MA15010H: Multi-variable Calculus

(Practice problem set 5 Hint/Model solutions: Riemann Integration, Fubini's Theorem)
September - November, 2025

1. Let $f : D = [a,b] \times [c,d] \to \mathbb{R}$ be defined by $f(x,y) = \varphi(x)\psi(y)$, where $\varphi : [a,b] \to \mathbb{R}$ and $\psi : [c,d] \to \mathbb{R}$ are continuous. Show that

$$\iint\limits_D f(x,y)dxdy = \left(\int_a^b \varphi(x)dx\right)\left(\int_c^d \psi(x)dx\right).$$

Solution: Easily followed by Fubini's theorem.

2. Let $f : D = [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}^c \cap [0,1]; \\ 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \text{ and } y \in \mathbb{Q}^c \cap [0,1]; \\ 1 - \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in lowest term and } y \in \mathbb{Q} \cap [0,1]. \end{cases}$$

Then f is integrable and $\iint\limits_D f(x,y)dxdy = 1$. Does repeated integral $\int\limits_0^1 \left(\int\limits_0^1 f(x,y)dy\right)dx$ exist?

Solution: It easy to see that $\int_0^1 f(x,y)dy = 1$ if $x \in \mathbb{Q}^c$ and it does not exist when $x \in \mathbb{Q}$. Hence the repeated integral $\int\limits_0^1 \left(\int\limits_0^1 f(x,y)dy\right)dx$ does not exist. Let $P_n = \{\frac{i}{n} : i = 0,1,\ldots,n\} \times \{\frac{j}{n} : j = 0,1,\ldots,n\}$. Note that $M_{ij} = 1$ and $m_{ij} \geq 1 - \frac{1}{n}$ for all i,j. Hence $U(P_n, f) = 1$. Now,

$$L(P_n, f) = \sum_{i=1}^n \sum_{j=1}^n m_{ij}\Delta x_i \Delta y_j \geq \sum_{i=1}^n \sum_{j=1}^n \left(1 - \frac{1}{n}\right)\frac{1}{n^2} = 1 - \frac{1}{n}.$$

This implies $U(P_n, f) - L(P_n, f) \leq \frac{1}{n} \to 0$. Thus the double integral f exists and $\iint\limits_D f(x,y)dxdy = 1$.

3. Find the volume of the tetrahedron T bounded by the planes $x = 0,\ y = 0,\ z = 0$, and $x - y - z = -1$.

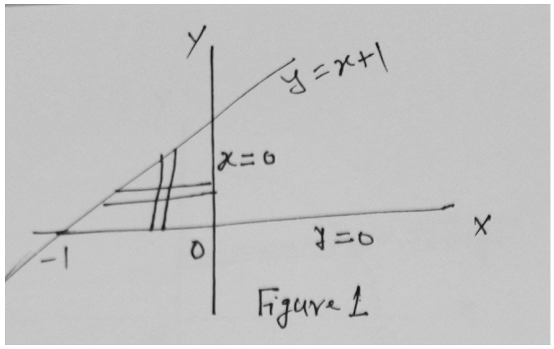
Solution: Let $z = f(x,y) = x - y + 1$. Let D be the projection of the plane $x - y - z = -1$ to the xy-plane. Please see Figure 1.

Then the volume of T is given by

$$\int\limits_D z dxdy = \int_{-1}^0 \left(\int_{y=0}^{x+1} (x - y + 1)dy\right)dx = \frac{1}{6}.$$

4. Evaluate the following iterated integrals applying Fubini's Theorem.

(a) $\int\limits_0^1 \int\limits_{x=y}^1 \cos(x^2)dxdy$.



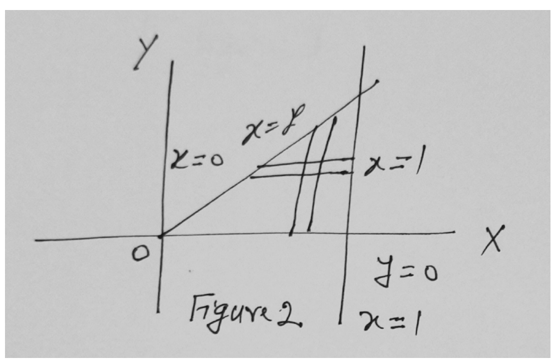
(b) $\int\limits_0^1 \int\limits_{y=\sqrt{x}}^1 e^{y^3} dydx.$

(c) $\int\limits_0^1 \int\limits_{y=x^2}^1 x^3 e^{y^3} dydx.$

(d) $\int\limits_0^1 \int\limits_{x=y}^1 \frac{1}{1+x^4} dxdy.$

(e) $\int_0^1 (\tan^{-1} \pi x - \tan^{-1} x)dx.$

Solution:(a) Please see Figure 2.

