MA15010H: Multi-variable Calculus

(Practice problem set 2: Hint/Model solution)

September - November, 2025

1. Examine whether the set $\{(x, y) \in \mathbb{R}^2 : 0 < x < y\}$ is (a) open (b) closed in $\mathbb{R}^2$.

   Solution: We have already shown in Ex.25 of Practice Problem Set - 1 that $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < y\}$ is an open set in $\mathbb{R}^2$.

   Again, since $\left(\frac{1}{2n}, \frac{1}{n}\right) \in S$ for all $n \in \mathbb{N}$ and $\left(\frac{1}{2n}, \frac{1}{n}\right) \to (0, 0) \notin S$, S is not a closed set in $\mathbb{R}^2$.

2. Examine whether the set $\{(x, x) : x \in \mathbb{R}\}$ is (a) open (b) closed in $\mathbb{R}^2$.

   Solution: We have $(0, 0) \in S = \{(x, x) : x \in \mathbb{R}\}$. If possible, let $(0, 0) \in S^0$. Then there exists $r > 0$ such that $B_r((0, 0)) \subset S$. Since $\left(\frac{r}{2}, 0\right) \in B_r((0, 0))$ but $\left(\frac{r}{2}, 0\right) \notin S$, we get a contradiction. Hence $(0, 0) \notin S^0$. Therefore S is not an open set in $\mathbb{R}^2$.

   Again, let $((x_n, x_n))$ be any sequence in S such that $(x_n, x_n) \to (x, y)$ in $\mathbb{R}^2$. Then $x_n \to x$ and $x_n \to y$. Hence $x = y$ and so $(x, y) \in S$. Therefore S is a closed set in $\mathbb{R}^2$.

3. Examine whether the set $\{(x, y) \in \mathbb{R}^2 : y \in \mathbb{Z}\}$ is (a) open (b) closed in $\mathbb{R}^2$.

   Solution: We have $(0, 0) \in S = \{(x, y) : y \in \mathbb{Z}\}$. If possible, let $(0, 0) \in S^0$. Then there exists $r > 0$ such that $B_r((0, 0)) \subset S$. If $s = \min\{\frac{1}{2}, \frac{r}{2}\}$, then $(0, s) \in B_r((0, 0))$ but $(0, s) \notin S$. Thus we get a contradiction. Hence $(0, 0) \notin S^0$ and therefore S is not an open set in $\mathbb{R}^2$.

   Again, let $((x_n, y_n))$ be any sequence in S such that $(x_n, y_n) \to (x, y)$ in $\mathbb{R}^2$. Then $y_n \to y$. There exists $N \in \mathbb{N}$ such that $|y_n - y| < \frac{1}{2}$ for all $n \geq N$ and hence $|y_n - y_0| \leq |y_n - y| + |y - y_0| < 1$ for all $n \geq n_0$. Since $y_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$, we get $y_n = y_0$ for all $n \geq n_0$ and so $y_n \to y_0$. Consequently $y = y_0 \in \mathbb{Z}$ and so $(x, y) \in S$. Therefore S is a closed set in $\mathbb{R}^2$.

4. Examine whether the set $(0, 1) \times \{0\}$ is (a) open (b) closed in $\mathbb{R}^2$.

   Solution: We have $\left(\frac{1}{2}, 0\right) \in (0, 1) \times \{0\}$. If possible, let $\left(\frac{1}{2}, 0\right) \in ((0, 1) \times \{0\})^0$. Then there exists $r > 0$ such that $B_r\left(\left(\frac{1}{2}, 0\right)\right) \subset (0, 1) \times \{0\}$. Since $\left(\frac{1}{2}, \frac{r}{2}\right) \in B_r\left(\left(\frac{1}{2}, 0\right)\right)$ but $\left(\frac{1}{2}, \frac{r}{2}\right) \notin (0, 1) \times \{0\}$, we get a contradiction. Hence $\left(\frac{1}{2}, 0\right) \notin ((0, 1) \times \{0\})^0$. Therefore $(0, 1) \times \{0\}$ is not an open set in $\mathbb{R}^2$.

   Again, since $\left(\frac{1}{n+1}, 0\right) \in (0, 1) \times \{0\}$ for all $n \in \mathbb{N}$ and $\left(\frac{1}{n+1}, 0\right) \to (0, 0) \notin (0, 1) \times \{0\}$, $(0, 1) \times \{0\}$ is not a closed set in $\mathbb{R}^2$.

5. If $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, then show that $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) > 0\}$ is an open set in $\mathbb{R}^m$.

