MA15010H: Multi-variable Calculus

(Practice problem set 1: Hint/Model solution)
September - November, 2025

Question 0.1. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, then show that $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$.

solution 0.2. *We have* $\|\mathbf{x}\| = \|\mathbf{x}-\mathbf{y}+\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}\|$ *and so* $\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}\|$. *Similarly,* $\|\mathbf{y}\| = \|\mathbf{y}-\mathbf{x}+\mathbf{x}\| \leq \|\mathbf{y}-\mathbf{x}\| + \|\mathbf{x}\| = \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{x}\|$ *and so* $\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{x}-\mathbf{y}\|$. *Therefore* $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$.

Question 0.3. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, then show that $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

solution 0.4. *For* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, *we have*

$$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) = \|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2.$$

By the Cauchy–Schwarz inequality, $\mathbf{x}\cdot\mathbf{y} \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$, *so*

$$\|\mathbf{x}+\mathbf{y}\|^2 \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

Taking square roots gives $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Question 0.5. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, then show that $\|\mathbf{x}\| \leq \max\{\|\mathbf{x}+\mathbf{y}\|, \|\mathbf{x}-\mathbf{y}\|\}$.

solution 0.6. *Suppose, for the sake of contradiction, that*

$$\|\mathbf{x}\| > \max\{\|\mathbf{x}+\mathbf{y}\|,\ \|\mathbf{x}-\mathbf{y}\|\}.$$

Then $\|\mathbf{x}+\mathbf{y}\| < \|\mathbf{x}\|$ *and* $\|\mathbf{x}-\mathbf{y}\| < \|\mathbf{x}\|$. *Note that*

$$\mathbf{x} = \frac{(\mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y})}{2}.$$

Taking norms and using the triangle inequality, we get

$$\|\mathbf{x}\| = \left\|\frac{(\mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y})}{2}\right\| \leq \frac{1}{2}\big(\|\mathbf{x}+\mathbf{y}\| + \|\mathbf{x}-\mathbf{y}\|\big) < \frac{1}{2}(2\|\mathbf{x}\|) = \|\mathbf{x}\|,$$

a contradiction. Hence,

$$\|\mathbf{x}\| \leq \max\{\|\mathbf{x}+\mathbf{y}\|,\ \|\mathbf{x}-\mathbf{y}\|\}.$$

Question 0.7. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Then show that $\|\mathbf{x}+\alpha\mathbf{y}\| \geq \|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ iff $\mathbf{x}\cdot\mathbf{y} = 0$.

solution 0.8. *First assume that* $\mathbf{x}\cdot\mathbf{y} = 0$. *If* $\alpha \in \mathbb{R}$, *then we have*

$$\|\mathbf{x}+\alpha\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\alpha\mathbf{x}\cdot\mathbf{y} + \alpha^2\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \alpha^2\|\mathbf{y}\|^2 \geq \|\mathbf{x}\|^2,$$

and hence $\|\mathbf{x}+\alpha\mathbf{y}\| \geq \|\mathbf{x}\|$.
Conversely, let $\|\mathbf{x}+\alpha\mathbf{y}\| \geq \|\mathbf{x}\|$ *for all* $\alpha \in \mathbb{R}$. *If possible, let* $\mathbf{x}\cdot\mathbf{y} \neq 0$. *Then for* $\alpha = -\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}$, *we have*

$$\begin{aligned}\|\mathbf{x}+\alpha\mathbf{y}\|^2 &= \|\mathbf{x}\|^2 + 2\alpha(\mathbf{x}\cdot\mathbf{y}) + \alpha^2\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{y}\|^2} + \frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{y}\|^2}\\ &= \|\mathbf{x}\|^2 - \frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{y}\|^2} < \|\mathbf{x}\|^2,\end{aligned}$$

which is a contradiction. Therefore $\mathbf{x}\cdot\mathbf{y} = 0$.

