MA1501H: Multi-variable Calculus

(Practice problem set 4: Hints / Model Solution)

July - November, 2025

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1. Let $f(\mathbf{x}) = \|\mathbf{x}\|^{\frac{5}{2}}$ for all $\mathbf{x} \in \mathbb{R}^m$. Using chain rule, show that $f : \mathbb{R}^m \to \mathbb{R}$ is differentiable and determine $f'(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$.

   Solution: Let $g(\mathbf{x}) = \|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbb{R}^m$ and let $\varphi(x) = x^{\frac{5}{4}}$ for all $x \in [0, \infty)$. Then we know that $g : \mathbb{R}^m \to \mathbb{R}$ and $\varphi : [0, \infty) \to \mathbb{R}$ are differentiable with $\nabla g(\mathbf{x}) = 2\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^m$ and $\varphi'(x) = \frac{5}{4}x^{\frac{1}{4}}$ for all $x \in [0, \infty)$. Since $f(\mathbf{x}) = \varphi(g(\mathbf{x}))$ for all $\mathbf{x} \in \mathbb{R}^m$, by chain rule, $f = \varphi \circ g$ is differentiable and for each $\mathbf{x} \in \mathbb{R}^m$, $\nabla f(\mathbf{x}) = \varphi'(g(\mathbf{x}))\nabla g(\mathbf{x}) = \frac{5}{2}\sqrt{\|\mathbf{x}\|}\,\mathbf{x}$. Therefore $f'(\mathbf{x}) = \frac{5}{2}\sqrt{\|\mathbf{x}\|}\begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}$ for all $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable and let $u(x, y, z) = f(x - y, y - z, z - x)$ for all $(x, y, z) \in \mathbb{R}^2$. Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ at each point of $\mathbb{R}^3$.

   Solution: Let $r(x, y, z) = x - y$, $s(x, y, z) = y - z$ and $t(x, y, z) = z - x$ for all $(x, y, z) \in \mathbb{R}^3$. Since all the partial derivatives of r, s, t are continuous on $\mathbb{R}^3$, r, s, $t : \mathbb{R}^3 \to \mathbb{R}$ are differentiable. Hence by chain rule, we get $\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t}$,
   $\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial f}{\partial s} - \frac{\partial f}{\partial r}$, and $\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial z} = \frac{\partial f}{\partial t} - \frac{\partial f}{\partial s}$, where each of the partial derivatives has been considered at the respective point. Therefore $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ at each point of $\mathbb{R}^3$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable and let $u(r, \theta) = f(r\cos\theta, r\sin\theta)$ for all $r > 0$, $\theta \in \mathbb{R}$. Show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ at each point $(x, y) = (r\cos\theta, r\sin\theta)$ of $\mathbb{R}^2 \setminus \{(0, 0)\}$.

   Solution: Since all the second order partial derivatives of f are continuous on $\mathbb{R}^2$, $\frac{\partial^2 f}{\partial x\partial y}(x, y) = \frac{\partial^2 f}{\partial y\partial x}(x, y)$ for all $(x, y) \in \mathbb{R}^2$.
   Let $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$ for all $(r, \theta) \in \mathbb{R}^2$. Then x, $y : \mathbb{R}^2 \to \mathbb{R}$ are twice continuously differentiable. Hence by chain rule, for all $(x, y) = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2$, we get
   $\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}$,
   $\frac{\partial^2 u}{\partial r^2} = \cos\theta\big(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial r}\big) + \sin\theta\big(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r}\big) = \cos^2\theta\frac{\partial^2 f}{\partial x^2} + \sin 2\theta\frac{\partial^2 f}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 f}{\partial y^2}$,
   $\frac{\partial u}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}$,
   and $\frac{\partial^2 u}{\partial \theta^2} = -r\cos\theta\frac{\partial f}{\partial x} - r\sin\theta\frac{\partial f}{\partial y} - r\sin\theta\big(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial \theta}\big) + r\cos\theta\big(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial \theta}\big)$
   $= r^2\sin^2\theta\frac{\partial^2 f}{\partial x^2} - r^2\sin 2\theta\frac{\partial^2 f}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2}$,
   where the partial derivatives have been considered at the respective points.
   Therefore $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = (\cos^2\theta + \sin^2\theta)\frac{\partial^2 f}{\partial x^2} + (\cos^2\theta + \sin^2\theta)\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ at each point $(x, y) = (r\cos\theta, r\sin\theta)$ of $\mathbb{R}^2 \setminus \{(0, 0)\}$.