   Solution: Let $(\mathbf{x}_n)$ be any sequence in $\mathbb{R}^m \setminus S$, where $S = \{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) > 0\}$ and let $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since f is continuous at $\mathbf{x}$, $f(\mathbf{x}_n) \to f(\mathbf{x})$. Also, since $\mathbf{x}_n \in \mathbb{R}^m \setminus S$ for all $n \in \mathbb{N}$, $f(\mathbf{x}_n) \leq 0$ for all $n \in \mathbb{N}$ and hence it follows that $f(\mathbf{x}) \leq 0$. Thus $\mathbf{x} \in \mathbb{R}^m \setminus S$ and therefore $\mathbb{R}^m \setminus S$ is a closed set in $\mathbb{R}^m$. Consequently S is an open set in $\mathbb{R}^m$.

6. If $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, then show that $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) \geq 0\}$ and $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) = 0\}$ are closed sets in $\mathbb{R}^m$.

Solution: Let $(\mathbf{x}_n)$ be any sequence in $S_1 = \{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) \geq 0\}$ and let $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since f is continuous at $\mathbf{x}$, $f(\mathbf{x}_n) \to f(\mathbf{x})$. Also, since $\mathbf{x}_n \in S_1$ for all $n \in \mathbb{N}$, $f(\mathbf{x}_n) \geq 0$ for all $n \in \mathbb{N}$ and hence it follows that $f(\mathbf{x}) \geq 0$. Thus $\mathbf{x} \in S_1$ and therefore S_1 is a closed set in $\mathbb{R}^m$.

Again, let $(\mathbf{x}_n)$ be any sequence in $S_2 = \{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) = 0\}$ and let $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since f is continuous at $\mathbf{x}$, $f(\mathbf{x}_n) \to f(\mathbf{x})$. Also, since $\mathbf{x}_n \in S_2$ for all $n \in \mathbb{N}$, $f(\mathbf{x}_n) = 0$ for all $n \in \mathbb{N}$ and hence it follows that $f(\mathbf{x}) = 0$. Thus $\mathbf{x} \in S_2$ and therefore S_2 is a closed set in $\mathbb{R}^m$.

7. Using Ex.2 in the Practice Problem Set - 2, show that $\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2z < 3|y|\}$ is an open set in $\mathbb{R}^3$ and $\{(x, y, z) \in \mathbb{R}^3 : \sin(xyz) = |xy|\}$ is a closed set in $\mathbb{R}^3$.

   Solution: If $f(x, y, z) = 3|y| - x^2 - 2z$ and $g(x, y, z) = \sin(xyz) - |xy|$ for all $(x, y, z) \in \mathbb{R}^3$, then we know that both $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are continuous. Hence by Ex.2(a) of Practice Problem Set - 2, $\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2z < 3|y|\} = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) > 0\}$ is an open set in $\mathbb{R}^3$ and by Ex.2(b) of Practice Problem Set - 2, $\{(x, y, z) \in \mathbb{R}^3 : \sin(xyz) = |xy|\} = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$ is a closed set in $\mathbb{R}^3$.

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{\sin(xy)}{xy} & \text{if } xy \neq 0, \\ 1 & \text{if } xy = 0. \end{cases}$$

   Show that f is continuous.

   Solution: If $\varphi(x, y) = xy$ and $\psi(x, y) = \sin(xy)$ for all $(x, y) \in \mathbb{R}^2$, then we know that $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ are continuous and $\varphi(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. Hence it follows that f is continuous at each point $(x, y) \in \mathbb{R}^2$ for which $xy \neq 0$.

   Let $(x, y) \in \mathbb{R}^2$ such that $xy = 0$ and let $((x_n, y_n))$ be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x, y)$. Then $x_n \to x, y_n \to y$ and so $x_n y_n \to xy = 0$. Now $f(x_n, y_n) = \frac{\sin(x_n y_n)}{x_n y_n}$ if $x_n y_n \neq 0$ and $f(x_n, y_n) = 1$ if $x_n y_n = 0$. Since $\lim_{t \to 0} \frac{\sin t}{t} = 1$, it follows that $f(x_n, y_n) \to 1 = f(x, y)$ and consequently f is continuous at (x, y). Therefore f is continuous.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x, y) = e^{-\frac{x^2 - 2xy + y^2}{|x-y|}}$ for all $(x, y) \in \mathbb{R}^2$ with $x \neq y$. If $x \in \mathbb{R}$, then find $f(x, x)$ such that f is continuous on $\mathbb{R}^2$.