Question 0.9. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and $a > 0$. Show that $|\mathbf{x} \cdot \mathbf{y}| \le a\|\mathbf{x}\|^2 + \frac{1}{4a}\|\mathbf{y}\|^2$.

solution 0.10. *By Cauchy-Schwartz inequality,*

$$\begin{aligned} |\mathbf{x} \cdot \mathbf{y}| &\le \|\mathbf{x}\|\|\mathbf{y}\| \\ &= 2\sqrt{a}\|\mathbf{x}\|\frac{1}{2\sqrt{a}}\|\mathbf{y}\| \\ &\le \left(\sqrt{a}\|\mathbf{x}\|\right)^2 + \left(\frac{1}{2\sqrt{a}}\|\mathbf{y}\|\right)^2 \\ &= a\|\mathbf{x}\|^2 + \frac{1}{4a}\|\mathbf{y}\|^2. \end{aligned}$$

Question 0.11. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Show that $|\|\mathbf{x}\| - \|\mathbf{y}\|| = \|\mathbf{x} - \mathbf{y}\|$ iff $\alpha\mathbf{x} = \beta\mathbf{y}$ for some $\alpha, \beta \ge 0$ with $(\alpha, \beta) \ne (0, 0)$.

solution 0.12. *We first assume that $|\|\mathbf{x}\| - \|\mathbf{y}\|| = \|\mathbf{x} - \mathbf{y}\|$. Then $|\|\mathbf{x}\| - \|\mathbf{y}\||^2 = \|\mathbf{x} - \mathbf{y}\|^2$, which gives $\|\mathbf{x}\|\,\|\mathbf{y}\| = |\mathbf{x} \cdot \mathbf{y}|$. By Cauchy–Schwarz equality, $\mathbf{y} = 0$ or $\mathbf{x} = t\mathbf{y}$ for some $t \in \mathbb{R}$. If $\mathbf{y} = 0$ we take $(\alpha, \beta) = (0, 1)$. If $\mathbf{y} \ne 0$ and $\mathbf{x} = t\mathbf{y}$, then*

$$\mathbf{x} \cdot \mathbf{y} = t\|\mathbf{y}\|^2 = \|\mathbf{x}\|\,\|\mathbf{y}\| = |t|\|\mathbf{y}\|^2,$$

so $t = |t| \ge 0$. Then $\alpha\mathbf{x} = \beta\mathbf{y}$ for some $\alpha, \beta \ge 0$ not both zero.

Conversely, suppose $\alpha\mathbf{x} = \beta\mathbf{y}$ for some $\alpha, \beta \ge 0$ with $(\alpha, \beta) \ne (0, 0)$. If $\mathbf{y} = 0$ (so $\mathbf{x} = 0$ as well) the equality is trivial. Otherwise $\mathbf{x} = t\mathbf{y}$ for some $t \ge 0$. Then

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| = |t - 1|\,\|\mathbf{y}\| \quad \textit{and} \quad \|\mathbf{x} - \mathbf{y}\| = \|t\mathbf{y} - \mathbf{y}\| = |t - 1|\,\|\mathbf{y}\|,$$

so the two sides are equal.

Question 0.13. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and $r > 0$ such that $\mathbf{y} \cdot \mathbf{z} = 0$ for all $\mathbf{z} \in B_r(\mathbf{x})$. Show that $\mathbf{y} = \mathbf{0}$.

solution 0.14. *If possible, let $\mathbf{y} \ne 0$. Then $\|\mathbf{y}\| \ne 0$. If $\mathbf{z} = \mathbf{x} + \frac{r}{2\|\mathbf{y}\|}\mathbf{y}$, then $\mathbf{z} \in \mathbb{R}^m$ and since $\|\mathbf{z} - \mathbf{x}\| = \frac{r}{2} < r$, $\mathbf{z} \in B_r(\mathbf{x})$. Hence $\mathbf{y} \cdot \mathbf{z} = 0$ and so $\mathbf{y} \cdot \mathbf{x} + \frac{r}{2\|\mathbf{y}\|}\|\mathbf{y}\|^2 = 0$. Since $\mathbf{x} \in B_r(\mathbf{x})$, $\mathbf{y} \cdot \mathbf{x} = 0$ and so from above, we get $\|\mathbf{y}\| = 0$, which is a contradiction. Therefore $\mathbf{y} = 0$.*

Question 0.15. If $\mathbf{x}_0 \in \mathbb{R}^m$ and $r > 0$, then determine $\sup\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in B_r(\mathbf{x}_0)\}$ with justification.

solution 0.16. *For all $\mathbf{x}, \mathbf{y} \in B_r(\mathbf{x}_0)$, $\|\mathbf{x} - \mathbf{y}\| \le \|\mathbf{x} - \mathbf{x}_0\| + \|\mathbf{x}_0 - \mathbf{y}\| < r + r = 2r$ and so $2r$ is an upper bound of $\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in B_r(\mathbf{x}_0)\}$. Let $\varepsilon > 0$ such that $\varepsilon < r$.*