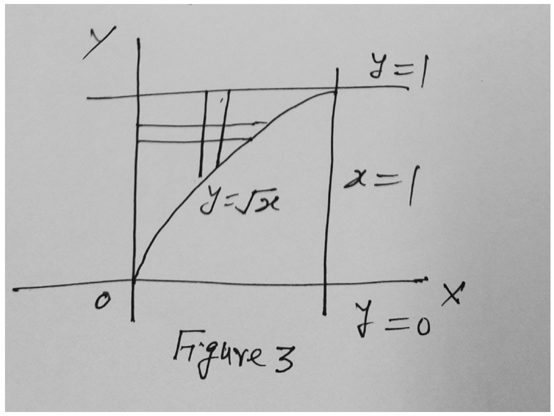


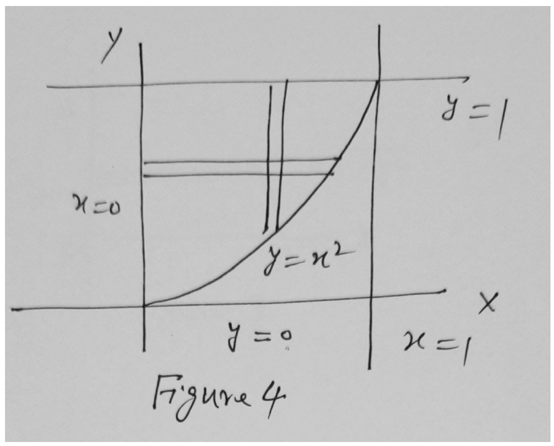
$$\int_0^1 \int_{x=y}^1 \cos(x^2) dxdy = \int_0^1 \left(\int_0^x \cos(x^2) dy \right) dx = \int_0^1 c \cos x^2 dx = \frac{1}{2} \sin 1.$$

(b) Please see Figure 3.

$$\int_0^1 \int_{y=\sqrt{x}}^1 e^{y^3} dydx = \int_0^1 \left(\int_0^{y^2} e^{y^3} dx \right) dy = \frac{1}{3} \int_0^1 e^u du = \frac{1}{3}(e-1).$$

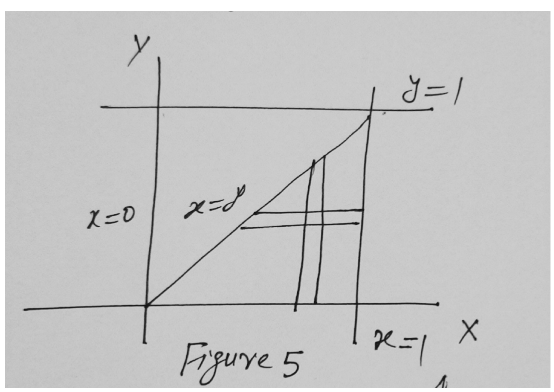
(c) Please see Figure 4.





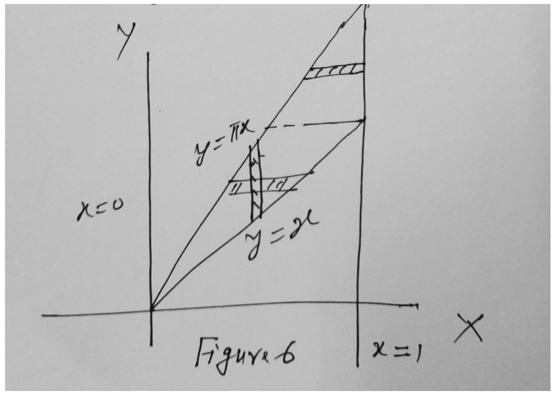
$$\int_0^1 \int_{y=x^2}^1 x^3 e^{y^3} dydx = \int_0^1 \int_0^{\sqrt{y}} x^3 e^{y^3} dxdy = \int_0^1 \frac{1}{4} y^2 e^{y^3} dy = \frac{1}{12}(e-1).$$

(d) Please see Figure 5.



$$\int_0^1 \int_{x=y}^1 \frac{1}{1+x^4} dxdy = \int_0^1 \left(\int_0^x \frac{1}{1+x^4} dy \right) dx = \frac{1}{2} \int_0^1 \frac{du}{1+u^2} = \frac{\pi}{8}.$$

(e) Please see Figure 6.

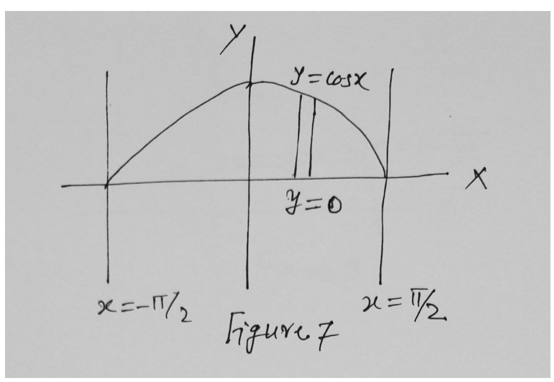


Note that

$$\begin{aligned}\int_0^1 (\tan^{-1}\pi x - \tan^{-1} x)dx &= \int_0^1 \int_{y=x}^{\pi x} \frac{1}{1+y^2} dydx \\ &= \int_0^1 \int_{x=\frac{y}{\pi}}^{y} \frac{1}{1+y^2} dxdy + \int_1^{x=\pi} \int_{\frac{y}{\pi}}^{1} \frac{1}{1+y^2} dxdy.\end{aligned}$$

5. Let D be the region lying below the curve $y = \cos x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and above the x-axis. Evaluate $\iint\limits_D \sin x dxdy$.