   Solution: Since $x^2 - 2xy + y^2 = |x - y|^2$ for all $x, y \in \mathbb{R}$, we find that $f(x, y) = e^{-|x-y|}$ for all $(x, y) \in \mathbb{R}^2$ with $x \neq y$. If $x \in \mathbb{R}$, then $(x + \frac{1}{n}, x) \to (x, x)$ and for f to be continuous at (x, x), we must have $f(x, x) = \lim_{n \to \infty} f(x + \frac{1}{n}, x) = \lim_{n \to \infty} e^{-\frac{1}{n}} = 1$. So, let $f(x, x) = 1$ for all $x \in \mathbb{R}$.

   If $g(x, y) = -|x - y|$ for all $(x, y) \in \mathbb{R}^2$ and $\varphi(t) = e^t$ for all $t \in \mathbb{R}$, then $f(x, y) = \varphi(g(x, y))$ for all $(x, y) \in \mathbb{R}^2$. Since we know that $g : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ are continuous, hence $f = \varphi \circ g$ is also continuous.

10. Let $f : S \subset \mathbb{R}^m \to \mathbb{R}^k$ be continuous and let $g : \mathbb{R}^m \to \mathbb{R}^k$ be such that $g(x) = f(x)$ for all $x \in S$.
(a) Show that g need not be continuous on S.
(b) If S is an open set in $\mathbb{R}^m$, then show that g is continuous on S.
Solution:
(a) Let $f(x, y) = 1$ for all $(x, y) \in S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and

$$g(x,y) = \begin{cases} 1 & \text{if } (x,y) \in S, \\ 2 & \text{if } (x,y) \in \mathbb{R}^2 \backslash S. \end{cases}$$

Then $f : S \to \mathbb{R}$ is continuous (as a constant function) and $f(x, y) = g(x, y)$ for all $(x, y) \in S$. However, g is not continuous at $(1, 0) \in S$, since $(1 + \frac{1}{n}, 0) \to (1, 0)$ but

$$g(1 + \frac{1}{n}, 0) = 2 \to 2 \neq 1 = g(1, 0).$$

(b) Let $x_0 \in S$ and $\varepsilon > 0$. Since S is an open set in $\mathbb{R}^m$, there exists $r > 0$ such that $B_r(x_0) \subset S$. Since f is continuous at x_0, there exists $s > 0$ such that $\|f(x) - f(x_0)\| < \varepsilon$ for all $x \in S \cap B_s(x_0)$. If $\delta = \min\{r, s\} > 0$, then $B_\delta(x_0) \subset B_r(x_0) \subset S$ and $B_\delta(x_0) \subset B_s(x_0)$. Hence for all $x \in B_\delta(x_0)$, we have $g(x) = f(x)$ and $\|g(x) - g(x_0)\| < \varepsilon$. Therefore g is continuous at x_0. Since $x_0 \in S$ is arbitrary, g is continuous on S.

11. Let $S_1 = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 < 4\}$ and $S_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 < 9\}$. Does there exist a continuous function from S_1 onto S_2? Justify.
Solution: Let $u = (1, 0), v = (0, 1)$ and let $f(x) = v + \frac{3}{2}(x - u) = \left(\frac{3x}{2} - \frac{3}{2}, 1 + \frac{3y}{2}\right)$ for all $x = (x, y) \in S_1$. If $x \in S_1$, then $\|f(x) - v\| = \frac{3}{2}\|x - u\| < 3$ and so $f(x) \in S_2$. Thus f maps S_1 to S_2 and clearly f is continuous (since both the component functions of f are continuous).
Again, if $y \in S_2$, then $x = u + \frac{2}{3}(y - v) \in \mathbb{R}^2$ and $\|x - u\| = \frac{2}{3}\|y - v\| < 2$, i.e. $x \in S_1$, and also $f(x) = y$. Thus $f : S_1 \to S_2$ is onto. Therefore there exists a continuous function from S_1 onto S_2.

12. If $S = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| < 1\}$, then does there exist a non-constant continuous function $f : \mathbb{R}^m \to \mathbb{R}$ such that $f(\mathbf{x}) = 5$ for all $\mathbf{x} \in S$? Justify.
Solution: There exists such a function as is shown by the following example. Let

$$f(\mathbf{x}) = \begin{cases} 5 & \text{if } \mathbf{x} \in S, \\ 5\|\mathbf{x}\| & \text{if } \mathbf{x} \in \mathbb{R}^m \setminus S. \end{cases}$$

If $(\mathbf{x}_n)$ is any sequence in $\mathbb{R}^m$ such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$, then using Ex.1(a) of Practice Problem Set - 1, we get $|\|\mathbf{x}_n\| - \|\mathbf{x}\|| \leq \|\mathbf{x}_n - \mathbf{x}\| \to 0$ and hence $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$. It follows that $f : \mathbb{R}^m \to \mathbb{R}$ is continuous. Clearly f is a non-constant function and $f(\mathbf{x}) = 5$ for all $\mathbf{x} \in S$.

13. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ such that $x \neq y$. Find a continuous function $f : \mathbb{R}^m \to \mathbb{R}$ such that $f(\mathbf{x}) = 1, f(\mathbf{y}) = 0$ and $0 \leq f(\mathbf{z}) \leq 1$ for all $\mathbf{z} \in \mathbb{R}^m$.

Solution: Let $f(\mathbf{z}) = \frac{\|\mathbf{z}-\mathbf{y}\|}{\|\mathbf{z}-\mathbf{x}\|+\|\mathbf{z}-\mathbf{y}\|}$ for all $\mathbf{z} \in \mathbb{R}^m$. If $(\mathbf{z}_n)$ is any sequence in $\mathbb{R}^m$ such that $\mathbf{z}_n \to \mathbf{z} \in \mathbb{R}^m$, then using Ex.1(a) of Practice Problem Set - 1, we find that $\|\mathbf{z}_n - \mathbf{x}\| \to \|\mathbf{z} - \mathbf{x}\|$, $\|\mathbf{z}_n - \mathbf{y}\| \to \|\mathbf{z} - \mathbf{y}\|$. Also, $\|\mathbf{v} - \mathbf{x}\| + \|\mathbf{v} - \mathbf{y}\| \neq 0$ for all $\mathbf{v} \in \mathbb{R}^m$. Hence it follows that $f(\mathbf{z}_n) \to f(\mathbf{z})$ and consequently $f : \mathbb{R}^m \to \mathbb{R}$ is continuous. Clearly $f(\mathbf{x}) = 1, f(\mathbf{y}) = 0$ and $0 \le f(\mathbf{z}) \le 1$ for all $\mathbf{z} \in \mathbb{R}^m$.

14. Let $f : \mathbb{R}^m \to \mathbb{R}$ be continuous such that $\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = 1$. Show that f is bounded on $\mathbb{R}^m$.

    Solution: Since $\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = 1$, there exists $r > 0$ such that $|f(\mathbf{x}) - 1| < 1$ for all $\mathbf{x} \in \mathbb{R}^m$ with $\|\mathbf{x}\| > r$. Hence $|f(\mathbf{x})| = |f(\mathbf{x}) - 1 + 1| \le |f(\mathbf{x}) - 1| + 1 < 2$ for all $\mathbf{x} \in \mathbb{R}^m$ with $\|\mathbf{x}\| > r$. Again, since $S = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| \le r\}$ is a closed and bounded subset of $\mathbb{R}^m$ and since $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, $f(S)$ is a bounded subset of $\mathbb{R}$. Hence there exists $K > 0$ such that $|f(\mathbf{x})| \le K$ for all $\mathbf{x} \in S$. If $M = \max\{2, K\}$, then $M > 0$ and $|f(\mathbf{x})| \le M$ for all $\mathbf{x} \in \mathbb{R}^m$. Consequently f is bounded on $\mathbb{R}^m$.

15. State TRUE or FALSE with justification: There exists $r > 0$ such that $\sin(xy) < \cos(xy)$ for all $x, y \in [-r, r]$.

    Solution: If $f(x, y) = \sin(xy) - \cos(xy)$ for all $(x, y) \in \mathbb{R}^2$, then we know that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0, 0)$ and $f(0, 0) = -1 < 0$. Hence there exists $\delta > 0$ such that $f(x, y) < 0$, i.e. $\sin(xy) < \cos(xy)$ for all $(x, y) \in B_\delta((0, 0))$. If $r = \frac{\delta}{2} > 0$, then $[-r, r] \times [-r, r] \subseteq B_\delta((0, 0))$ and hence for all $x, y \in [-r, r]$, we have $(x, y) \in B_\delta((0, 0))$ and consequently $\sin(xy) < \cos(xy)$. Therefore the given statement is TRUE.

16. State TRUE or FALSE with justification: There exists a continuous function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(\cos n) = \left(n, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$.

    Solution: Since $(\cos n)$ is a bounded sequence in $\mathbb{R}$, by Bolzano-Weierstrass theorem in $\mathbb{R}$, there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ and $\alpha \in \mathbb{R}$ such that $\cos n_k \to \alpha$. If $f : \mathbb{R} \to \mathbb{R}^2$ is continuous, then $\left(n_k, \frac{1}{n_k}\right) = f(\cos n_k) \to f(\alpha)$ in $\mathbb{R}^2$ and consequently the sequence $\left(n_k, \frac{1}{n_k}\right)$ converges in $\mathbb{R}^2$, which is not true, since (n_k) is unbounded. Hence it follows that no continuous function $f : \mathbb{R} \to \mathbb{R}^2$ can exist satisfying $f(\cos n) = \left(n, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$. Therefore the given statement is FALSE.