4. Show that a differentiable function $f : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}$ is homogeneous of degree $\alpha \in \mathbb{R}$ (*i.e.* $f(t\mathbf{x}) = t^\alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and for all $t > 0$) iff $\nabla f(\mathbf{x}) \cdot \mathbf{x} = \alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$.

(The only if part of this result is known as Euler's theorem on homogeneous functions.)

Solution: We first assume that f is homogeneous of degree α. Let $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and let $\varphi(t) = f(t\mathbf{x}) - t^\alpha f(\mathbf{x})$ for all $t > 0$. Then $\varphi(t) = 0$ for all $t > 0$ and since f is differentiable, by chain rule, we get $\varphi'(t) = \nabla f(t\mathbf{x}) \cdot \mathbf{x} - \alpha t^{\alpha-1} f(\mathbf{x}) = 0$ for all $t > 0$. Putting $t = 1$, we obtain $\nabla f(\mathbf{x}) \cdot \mathbf{x} = \alpha f(\mathbf{x})$.

Conversely, let $\nabla f(\mathbf{x}) \cdot \mathbf{x} = \alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$. Let $\mathbf{x} \in \mathbb{R}^m \setminus \{0\}$ and let $g(t) = t^{-\alpha} f(t\mathbf{x})$ for all $t > 0$. Since f is differentiable, by chain rule, $g : (0, \infty) \to \mathbb{R}$ is differentiable and $g'(t) = t^{-\alpha}\, \nabla f(t\mathbf{x}) \cdot \mathbf{x} - \alpha t^{-\alpha-1} f(t\mathbf{x}) = \alpha t^{-\alpha-1}\, \nabla f(t\mathbf{x}) \cdot \mathbf{x} - \alpha t^{-\alpha-1} f(t\mathbf{x}) = 0$ for all $t > 0$. Hence g is a constant function and so $g(t) = g(1) = f(\mathbf{x})$ for all $t > 0$. Consequently $f(t\mathbf{x}) = t^\alpha f(\mathbf{x})$ for all $t > 0$ and therefore f is a homogeneous function of degree α.

5. If $f(x, y) = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ for all $(x, y) \in \mathbb{R}^2 \setminus S$, where $S = \{(x, x) : x \in \mathbb{R}\}$, then using Euler's theorem on homogeneous functions, show that $x f_x(x, y) + y f_y(x, y) = \sin(2f(x, y))$ for all $(x, y) \in \mathbb{R}^2 \setminus S$.

   Solution: If $g(x, y) = \tan(f(x, y)) = \frac{x^3+y^3}{x-y}$ for all $(x, y) \in \mathbb{R}^2 \setminus S$, then $g(tx, ty) = t^2 g(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus S$ and for all $t > 0$. Hence $g : \mathbb{R}^2 \setminus S \to \mathbb{R}$ is a homogeneous function of degree 2 and therefore by Euler's theorem on homogeneous functions, $xg(x, y) + yg(x, y) = 2g(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus S$. Again, by chain rule, $g_x(x, y) = \sec^2(f(x, y)) f_x(x, y)$ and $g_y(x, y) = \sec^2(f(x, y)) f_y(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus S$. Hence we get $x f_x(x, y) + y f_y(x, y) = 2\tan(f(x, y)) \cos^2(f(x, y)) = \sin(2f(x, y))$ for all $(x, y) \in \mathbb{R}^2 \setminus S$.