Then $\mathbf{x}_0 + (r - \frac{\varepsilon}{3})\mathbf{e}_1$, $\mathbf{x}_0 - (r - \frac{\varepsilon}{3})\mathbf{e}_1 \in \mathbb{R}^m$ and since $\|\mathbf{x}_0 + (r - \frac{\varepsilon}{3})\mathbf{e}_1 - \mathbf{x}_0\| = r - \frac{\varepsilon}{3} < r$, we have $\mathbf{x}_0 + (r - \frac{\varepsilon}{3})\mathbf{e}_1, \mathbf{x}_0 - (r - \frac{\varepsilon}{3})\mathbf{e}_1 \in B_r(\mathbf{x}_0)$. Also, $\|(\mathbf{x}_0 + (r - \frac{\varepsilon}{3})\mathbf{e}_1) - (\mathbf{x}_0 - (r - \frac{\varepsilon}{3})\mathbf{e}_1)\| = 2r - \frac{2\varepsilon}{3} > 2r - \varepsilon$ and hence $2r - \varepsilon$ is not an upper bound of $\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in B_r(\mathbf{x}_0)\}$. Therefore $\sup\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in B_r(\mathbf{x}_0)\} = 2r$.

Question 0.17. Let $S \subseteq \mathbb{R}^m$ such that $S \subseteq B_r[\mathbf{x}_0]$ for some $\mathbf{x}_0 \in \mathbb{R}^m$ and for some $r > 0$. Show that S is a bounded set.

solution 0.18. *If* $\mathbf{x} \in S$*, then* $\mathbf{x} \in B_r[\mathbf{x}_0]$ *and hence*

$$\|\mathbf{x}\| = \|\mathbf{x} - \mathbf{x}_0 + \mathbf{x}_0\| \leq \|\mathbf{x} - \mathbf{x}_0\| + \|\mathbf{x}_0\| \leq r + \|\mathbf{x}_0\|.$$

Therefore S *is a bounded set in* $\mathbb{R}^m$*.*

Question 0.19. Let $\alpha \in (0,1)$ and let $\mathbf{x}_n = (n^3\alpha^n, \frac{1}{n}\lfloor n\alpha \rfloor)$ for all $n \in \mathbb{N}$ (For each $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x). Examine whether the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^2$. Also, find $\lim\limits_{n\to\infty} \mathbf{x}_n$ if the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^2$.

solution 0.20. *Let* $x_n = n^3\alpha^n$ *and* $y_n = \frac{1}{n}\lfloor n\alpha \rfloor$ *for all* $n \in \mathbb{N}$*. Using the ratio test, the sequence* (x_n) *converges in* $\mathbb{R}$ *to* 0*. Again, since* $n\alpha < \lfloor n\alpha \rfloor + 1$ *for all* $n \in \mathbb{N}$*, we have* $n\alpha - 1 < \lfloor n\alpha \rfloor \leq n\alpha$ *for all* $n \in \mathbb{N}$ *and so it follows that* $\alpha - \frac{1}{n} < y_n \leq \alpha$ *for all* $n \in \mathbb{N}$*. Hence by the sandwich theorem, the sequence* (y_n) *converges in* $\mathbb{R}$ *to* α*. Therefore the sequence* $(\mathbf{x}_n)$ *converges in* $\mathbb{R}^2$ *and* $\lim\limits_{n\to\infty} \mathbf{x}_n = (0, \alpha)$*.*

Question 0.21. Let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^m$ such that the series $\sum\limits_{n=1}^{\infty} 2\|\mathbf{x}_n\|^2$ is convergent. Show that the series $\sum\limits_{n=1}^{\infty} \|\mathbf{x}_n\|$ is convergent.

solution 0.22. *For all* $n \in \mathbb{N}$*, using the Cauchy–Schwarz inequality, we have*

$$\begin{aligned}\sum_{k=1}^{n} \|\mathbf{x}_k\| &= \sum_{k=1}^{n} k \cdot \frac{\|\mathbf{x}_k\|}{k} \\ &\leq \left(\sum_{k=1}^{n} k^2\|\mathbf{x}_k\|^2\right)^{1/2} \left(\sum_{k=1}^{n} \frac{1}{k^2}\right)^{1/2} \\ &\leq \left(\sum_{k=1}^{\infty} k^2\|\mathbf{x}_k\|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{1/2} < \infty.\end{aligned}$$