17. State TRUE or FALSE with justification: There exists a continuous function from $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ onto $\mathbb{R}^2$.

    Solution: We know that $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\} = B_1[(0, 0)]$ is a closed and bounded set in $\mathbb{R}^2$ and $\mathbb{R}^2$ is not bounded. Hence there cannot exist any continuous function from $B_1[(0, 0)]$ onto $\mathbb{R}^2$.

18. State TRUE or FALSE with justification: There exists a one-one continuous function from $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ onto $\mathbb{R}^2$.

    Solution: Let $S = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ and let $f(\mathbf{x}) = \frac{1}{1-\|\mathbf{x}\|}\mathbf{x} = \left(\frac{x}{1-\sqrt{x^2+y^2}}, \frac{y}{1-\sqrt{x^2+y^2}}\right)$ for all $\mathbf{x} = (x, y) \in S$. If $\mathbf{x} \in S$ and $(\mathbf{x}_n)$ is any sequence in S such that $\mathbf{x}_n \to \mathbf{x}$, then using Ex.1(a) of Practice Problem Set - 1, we get $\|\mathbf{x}_n\| - \|\mathbf{x}\| \le \|\mathbf{x}_n - \mathbf{x}\| \to 0$ and so

$\|\mathbf{x}_n\| \to \|\mathbf{x}\|$. Hence $1 - \|\mathbf{x}_n\| \to 1 - \|\mathbf{x}\|$ and since $1 - \|\mathbf{x}\| \neq 0$ and $1 - \|\mathbf{x}_n\| \neq 0$ for all $n \in \mathbb{N}$, it follows that $f(\mathbf{x}_n) \to f(\mathbf{x})$. Therefore $f : S \to \mathbb{R}^2$ is continuous at $\mathbf{x}$ and since $\mathbf{x} \in S$ is arbitrary, f is continuous.

Let $\mathbf{x}_1, \mathbf{x}_2 \in S$ such that $f(\mathbf{x}_1) = f(\mathbf{x}_2)$. Then $\|f(\mathbf{x}_1)\| = \|f(\mathbf{x}_2)\|$, i.e. $\frac{\|\mathbf{x}_1\|}{1-\|\mathbf{x}_1\|} = \frac{\|\mathbf{x}_2\|}{1-\|\mathbf{x}_2\|}$, which gives $\|\mathbf{x}_1\| = \|\mathbf{x}_2\|$. Consequently from $\frac{1}{1-\|\mathbf{x}_1\|}\mathbf{x}_1 = \frac{1}{1-\|\mathbf{x}_2\|}\mathbf{x}_2$, we get $\mathbf{x}_1 = \mathbf{x}_2$. Hence f is one-one.

Again, if $\mathbf{y} \in \mathbb{R}^2$, then taking $\mathbf{x} = \frac{1}{1+|\mathbf{y}|}\mathbf{y}$, we find that $\|\mathbf{x}\| < 1$, i.e. $\mathbf{x} \in S$ and $f(\mathbf{x}) = \mathbf{y}$. Hence f is onto.

Thus $f : S \to \mathbb{R}^2$ is the required function and therefore the given statement is TRUE.

19. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, then does there exist a sequence $((x_n, y_n))$ in $\mathbb{R}^2$ such that $x_n^2 + y_n^2 = \frac{1}{2}$ and $f(x_n, y_n) = \left(n, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$? Justify.

Solution: If possible, let there exist a sequence $((x_n, y_n))$ in $\mathbb{R}^2$ such that $x_n^2 + y_n^2 = \frac{1}{2}$ and $f(x_n, y_n) = \left(n, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$. Then $\|(x_n, y_n)\| = \sqrt{x_n^2 + y_n^2} = \frac{1}{\sqrt{2}}$ for all $n \in \mathbb{N}$ and so $((x_n, y_n))$ is a bounded sequence in $\mathbb{R}^2$. Hence by the Bolzano-Weierstrass theorem in $\mathbb{R}^2$, there exist $(x, y) \in \mathbb{R}^2$ and a convergent subsequence $((x_{n_k}, y_{n_k}))$ of $((x_n, y_n))$ such that $(x_{n_k}, y_{n_k}) \to (x, y)$. Since f is continuous at (x, y), $\left(n_k, \frac{1}{n_k}\right) = f(x_{n_k}, y_{n_k}) \to f(x, y) \in \mathbb{R}^2$. Consequently, the sequence (n_k) converges in $\mathbb{R}$, which is not true, since (n_k) is unbounded. Hence it follows that there cannot exist any sequence $((x_n, y_n))$ in $\mathbb{R}^2$ such that $x_n^2 + y_n^2 = \frac{1}{2}$ and $f(x_n, y_n) = \left(n, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$.