6. If $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ is a twice continuously differentiable homogeneous function of degree $n \in \mathbb{N}$, then show that $\left(x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2}\right)(x, y) = n(n-1) f(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

   Solution: By Euler's theorem on homogeneous functions, we get $x \frac{\partial f}{\partial x}(x, y) + y \frac{\partial f}{\partial y}(x, y) = nf(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Differentiating this partially with respect to x and y respectively, we get $x \frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial f}{\partial x}(x, y) + y \frac{\partial^2 f}{\partial x \partial y}(x, y) = nf_x(x, y)$ and $x \frac{\partial^2 f}{\partial y \partial x}(x, y) + y \frac{\partial^2 f}{\partial y^2}(x, y) + \frac{\partial f}{\partial y}(x, y) = nf_y(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Since the second order partial derivatives of f are continuous, we have $\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and hence by multiplying the above two relations by x and y respectively and then adding, we get $\left(x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2}\right)(x, y) + \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}\right)(x, y) = n\left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}\right)(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Therefore $\left(x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2}\right)(x, y) = n(n-1) f(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable such that $f_x(a, b) = f_y(a, b)$ for all $(a, b) \in \mathbb{R}^2$ and $f(a, 0) > 0$ for all $a \in \mathbb{R}$. Show that $f(a, b) > 0$ for all $(a, b) \in \mathbb{R}^2$.

   Solution: Let $(a, b) \in \mathbb{R}^2$ and let $g(t) = f(a + bt, b - bt)$ for all $t \in [0, 1]$. Then $g : [0, 1] \to \mathbb{R}$ is continuously differentiable. By the mean value theorem of single variable calculus, there exists $t_0 \in (0, 1)$ such that $g(1) - g(0) = g'(t_0) = \nabla f(a + bt_0, b - bt_0) \cdot (b, -b)$ (by chain rule) and hence $f(a + b, 0) - f(a, b) = bf_x(a + bt_0, b - bt_0) - bf_y(a + bt_0, b - bt_0) = 0$. Therefore

$f(a,b) = f(a+b,0) > 0$.

8. Let $\alpha > 0$ and let $f : \mathbb{R}^m \to \mathbb{R}$ satisfy $|f(\mathbf{x}) - f(\mathbf{y})| \leq \alpha \|\mathbf{x} - \mathbf{y}\|^2$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Show that f is a constant function.

   Solution: Let $\mathbf{x}_0, \mathbf{h} \in \mathbb{R}^m$. By the given condition $|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - \mathbf{0} \cdot \mathbf{h}| \leq \alpha \|\mathbf{h}\|^2$ and so $\lim\limits_{\mathbf{h} \to \mathbf{0}} \frac{|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - \mathbf{0} \cdot \mathbf{h}|}{\|\mathbf{h}\|} = 0$. Hence f is differentiable at $\mathbf{x}_0$ and $\nabla f(\mathbf{x}_0) = \mathbf{0}$. Since $\mathbf{x}_0 \in \mathbb{R}^m$ is arbitrary, f is differentiable and $\nabla f(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^m$. If $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$, then $L = \{(1-t)\mathbf{x}_1 + t\mathbf{x}_2 : t \in [0,1]\} \subseteq \mathbb{R}^m$ and hence by the mean value theorem, there exists $\mathbf{c} \in L$ such that $f(\mathbf{x}_2) - f(\mathbf{x}_1) = \nabla f(\mathbf{c}) \cdot (\mathbf{x}_2 - \mathbf{x}_1) = 0$. Thus $f(\mathbf{x}_1) = f(\mathbf{x}_2)$ and therefore f is a constant function.