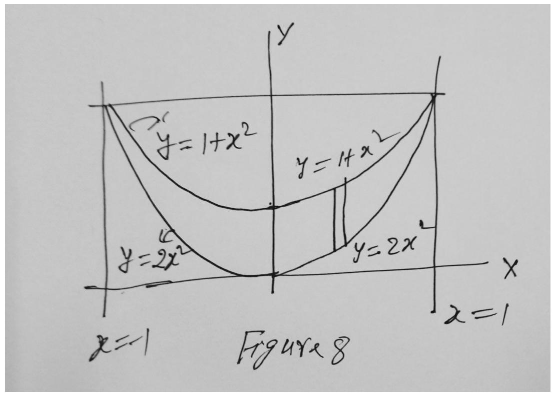
Solution: Please see Figure 7.



$$\iint\limits_D \sin x dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{y=0}^{\cos x} \sin x dydx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos x dx = 0.$$

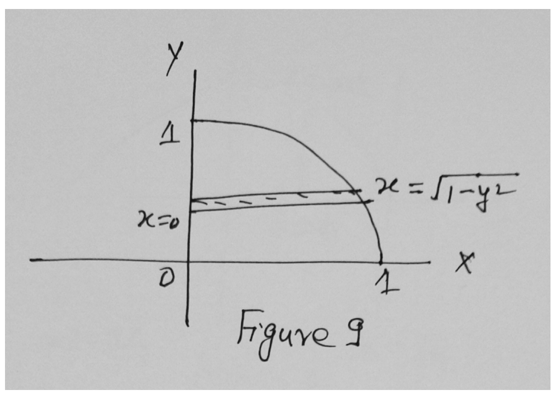
6. Let D be the region in $\mathbb{R}^2$ bounded by the curves $y = 2x^2$ and $y = 1 + x^2$. Evaluate the double integral $\iint\limits_D (2x^2 + y)dxdy$.

Solution: Please see Figure 8.



$$\iint_D (2x^2 + y)dxdy = \int_{-1}^{1} \left(\int_{y=2x^2}^{1+x^2} (3x^2 + y)dy \right) dx.$$

7. Evaluate $\iint\limits_D x \cos\left(y - \frac{y^3}{3}\right) dxdy$, where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$.
Solution: Please see Figure 9.



$$\begin{aligned}\iint_D x \cos\left(y - \frac{y^3}{3}\right) dxdy &= \int_0^1 \int_{x=0}^{\sqrt{1-y^2}} \cos\left(y - \frac{y^3}{3}\right) dxdy \\ &= \frac{1}{2} \int_0^1 (1 - y^2) \cos\left(y - \frac{y^3}{3}\right) dy = \frac{1}{2} \int_0^{\frac{3}{2}} \cos t dt.\end{aligned}$$

8. Find the volume of the solid D enclosed by the surfaces $z = 6 - x^2 - y^2,\ z = 2x^2 + y^2 - 1,\ x = -1,\ x = 1,\ y = -1$ and $y = 1$.

Solution: Note that $(6-x^2-y^2)-(2x^2+y^2-1) \geq 0$ for all $(x,y) \in [-1.1] \times [-1,1]$. The volume of D is given by

$$\int_{-1}^{1}\int_{-1}^{1}(6-x^2-y^2)-(2x^2+y^2-1)dydx.$$

9. Let D be the solid bounded by the surfaces $y = x^2,\ y = 3x,\ z = 0$ and $z = x^2 + y^2$. Find the volume of the solid.
Solution: Let R be the region in $\mathbb{R}^2$ bounded by the curves $y = x^2$ and $y = 3x$. Then the volume of D is

$$\iint\limits_R (x^2+y^2)dxdy = \int_0^3 \int_{y=x^2}^{3x} (x^2+y^2)dydx.$$

10. Let D be the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y + z = 1$ and $z = 0$. Find the volume of the solid.
Solution: Let $R = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Then the solid D lies above the region R and below the graph $z = 1 - y$. The volume of D is

$$\iint\limits_R (1-y)dxdy = \iint\limits_R dxdy - \iint\limits_R ydxdy.$$

Note that

$$\iint\limits_R ydxdy = \int_{-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} ydydx = 0.$$

Hence required volume is π.