This shows that the sequence $(\sum_{k=1}^{n} \|\mathbf{x}_k\|)$ *of partial sums of the series* $\sum_{k=1}^{\infty} \|\mathbf{x}_k\|$ *of non-negative real numbers is bounded above and hence the sequence* $(\sum_{k=1}^{n} \|\mathbf{x}_k\|)$ *converges in* $\mathbb{R}$*. Consequently the series* $\sum_{n=1}^{\infty} \|\mathbf{x}_n\|$ *is convergent in* $\mathbb{R}$*.*

Question 0.23. Let $(\mathbf{x}_n)$ and $(\mathbf{y}_n)$ be sequences in $\mathbb{R}^m$ such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y}_n \to \mathbf{y} \in \mathbb{R}^m$. Show that $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$ and $\mathbf{x}_n \cdot \mathbf{y}_n \to \mathbf{x} \cdot \mathbf{y}$.

solution 0.24. *Since* $\mathbf{x}_n \to \mathbf{x}$ *and* $\mathbf{y}_n \to \mathbf{y}$*,* $\|\mathbf{x}_n - \mathbf{x}\| \to 0$ *and* $\|\mathbf{y}_n - \mathbf{y}\| \to 0$*. Hence*

$$\|(\mathbf{x}_n + \mathbf{y}_n) - (\mathbf{x} + \mathbf{y})\| \leq \|\mathbf{x}_n - \mathbf{x}\| + \|\mathbf{y}_n - \mathbf{y}\| \to 0.$$

Therefore $\|(\mathbf{x}_n + \mathbf{y}_n) - (\mathbf{x} + \mathbf{y})\| \to 0$ *and so* $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$*.*

Again,

$$\begin{aligned}|\mathbf{x}_n \cdot \mathbf{y}_n - \mathbf{x} \cdot \mathbf{y}| &= |\mathbf{x}_n \cdot \mathbf{y}_n - \mathbf{x}_n \cdot \mathbf{y} + \mathbf{x}_n \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y}| = |\mathbf{x}_n \cdot (\mathbf{y}_n - \mathbf{y}) + (\mathbf{x}_n - \mathbf{x}) \cdot \mathbf{y}| \\ &\leq |\mathbf{x}_n \cdot (\mathbf{y}_n - \mathbf{y})| + |(\mathbf{x}_n - \mathbf{x}) \cdot \mathbf{y}| \leq \|\mathbf{x}_n\|\|\mathbf{y}_n - \mathbf{y}\| + \|\mathbf{x}_n - \mathbf{x}\|\|\mathbf{y}\| \quad \textit{for all } n \in \mathbb{N}.\end{aligned}$$

Since $(\mathbf{x}_n)$ *is a convergent sequence in* $\mathbb{R}^m$*,* $(\mathbf{x}_n)$ *is bounded in* $\mathbb{R}^m$*. Hence there exists* $r > 0$ *such that* $\|\mathbf{x}_n\| \le r$ *for all* $n \in \mathbb{N}$*. Therefore*

$$|\mathbf{x}_n \cdot \mathbf{y}_n - \mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}_n\|\|\mathbf{y}_n - \mathbf{y}\| + \|\mathbf{x}_n - \mathbf{x}\|\|\mathbf{y}\| \to 0$$

and so $|\mathbf{x}_n \cdot \mathbf{y}_n - \mathbf{x} \cdot \mathbf{y}| \to 0$*. Hence* $\mathbf{x}_n \cdot \mathbf{y}_n \to \mathbf{x} \cdot \mathbf{y}$*.*

Question 0.25. Let $\mathbf{x} \in \mathbb{R}^m$ and let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^m$ such that $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$ and $\mathbf{x}_n \cdot \mathbf{x} \to \mathbf{x} \cdot \mathbf{x}$. Show that $(\mathbf{x}_n)$ is convergent.

solution 0.26. *Since*

$$\|\mathbf{x}_n - \mathbf{x}\|^2 = \|\mathbf{x}_n\|^2 - 2\mathbf{x}_n \cdot \mathbf{x} + \|\mathbf{x}\|^2 \to \|\mathbf{x}\|^2 - 2\mathbf{x} \cdot \mathbf{x} + \|\mathbf{x}\|^2 = 2\|\mathbf{x}\|^2 - 2\|\mathbf{x}\|^2 = 0,$$

we have that $\|\mathbf{x}_n - \mathbf{x}\| \to 0$ *and hence* $\mathbf{x}_n \to \mathbf{x}$*. Therefore* $(\mathbf{x}_n)$ *is convergent in* $\mathbb{R}^m$*.*