9. Let S be a nonempty open and convex set in $\mathbb{R}^2$ and let $f : S \to \mathbb{R}$ be such that $f_x(x,y) = 0 = f_y(x,y)$ for all $(x,y) \in S$. Show that f is a constant function.
   (A set $S \subseteq \mathbb{R}^m$ is called convex if $(1-t)\mathbf{x} + t\mathbf{y} \in S$ for all $\mathbf{x}, \mathbf{y} \in S$ and for all $t \in [0,1]$.)

   Solution: Since $f_x(x,y) = 0 = f_y(x,y)$ for all $(x,y) \in S$, $f_x, f_y : S \to \mathbb{R}$ are continuous and hence f is differentiable. If $\mathbf{x}_1, \mathbf{x}_2 \in S$, then $L = \{(1-t)\mathbf{x}_1 + t\mathbf{x}_2 : t \in [0,1]\} \subseteq S$ (since S is convex) and hence by the mean value theorem, there exists $\mathbf{c} \in L$ such that $f(\mathbf{x}_2) - f(\mathbf{x}_1) = \nabla f(\mathbf{c}) \cdot (\mathbf{x}_2 - \mathbf{x}_1) = 0$, since $\nabla f(\mathbf{c}) = (f_x(\mathbf{c}), f_y(\mathbf{c})) = (0,0)$. Thus $f(\mathbf{x}_1) = f(\mathbf{x}_2)$ and therefore f is a constant function.

10. Find the equations of the tangent plane and the normal line to the surface given by $z = x^2 + y^2 - 2xy + 3y - x + 4$ at the point $(2, -3, 18)$.

    Solution: Let $f(x,y,z) = x^2 + y^2 - 2xy - x + 3y - z + 4$ for all $(x,y,z) \in \mathbb{R}^3$. Then $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable and $f_x(x,y,z) = 2x - 2y - 1$, $f_y(x,y,z) = 2y - 2x + 3$ and $f_z(x,y,z) = -1$ for all $(x,y,z) \in \mathbb{R}^3$. Hence the equation of the tangent plane to the given surface $f(x,y,z) = 0$ at $\mathbf{x}_0 = (2,-3,18)$ is $f_x(\mathbf{x}_0)(x-2) + f_y(\mathbf{x}_0)(y+3) + f_z(\mathbf{x}_0)(z-18) = 0$, *i.e.* $10(x-2) - 7(y+3) - (z-18) = 0$, which simplifies to $10x - 7y - z = 23$.
    Again, the equation of the normal line to the given surface $f(x,y,z) = 0$ at $\mathbf{x}_0$ is $\frac{x-2}{f_x(\mathbf{x}_0)} = \frac{x-2}{f_y(\mathbf{x}_0)} = \frac{x-2}{f_z(\mathbf{x}_0)}$, *i.e.* $\frac{x-2}{10} = \frac{y+3}{-7} = \frac{z-18}{-1}$.

11. Find all points on the paraboloid $z = x^2 + y^2$ at which the tangent plane to the paraboloid is parallel to the plane $x + y + z = 1$. Also, determine the equations of the corresponding tangent planes.

    Solution: Let $(x_0, y_0, z_0) \in \mathbb{R}^3$ be a point on the paraboloid $z = x^2 + y^2$ at which the tangent plane to the paraboloid is parallel to the plane $x + y + z = 1$. If $g(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$, then $g : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and $g_x(x,y) = 2x$, $g_y(x,y) = 2y$ for all $(x,y) \in \mathbb{R}^2$. Hence the equation of the tangent plane to the paraboloid $z = g(x,y)$ at (x_0, y_0, z_0) is $z = g(x_0,y_0) + g_x(x_0,y_0)(x - x_0) + g_y(x_0,y_0)(y - y_0)$, $z = z_0 + 2x_0(x - x_0) + 2y_0(y - y_0)$. Since this plane is parallel to the plane $z = 1 - x - y$, we must have that $2x_0 = -1$ and $2y_0 = -1$ and hence the required point is $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$.