20. Examine whether

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4 + y^2}$$

exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $((x_n, y_n))$ be any sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n, y_n) \to (0, 0)$. Then $x_n \to 0$ and $y_n \to 0$. Since

$$\left| \frac{x_n^3 y_n}{x_n^4 + y_n^2} \right| \leq \frac{|x_n|^3 |y_n|}{x_n^4} \leq |x_n| + |y_n| \to 0,$$

it follows that $\frac{x_n^3 y_n}{x_n^4 + y_n^2} \to 0$. Therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4 + y^2} = 0.$$

21. Examine whether

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2}$$

exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $((x_n, y_n))$ be any sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n, y_n) \to (0,0)$. Then $x_n \to 0$, $y_n \to 0$, and hence

$$\frac{x_n^3 - y_n^3}{x_n^2 + y_n^2} = \frac{x_n^3}{x_n^2 + y_n^2} - \frac{y_n^3}{x_n^2 + y_n^2} \le |x_n| + |y_n| \to 0.$$

Consequently $\frac{x_n^3 - y_n^3}{x_n^2 + y_n^2} \to 0$, and therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2} = 0.$$

22. Examine whether

$$\lim_{(x,y)\to(0,0)} |x| e^{-|x|/y^2}$$

exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $f(x, y) = |x| e^{-|x|/y^2}$ for all $(x, y) \in \mathbb{R}^2$ with $y \neq 0$. We have $(0, \frac{1}{n}) \to (0,0)$ and $(\frac{1}{n^2}, \frac{1}{n}) \to (0,0)$. Also, $f(0, \frac{1}{n}) \to 0$ and $f(\frac{1}{n^2}, \frac{1}{n}) \to \frac{1}{e}$. Since $\lim\limits_{n\to\infty} f(0, \frac{1}{n}) \neq \lim\limits_{n\to\infty} f(\frac{1}{n^2}, \frac{1}{n})$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

23. Examine whether

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^2 - 1}{x^2 + y^3}$$

exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $f(x, y) = \frac{x^3 + y^2 - 1}{x^2 + y^3}$ for all $(x, y) \in \mathbb{R}^2$ with $x^2 + y^3 \neq 0$. We have $(\frac{1}{n}, 0) \to (0,0)$ and $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$. Also, $f(\frac{1}{n}, 0) = \frac{1/n^3 - 1}{1/n^2} = \frac{1}{n} - n^2 \to -\infty$ and $f(\frac{1}{n}, \frac{1}{n}) = \frac{1/n^3 + 1/n^2 - 1}{1/n^2 + 1/n^3} = 1 + \frac{1/n - 1}{1 + 1/n} \to 1$. Since $f(\frac{1}{n}, 0) \neq f(\frac{1}{n}, \frac{1}{n})$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

24. Examine whether

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^2 + x^2 - 1}{x^2 + y^3}$$

exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $((x_n, y_n))$ be any sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n, y_n) \to (0,0)$. Then $x_n \to 0$ and $y_n \to 0$. Since $0 \le \frac{\sqrt{x_n^2 y_n^4 + 1} - 1}{x_n^2 + y_n^3} \le \frac{x_n^2 y_n^4}{x_n^2 + y_n^3} \le x_n^2$ if $y_n^2 \to 0$, it follows that

$$\frac{\sqrt{x_n^2 y_n^4 + 1} - 1}{x_n^2 + y_n^3} \to 0.$$

Therefore

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2 y^4 + 1} - 1}{x^2 + y^3} = 0.$$

25. Examine whether
$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2+x^2-1}{x^2+y^3}$$
exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $f(x,y)=\frac{x^2y^2+x^2-1}{x^2+y^3}$ for all $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$. We have $(\frac{1}{\sqrt{n}},\frac{1}{n})\to(0,0)$ and $(\frac{1}{n},0)\to(0,0)$. Also, $f(\frac{1}{n},0)\to-1$ and $f(\frac{1}{\sqrt{n}},\frac{1}{n})\to\frac{1}{2}$. Since
$$\lim_{n\to\infty} f\left(\frac{1}{n},0\right)\neq\lim_{n\to\infty} f\left(\frac{1}{\sqrt{n}},\frac{1}{n}\right),$$
$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

26. Examine whether
$$\lim_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2}$$
exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $f(x,y,z)=\frac{(x+y+z)^2}{x^2+y^2+z^2}$ for all $(x,y,z)\in\mathbb{R}^3\setminus\{(0,0,0)\}$. We have $(\frac{1}{n},0,0)\to(0,0,0)$ and $(\frac{1}{n},\frac{1}{n},0)\to(0,0,0)$. Also, $f(\frac{1}{n},0,0)=1$ and $f(\frac{1}{n},\frac{1}{n},0)=2\to2$. Since $\lim_{n\to\infty} f\left(\frac{1}{n},0,0\right)\neq\lim_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n},0\right)$,
$$\lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$$
does not exist (in $\mathbb{R}$).

27. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by
$$f(x,y)=\begin{cases} x+y & \text{if } x\neq y,\\ 1 & \text{if } x=y.\end{cases}$$
Examine whether
$$\lim_{(x,y)\to(0,0)} f(x,y)$$
exists (in $\mathbb{R}$).

Solution: We have $(\frac{1}{n},0)\to(0,0)$ and $(\frac{1}{n},\frac{1}{n})\to(0,0)$. Also, $f(\frac{1}{n},0)=\frac{1}{n}\to0$ and $f(\frac{1}{n},\frac{1}{n})=1\to1$. Since
$$\lim_{n\to\infty} f\left(\frac{1}{n},0\right)\neq\lim_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n}\right),$$
$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

28. Let $S\subseteq\mathbb{R}^2$, $(x_0,y_0)\in\mathbb{R}^2$ and $r>0$ be such that $(B_r(x_0)\times B_r(y_0))\setminus\{(x_0,y_0)\}\subseteq S$. Let $\lim_{x\to x_0} f(x,y)$ exist (in $\mathbb{R}$) for each $y\in B_r(y_0)\setminus\{y_0\}$, $\lim_{y\to y_0} f(x,y)$ exist (in $\mathbb{R}$) for each $x\in B_r(x_0)\setminus\{x_0\}$ and $\lim_{(x,y)\to(x_0,y_0)} f(x,y)=\ell\in\mathbb{R}$. Show that
$$\lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right)=\lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right)=\ell.$$

[$\lim_{x\to x_0}(\lim_{y\to y_0} f(x,y))$ and $\lim_{y\to y_0}(\lim_{x\to x_0} f(x,y))$ are called the iterated limits of f at (x_0, y_0).]

Solution: Let $\varepsilon > 0$. Since $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell$, there exists $\delta \in (0, r)$ such that
$$|f(x,y) - \ell| < \frac{\varepsilon}{2}$$
for all $(x,y) \in B_\delta((x_0,y_0)) \setminus \{(x_0,y_0)\}$. Let $g(x) = \lim_{y\to y_0} f(x,y)$ for all $x \in B_r(x_0) \setminus \{x_0\}$ and let $x \in B_{\delta/2}(x_0)\setminus\{x_0\}$. Then there exists $s \in (0, \frac{\delta}{2})$ such that $|f(x,y)-g(x)| < \frac{\varepsilon}{2}$ for all $y \in B_s(y_0) \setminus \{y_0\}$. We choose any $y \in B_s(y_0) \setminus \{y_0\}$. Then
$$0 < \|(x,y) - (x_0,y_0)\| = \sqrt{(x-x_0)^2 + (y-y_0)^2} < \sqrt{\frac{\delta^2}{4} + s^2} < \delta,$$
i.e. $(x,y) \in B_\delta((x_0,y_0)) \setminus \{(x_0,y_0)\}$ and hence
$$|g(x) - \ell| \le |g(x) - f(x,y)| + |f(x,y) - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
Therefore $\lim_{x\to x_0} g(x) = \ell$, i.e.
$$\lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right) = \ell.$$
Similarly, we can show that
$$\lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right) = \ell.$$

29. Show that
$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2}{x^2+y^2}\right) \neq \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2}{x^2+y^2}\right)$$
and hence conclude that
$$\lim_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}$$
does not exist (in $\mathbb{R}$).

Solution: For each $x \in \mathbb{R} \setminus \{0\}$,
$$\lim_{y\to 0}\frac{x^2}{x^2+y^2} = \frac{x^2}{x^2} = 1$$
and for each $y \in \mathbb{R} \setminus \{0\}$,
$$\lim_{x\to 0}\frac{x^2}{x^2+y^2} = 0.$$
Hence
$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2}{x^2+y^2}\right) = \lim_{x\to 0} 1 = 1 \neq 0 = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2}{x^2+y^2}\right).$$
Using Ex. 15 of Practice Problem Set 2, we can conclude that
$$\lim_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}$$

does not exist (in $\mathbb{R}$).

30. Show that

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{-x^2y^2}{x^4+y^4+(x-y)^2}\right)=0=\lim_{y\to 0}\left(\lim_{x\to 0}\frac{-x^2y^2}{x^4+y^4+(x-y)^2}\right)$$

but that

$$\lim_{(x,y)\to(0,0)}\frac{-x^2y^2}{x^4+y^4+(x-y)^2}$$

does not exist (in $\mathbb{R}$).