Question 0.27. State TRUE or FALSE with justification: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{x} \neq \mathbf{y}$ and $\|\mathbf{x}\| = 1 = \|\mathbf{y}\|$, then it is necessary that $\|\mathbf{x} + \mathbf{y}\| < 2$.

solution 0.28. *We have*

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\mathbf{x} \cdot \mathbf{y} = 2 + 2\mathbf{x} \cdot \mathbf{y}$$

and

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x} \cdot \mathbf{y} = 2 - 2\mathbf{x} \cdot \mathbf{y}.$$

Hence

$$\|\mathbf{x} + \mathbf{y}\|^2 = 2 + 2 - \|\mathbf{x} - \mathbf{y}\|^2 < 4,$$

since $\|\mathbf{x} - \mathbf{y}\| > 0$*. So* $\|\mathbf{x} + \mathbf{y}\| < 2$*. Therefore the given statement is TRUE.*

Question 0.29. State TRUE or FALSE with justification: If $(\mathbf{x}_n)$ is a sequence in $\mathbb{R}^m$ such that for each $\mathbf{x} \in \mathbb{R}^m$,

$$\lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{x}$$

exists (in $\mathbb{R}$), then

$$\lim_{n\to\infty} \|\mathbf{x}_n\|^2$$

must exist (in $\mathbb{R}$).

solution 0.30. *For each* $n \in \mathbb{N}$*, let* $\mathbf{x}_n = \left(x_1^{(n)}, \dots, x_m^{(n)}\right)$*.*

By the given condition,

$$\lim_{n\to\infty} x_j^{(n)} = \lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{e}_j$$

exists (in $\mathbb{R}$*) for* $j = 1, \dots, m$*. Consequently*

$$\lim_{n\to\infty} \|\mathbf{x}_n\|^2 = \lim_{n\to\infty} \left((x_1^{(n)})^2 + \dots + (x_m^{(n)})^2\right)$$

exists (in $\mathbb{R}$*). Therefore the given statement is TRUE.*

Question 0.31. State TRUE or FALSE with justification: There exists an unbounded sequence (x_n) of distinct real numbers such that the sequence $\big((x_n, \cos x_n)\big)$ in $\mathbb{R}^2$ has a convergent subsequence.

solution 0.32. *The sequence*

$$(x_n) = \left(1, \tfrac{1}{2}, 2, \tfrac{1}{3}, 3, \tfrac{1}{4}, \dots\right)$$

in $\mathbb{R}$ *is unbounded and its subsequence*

$$(x_{2n}) = \left(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots\right)$$

converges in $\mathbb{R}$*. By continuity of the cosine function, the sequence* $\cos x_{2n}$ *also converges in* $\mathbb{R}$*. Hence the subsequence*

$$\big((x_{2n}, \cos x_{2n})\big)$$

of the sequence $\big((x_n, \cos x_n)\big)$ *converges in* $\mathbb{R}^2$*. Therefore the given statement is TRUE.*

Question 0.33. Let $S = \{(x, y) \in \mathbb{R}^2 : x \neq y\}$ and let $f : S \to \mathbb{R}$ be defined by $f(x, y) = \frac{x+y}{x-y}$ for all $(x, y) \in S$. Show by using the definition of continuity that f is continuous at $(1, 2)$.