Also, the equation of the tangent plane to the paraboloid at the point $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ is $z = \frac{1}{2} - (x + \frac{1}{2}) - (y + \frac{1}{2})$, *i.e.* $2x + 2y + 2z + 1 = 0$.

12. If $f(x, y) = x^3 + y^3 - 63x - 63y + 12xy$ for all $(x, y) \in \mathbb{R}^2$, then determine all the points of local maximum, local minimum and all the saddle points of $f : \mathbb{R}^2 \to \mathbb{R}$.

    Solution: We have $f_x(x, y) = 3x^2 - 63 + 12y$, $f_y(x, y) = 3y^2 - 63 + 12x$, $f_{xx}(x, y) = 6x$, $f_{yy}(x, y) = 6y$ and $f_{xy}(x, y) = 12$ for all $(x, y) \in \mathbb{R}^2$. We solve the system of equations $f_x(x, y) = 0$, $f_y(x, y) = 0$. Considering $f_x(x, y) - f_y(x, y) = 0$, we obtain $(x - y)(x + y - 4) = 0$ and hence $x = y$ or $x + y = 4$. If $x = y$, then from $f_x(x, y) = 0$, we get $x^2 + 4x - 21 = 0$ and so $x = 3, -7$. Hence in this case we get total two critical points $(3, 3)$ and $(-7, -7)$. Again, if $x + y = 4$, then $f_x(x, y) = 0$ gives $x^2 - 4x - 5 = 0$ and so $x = 5, -1$. Hence in this case we again get total two critical points $(5, -1)$ and $(-1, 5)$.
    Since $f_{xx}(3, 3)f_{yy}(3, 3) - f_{xy}(3, 3)^2 = 180 > 0$ and $f_{xx}(3, 3) = 18 > 0$, f has a local minimum at $(3, 3)$.
    Since $f_{xx}(-7, -7)f_{yy}(-7, -7) - f_{xy}(-7, -7)^2 = 1620 > 0$ and $f_{xx}(-7, -7) = -42 < 0$, f has a local maximum at $(-7, -7)$.
    Again, since $f_{xx}f_{yy} - f_{xy}^2 = -324 < 0$ for each of $(5, -1)$ and $(-1, 5)$, both $(5, -1)$ and $(-1, 5)$ are saddle points of f.

13. If $f(x, y) = 2x^4 + 2x^2y + y^2$ for all $(x, y) \in \mathbb{R}^2$, then determine all the points of local maximum, local minimum and all the saddle points of $f : \mathbb{R}^2 \to \mathbb{R}$.

    Solution: Solving $f_x(x, y) = 8x^3 + 4xy = 0$ and $f_y(x, y) = 2x^2 + 2y = 0$, we get $(x, y) = (0, 0)$ and hence $(0, 0)$ is the only critical point of f. Now, $f_{xx}(x, y) = 24x^2 + 4y$, $f_{yy}(x, y) = 2$ and $f_{xy}(x, y) = 4x$ for all $(x, y) \in \mathbb{R}^2$ and hence $f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 0$. Therefore no definite conclusion (regarding local extremum and saddle point) of f at $(0, 0)$ can be obtained from the second order partial derivatives of f.
    However, since $f(x, y) = (x^2 + y)^2 + x^4 \geq 0 = f(0, 0)$ for all $(x, y) \in \mathbb{R}^2$, f has a local (in fact, absolute) minimum at $(0, 0)$.

14. If $f(x, y) = 4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$ for all $(x, y) \in \mathbb{R}^2$, then determine all the points of local maximum, local minimum and all the saddle points of $f : \mathbb{R}^2 \to \mathbb{R}$.

