (y, xz, 1)$.
**Solution:** Let $f(x, y, x) = x^2 + y^2 - z$. Then $S = \{(x, y, z) : \ f(x, y, x) = 0\}$ will be the surface of the domain. The unit normal to $S$ will be $\hat{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{(2x, 2y, -1)}{\sqrt{4(x^2+y^2)+1}}$. Let $z = g(x, y) = x^2 + y^2$. Then $d\sigma = \sqrt{g_x^2 + g_y^2 + 1} dxdy$. By Stoke's Theorem,

$$\int_C \vec{F}.d\vec{R} = \iint_S \operatorname{curl} \vec{F} \cdot \hat{n} \ d\sigma = \iint_R \operatorname{curl} \vec{F} \cdot \hat{n} \sqrt{1 + g_x^2 + g_y^2} dxdy,$$

where $R = \{(x, y) : \ x^2 + y^2 \le 1\}$.

10. Let $\vec{F} = (xy, yz, zx)$ and $S$ be the surface $z = 4 - x^2 - y^2$ with $2 \le z \le 4$. Use divergence theorem to find the surface integral $\iint_S \vec{F}.\vec{n} \ d\sigma$.
**Solution:** Let $S_1 = \{(x, y, 2) : x^2 + y^2 \le 2\}$. Please refer to Figure 6.




By divergence theorem,

$$\iint_S \vec{F}.\vec{n} \ d\sigma + \iint_{S_1} \vec{F}.\vec{n_1} \ d\sigma_1 = \iiint_D \operatorname{div} \vec{F} \ dxdydz.$$

Here

$$\iint_{S_1} \vec{F}.\vec{n}\ d\sigma_1 = \iint_{x^2+y^2\leq 2} (xy, yz, zx).(-k)dxdy$$

11. Let $S$ be the sphere $x^2+y^2+z^2=1$. If some $\alpha \in \mathbb{R}$ satisfies $\iint_S (zx+\alpha y^2+xz)d\sigma = \frac{4\pi}{3}$, then find $\alpha$.

**Solution:**

Let $D$ denote the solid enclosed by the surface $S$. By divergence theorem

$$\iint_S (z, \alpha y, x).(x, y, z)d\sigma = \iiint_D \alpha dxdydz = \alpha\frac{4\pi}{3}.$$

Hence $\alpha = 1$.