Solution: Let $f(x,y)=\frac{-x^2y^2}{x^4+y^4+(x-y)^2}$ for all $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$. Then

$$\lim_{y\to 0}f(x,y)=0$$

for each $x\in\mathbb{R}\setminus\{0\}$ and $\lim_{x\to 0}f(x,y)=0$ for each $y\in\mathbb{R}\setminus\{0\}$. Consequently,

$$\lim_{x\to 0}\left(\lim_{y\to 0}f(x,y)\right)=0=\lim_{y\to 0}\left(\lim_{x\to 0}f(x,y)\right).$$

Again, we have $(\frac{1}{n},0)\to(0,0)$ and $(\frac{1}{n},\frac{1}{n})\to(0,0)$. Also, $f(\frac{1}{n},0)\to 0$ and $f(\frac{1}{n},\frac{1}{n})\to -1$. Since $\lim_{n\to\infty}f(\frac{1}{n},0)\neq\lim_{n\to\infty}f(\frac{1}{n},\frac{1}{n})$, $\lim_{(x,y)\to(0,0)}f(x,y)$ does not exist (in $\mathbb{R}$).

31. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=\begin{cases}x\sin\frac{1}{y} & \text{if } y\neq 0\\ 0 & \text{if } y=0\end{cases}.$

Show that $\lim_{(x,y)\to(0,0)}f(x,y)=0$ and $(\lim_{y\to 0}f(x,y))=0$ but that $\lim_{x\to 0}f(x,y)$ does not exist (in $\mathbb{R}$) if $x\in\mathbb{R}\setminus\{0\}$ and so $\lim_{x\to 0}(\lim_{y\to 0}f(x,y))$ is not defined.

Solution: If $((x_n,y_n))$ is any sequence in $\mathbb{R}^2\setminus\{(0,0)\}$ such that $(x_n,y_n)\to(0,0)$, then $x_n\to 0$ and hence $|f(x_n,y_n)|\leq|x_n|\to 0$. Therefore $f(x_n,y_n)\to 0$ and so $\lim_{(x,y)\to(0,0)}f(x,y)=0$.

Again, for each $y\in\mathbb{R}\setminus\{0\}$, $\lim_{x\to 0}f(x,y)=\lim_{x\to 0}x\sin\frac{1}{y}=0$ and so $\lim_{x\to 0}(\lim_{y\to 0}f(x,y))=\lim_{x\to 0}0=0$.

If $x\in\mathbb{R}\setminus\{0\}$, then $\lim_{y\to 0}f(x,y)=\lim_{y\to 0}x\sin\frac{1}{y}$ which does not exist (in $\mathbb{R}$) and so $\lim_{x\to 0}(\lim_{y\to 0}f(x,y))$ is not defined.

32. Show that

$$\lim_{(x,y)\to(0,0)}\frac{1}{3x^2+y^4}=\infty.$$

Solution: Let $((x_n,y_n))$ be any sequence in $\mathbb{R}^2\setminus\{(0,0)\}$ such that $(x_n,y_n)\to(0,0)$. Then $x_n\to 0$, $y_n\to 0$ and hence $3x_n^2+y_n^4\to 0$. If $r>0$, then there exists $n_0\in\mathbb{N}$ such that $3x_n^2+y_n^4<\frac{1}{r}$ for all $n\geq n_0$ and so $\frac{1}{3x_n^2+y_n^4}>r$ for all $n\geq n_0$. Therefore $\frac{1}{3x_n^2+y_n^4}\to\infty$ and consequently

$$\lim_{(x,y)\to(0,0)}\frac{1}{3x^2+y^4}=\infty.$$

33. Let I be an open interval in $\mathbb{R}$ and let $F : I \to \mathbb{R}^m$ be a differentiable function such that $F(t) \cdot F'(t) = 0$ for all $t \in I$. Show that $\|F(t)\|$ is constant for all $t \in I$.

Solution: Since F is differentiable, the function $t \mapsto \|F(t)\|^2 = F'(t) \cdot F(t)$ from I to $\mathbb{R}$ is also differentiable, and

$$\frac{d}{dt}\|F(t)\|^2 = F'(t) \cdot F(t) + F(t) \cdot F'(t) = 2F(t) \cdot F'(t) = 0 \quad \text{for all } t \in I.$$

Hence, there exists $c \in \mathbb{R}$ such that

$$\|F(t)\|^2 = c \quad \text{for all } t \in I.$$

Clearly, $c \geq 0$, and so

$$\|F(t)\| = \sqrt{c} \quad \text{for all } t \in I.$$