    Solution: We have $f_x(x, y) = 8x - y + 3x^2y + y^3$, $f_y(x, y) = -x + 8y + x^3 + 3xy^2$, $f_{xx}(x, y) = 8 + 6xy$, $f_{yy}(x, y) = 8 + 6xy$ and $f_{xy}(x, y) = -1 + 3x^2 + 3y^2$ for all $(x, y) \in \mathbb{R}^2$. We solve the system of equations $f_x(x, y) = 0$, $f_y(x, y) = 0$. Considering $f_x(x, y) + f_y(x, y) = 0$, we obtain $(x + y)[(x + y)^2 + 7] = 0$ and hence $x + y = 0$. Now, $f_x(x, y) = 0$ gives $x(9 - 4x^2) = 0$ and so $x = 0, \frac{3}{2}, -\frac{3}{2}$. Hence we get total three critical points $(0, 0)$, $(\frac{3}{2}, -\frac{3}{2})$ and $(-\frac{3}{2}, \frac{3}{2})$.
    Since $f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 63 > 0$ and $f_{xx}(0, 0) = 8 > 0$, f has a local minimum at $(0, 0)$.
    Again, since $f_{xx}f_{yy} - f_{xy}^2 = -324 < 0$ for each of $(\frac{3}{2}, -\frac{3}{2})$ and $(-\frac{3}{2}, \frac{3}{2})$, both $(\frac{3}{2}, -\frac{3}{2})$ and $(-\frac{3}{2}, \frac{3}{2})$ are saddle points of f.

15. If $f(x, y, z) = x^2 + y^2 + z^2 + 2xyz - 4zx - 2yz - 2x - 4y + 4z$ for all $(x, y, z) \in \mathbb{R}^3$, then find all the points of local maximum, local minimum and all the saddle points of $f : \mathbb{R}^3 \to \mathbb{R}$.

Solution: We have $f_x(x, y, z) = 2yz - 4z + 2x - 2$, $f_y(x, y, z) = 2zx - 2z + 2y - 4$ and $f_z(x, y, z) = 2xy - 4x - 2y + 2z + 4$ for all $(x, y, z) \in \mathbb{R}^3$. In order to solve the system of equations $f_x(x, y, z) = 0$, $f_y(x, y, z) = 0$, $f_z(x, y, z) = 0$, we add the last two equations to obtain $x(y + z - 2) = 0$, and so $x = 0$ or $y + z = 2$.
Case 1: $x = 0$
In this case $y - z = 2$ and $yz - 2z = 1$, from which we get $z = 1, -1$. Hence in this case we obtain total two critical points of f, which are $(0, 3, 1)$ and $(0, 1, -1)$.
Case 2: $y + z = 2$
In this case $-z^2 + x - 1 = 0$ and so $(z^2 + 1)z - 2z = 0$, which gives $z = 0,\ 1,\ -1$. Hence in this case we obtain total three critical points of f, which are $(1, 2, 0)$, $(2, 1, 1)$ and $(2, 3, -1)$.
Now, $f_{xx}(x, y, z) = 2$, $f_{yy}(x, y, z) = 2$, $f_{zz}(x, y, z) = 2$, $f_{xy}(x, y, z) = 2z$, $f_{yz}(x, y, z) = 2x - 2$ and $f_{zx}(x, y, z) = 2y - 4$ for all $(x, y, z) \in \mathbb{R}^3$. Hence $H_f(x, y, z) = \begin{bmatrix} 2 & 2z & 2y-4 \\ 2z & 2 & 2x-2 \\ 2y-4 & 2x-2 & 2 \end{bmatrix}$ for all $(x, y, z) \in \mathbb{R}^3$.
The leading principal minors of $H_f(1, 2, 0)$ are 2, 4 and 8 (all of which are positive), and therefore f has a local minimum at $(1, 2, 0)$.
It can also be easily seen that $\det(H_f(x, y, z)) = -32 < 0$ for each of the remaining four critical points of f and $f_{xx}(x, y, z) = 2 > 0$ for each of these points. Therefore each of these remaining four critical points of f are saddle points of f.

16. If $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, then determine $\max\{x^2 + 2x + y^2 : (x, y) \in S\}$ and $\min\{x^2 + 2x + y^2 : (x, y) \in S\}$.

