MA15010H: Multi-variable Calculus

(Practice problem set 2: Sequential criteria for continuity and vector differentiability)
July - November, 2025

1. Examine whether the following sets are (a) open (b) closed in $\mathbb{R}^2$.
   (a) $\{(x, y) \in \mathbb{R}^2 : 0 < x < y\}$
   (b) $\{(x, x) : x \in \mathbb{R}\}$
   (c) $\{(x, y) \in \mathbb{R}^2 : y \in \mathbb{Z}\}$
   (d) $(0, 1) \times \{0\}$

2. If $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, then show that
   (a) $\{x \in \mathbb{R}^m : f(x) > 0\}$ is an open set in $\mathbb{R}^m$.
   (b) $\{x \in \mathbb{R}^m : f(x) \geq 0\}$ and $\{x \in \mathbb{R}^m : f(x) = 0\}$ are closed sets in $\mathbb{R}^m$.

3. Using Ex. 2 above, show that $\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2z < 3|y|\}$ is an open set in $\mathbb{R}^3$ and $\{(x, y, z) \in \mathbb{R}^3 : \sin(xyz) = |xy|\}$ is a closed set in $\mathbb{R}^3$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} \dfrac{\sin(xy)}{xy} & \text{if } xy \neq 0, \\ 1 & \text{if } xy = 0. \end{cases}$

   Show that f is continuous.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x, y) = e^{-\frac{x^2 - 2xy + y^2}{x^2 - y^2}}$ for all $(x, y) \in \mathbb{R}^2$ with $x \neq y$. If $x \in \mathbb{R}$, then find $f(x, x)$ such that f is continuous on $\mathbb{R}^2$.

6. Let $f : S \subset \mathbb{R}^m \to \mathbb{R}^k$ be continuous and let $g : \mathbb{R}^m \to \mathbb{R}^k$ be such that $g(x) = f(x)$ for all $x \in S$.
   (a) Show that g need not be continuous on S.
   (b) If S is an open set in $\mathbb{R}^m$, then show that g is continuous on S.

7. Let $S_1 = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 < 4\}$ and $S_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 < 9\}$. Does there exist a continuous function from S_1 onto S_2? Justify.

8. If $S = \{x \in \mathbb{R}^m : \|x\| < 1\}$, then does there exist a non-constant continuous function $f : \mathbb{R}^m \to \mathbb{R}$ such that $f(x) = 5$ for all $x \in S$? Justify.

9. Let $x, y \in \mathbb{R}^m$ such that $x \neq y$. Find a continuous function $f : \mathbb{R}^m \to \mathbb{R}$ such that $f(x) = 1$, $f(y) = 0$ and $0 \leq f(z) \leq 1$ for all $z \in \mathbb{R}^m$.

10. Let $f : \mathbb{R}^m \to \mathbb{R}$ be continuous such that $\lim_{\|x\| \to \infty} f(x) = 1$. Show that f is bounded on $\mathbb{R}^m$.

11. State TRUE or FALSE with justification for each of the following statements.
   (a) There exists $r > 0$ such that $\sin(xy) < \cos(xy)$ for all $x, y \in [-r, r]$.
   (b) There exists a continuous function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(\cos n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$.
   (c) There exists a continuous function from $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ onto $\mathbb{R}^2$.

(d) There exists a one-one continuous function from $\{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$ onto $\mathbb{R}^2$.

12. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, then does there exist a sequence (x_n, y_n) in $\mathbb{R}^2$ such that $x_n^2 + y_n^2 = \frac{1}{2}$ and $f(x_n, y_n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$? Justify.

13. Examine whether the following limits exist (in $\mathbb{R}$) and find their values if they exist (in $\mathbb{R}$).

    (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^2+y^2}$

    (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2+y^2}$

    (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x|y^2}{x^2/|y^3|}$

    (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+y}$

    (e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y^4 + 1}{x^2+y^2}$

    (f) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y^2 + y^5}{x^4+y^4}$

    (g) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2}$

14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} x+y & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$

    Examine whether $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exists (in $\mathbb{R}$).

15. Let $S \subseteq \mathbb{R}^2$, $(x_0, y_0) \in \mathbb{R}^2$ and $r > 0$ be such that $[B_r(x_0) \times B_r(y_0)] \setminus \{(x_0, y_0)\} \subseteq S$. Let $\displaystyle\lim_{x\to x_0} f(x,y)$ exist (in $\mathbb{R}$) for each $y \in B_r(y_0) \setminus \{y_0\}$, $\displaystyle\lim_{y\to y_0} f(x,y)$ exist (in $\mathbb{R}$) for each $x \in B_r(x_0) \setminus \{x_0\}$ and $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell \in \mathbb{R}$.

    Show that $\displaystyle\lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right) = \lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right) = \ell.$

$$\left[\lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right) \text{ and } \lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right) \text{ are called the iterated limits of } f \text{ at } (x_0,y_0)\right].$$

16. Show that $\lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^2}{x^2+y^2}\right) \neq \lim_{y\to 0}\left(\lim_{x\to 0} \frac{x^2}{x^2+y^2}\right)$ and hence conclude that $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does not exist (in $\mathbb{R}$).

17. Show that $\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2y^2}{x^2+y^2+(x-y)^2}\right) = 0 = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2y^2}{x^2+y^2+(x-y)^2}\right)$ but that $\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2+y^2+(x-y)^2}$ does not exist (in $\mathbb{R}$).

18. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} x\sin\frac{1}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$
Show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$ and $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0$ but that $\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$ is not defined.

19. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{3x^2+y^4} = \infty$.

20. Let I be an open interval in $\mathbb{R}$ and let $F : I \to \mathbb{R}^m$ be a differentiable function such that $F(t)\cdot F'(t) = 0$ for all $t \in I$. Show that $\|F(t)\|$ is constant for all $t \in I$.