Solution: Let $f(x, y) = x^2 + 2x + y^2$ for all $(x, y) \in S$. Since S is a closed and bounded set in $\mathbb{R}^2$ and $f : S \to \mathbb{R}$ is continuous, both $\max\{f(x, y) : (x, y) \in S\}$ and $\min\{f(x, y) : (x, y) \in S\}$ exist (in $\mathbb{R}$).
We first look for local extrema of f in $S^0 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Solving the system of equations $f_x(x, y) = 2x + 2 = 0$, $f_y(x, y) = 2y = 0$, we get $(x, y) = (-1, 0)$, which does not belong to S^0. Hence f does not have any local extremum in S^0.
Again, the boundary of S consists of all the points on the circle $x^2 + y^2 = 1$. Taking the parametric representation of the circle $x^2 + y^2 = 1$ as $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, we look for local extrema of $\varphi : [0, 2\pi] \to \mathbb{R}$, where $\varphi(t) = f(\gamma(t)) = 1 + 2\cos t$ for all $t \in [0, 2\pi]$. Clearly φ has local (in fact, absolute) maxima only at $t = 0,\ 2\pi$ and local (in fact, absolute) minimum at $t = \pi$. These points correspond to the points $(1, 0)$ and $(-1, 0)$ of S.
Since $f(1, 0) = 3$ and $f(-1, 0) = -1$, it follows that $\max\{f(x, y) : (x, y) \in S\} = 3$,
$\min\{f(x, y) : (x, y) \in S\} = -1$ and these values are attained by f at $(1, 0)$ and $(-1, 0)$ respectively.

17. Find the (absolute) maximum value of $f(x,y,z) = 8xyz^2 - 200(x+y+z)$ subject to the constraint $x+y+z = 100$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

Solution: Let $S = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$ and let $f(x,y,z) = 8xyz^2 - 200(x+y+z)$, $g(x,y,z) = x+y+z-100$ for all $(x,y,z) \in S$. If either of x, y, or z is 0, then $f(x,y,z) = -200(x+y+z)$ and so under the constraint $x+y+z = 100$, $f(x,y,z) = -20000$, which is clearly not the maximum value of $f(x,y,z)$ under the given conditions. Hence in order to find the maximum value of $f(x,y,z)$ subject to the given constraint, we may assume that $x > 0$, $y > 0$, and $z > 0$. Clearly f, $g : S \to \mathbb{R}$ are continuously differentiable on $S^0 = \{(x,y,z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0\}$ and $\nabla g(x,y,z) = (1,1,1) \neq (0,0,0)$ for all $(x,y,z) \in S^0$. Let $(x_0,y_0,z_0) \in \Omega = \{(x,y,z) \in S : g(x,y,z) = 0\}$ and let $\lambda \in \mathbb{R}$ such that $\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0)$. Then $(8y_0z_0^2 - 200, 8x_0z_0^2 - 200, 16x_0y_0z_0 - 200) = \lambda(1,1,1)$ and hence $8y_0z_0^2 - 200 = \lambda$, $8x_0z_0^2 - 200 = \lambda$, $16x_0y_0z_0 - 200 = \lambda$. So, we get $8y_0z_0^2 = 8x_0z_0^2$ and hence $x_0 = y_0$. Consequently $8x_0z_0^2 = 16x_0^2z_0$ and so $z_0 = 2x_0$. Since $x_0 + y_0 + z_0 = 100$, we get $x_0 = 25$, $y_0 = 25$, $z_0 = 50$. Hence by Lagrange multiplier method, $(25,25,50)$ is the only possible point in S^0 where $f\big|_\Omega$ has a local extremum. Again, since Ω is a closed and bounded set in $\mathbb{R}^3$ and since f is continuous on Ω, $\max\{f(x,y,z) : (x,y,z) \in \Omega\}$ must exist (in $\mathbb{R}$). Consequently $f(25,25,50) = 12480000$ is the required maximum value.