solution 0.34. *Let* $\varepsilon > 0$*. For all* $(x, y) \in S$*, we have* $|f(x, y) - f(1, 2)| = \left|\frac{x+y}{x-y} + 3\right| = 2\left|\frac{2x-y}{x-y}\right|$*. If* $(x, y) \in S$ *and* $\|(x, y) - (1, 2)\| = \sqrt{(x-1)^2 + (y-2)^2} < \frac{1}{4}$*, then* $|x - 1| < \frac{1}{4}$ *and* $|y - 2| < \frac{1}{4}$*, and so* $|x - y| = |1 - ((2 - y) + (x - 1))| \geq 1 - |(2 - y) + (x - 1)| \geq 1-(|2-y|+|x-1|) \geq 1-(|2-y|+|x-1|) \geq 1-\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2}$*. Again, if* $r > 0$ *and* $(x, y) \in S$ *such that* $\|(x, y)-(1, 2)\| = \|(x, y)-(1, 2)\| = \sqrt{(x-1)^2 + (y-2)^2} < r$*, then* $|x-1| < r$ *and* $|y-2| < r$*, and so* $|2x-y| = |2(x-1)+2-y| \leq |2(x-1)|+|y-2| < 2r+r = 3r$*. Hence if we choose* $\delta = \min\left\{\frac{1}{4}, \frac{\varepsilon}{12}\right\}$*, then* $\delta > 0$ *and for all* $(x, y) \in S$ *satisfying* $\|(x, y)-(1, 2)\| < \delta$*, we have* $|f(x, y) - f(1, 2)| < 12\delta \leq \varepsilon$*. Therefore* f *is continuous at* $(1, 2)$*.*

Question 0.35. If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $f(x, y) = x^2 + y^2$ for all $x \in \mathbb{Q}$ and for all $y \in \mathbb{R} \setminus \mathbb{Q}$, then determine $f(\sqrt{2}, 2)$.

solution 0.36. *We know that there exist sequences* (x_n) *in* $\mathbb{Q}$ *and* (y_n) *in* $\mathbb{R} \setminus \mathbb{Q}$ *such that* $x_n \to \sqrt{2}$ *and* $y_n \to 2$*. Hence* $(x_n, y_n) \to (\sqrt{2}, 2)$*. Since* f *is continuous at* $(\sqrt{2}, 2)$*, we have* $f(\sqrt{2}, 2) = \lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty}(x_n^2 + y_n^2) = \lim_{n\to\infty} x_n^2 + \lim_{n\to\infty} y_n^2 = (\sqrt{2})^2 + 2^2 = 2 + 4 = 6$*.*

Question 0.37. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0, 0)$, where for all $(x, y) \in \mathbb{R}^2$,

$$f(x, y) = \begin{cases} xy & \text{if } xy \geq 0, \\ -xy & \text{if } xy < 0. \end{cases}$$

solution 0.38. *Let* $(x_n, y_n))$ *be any sequence in* $\mathbb{R}^2$ *such that* $(x_n, y_n) \to (0, 0)$*. Then* $x_n \to 0$ *and* $y_n \to 0$*. We have* $|f(x_n, y_n)| = |x_n y_n| \to 0$ *and hence* $f(x_n, y_n) \to 0 = f(0, 0)$*. Therefore* f *is continuous at* $(0, 0)$*.*

Question 0.39. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0, 0)$, where for all $(x, y) \in \mathbb{R}^2$,

$$f(x, y) = \begin{cases} \frac{xy^3}{x^2+y^4} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

solution 0.40. *Let $\varepsilon > 0$. Then for all $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have*

$$|f(x,y) - f(0,0)| = \left|\frac{xy^3}{x^2+y^4}\right| \leq |y| \leq \frac{1}{2}\sqrt{x^2+y^2}.$$

Let $\delta = 2\varepsilon$. Then $\delta > 0$ and for all $(x, y) \in \mathbb{R}^2$ with $\|(x,y) - (0,0)\| = \sqrt{x^2+y^2} < \delta$, we have $|f(x,y) - f(0,0)| < \varepsilon$. Therefore f is continuous at $(0,0)$.

Question 0.41. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$, where for all $(x, y) \in \mathbb{R}^2$,

$$f(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } 0 < y < x^2, \\ 0 & \text{otherwise.} \end{cases}$$

solution 0.42. *Since $(\frac{1}{n}, \frac{1}{2n^2}) \to (0,0)$ but $f(\frac{1}{n}, \frac{1}{2n^2}) = 1 \to 1 \neq 0 = f(0,0)$, f is not continuous at $(0,0)$.*

Question 0.43. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, where for all $(x, y) \in \mathbb{R}^2$,

$$f(x,y) = \begin{cases} \frac{xy}{x-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

solution 0.44. *If $\varphi(x,y) = xy$ and $\psi(x,y) = x - y$ for all $(x, y) \in \mathbb{R}^2$, then as polynomial functions, $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ are continuous and $\psi(x,y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$ with $x \neq y$. Hence f is continuous at each $(x, y) \in \mathbb{R}^2$ with $x \neq y$. Let $x \in \mathbb{R} \setminus \{0\}$. Then $(x + \frac{1}{n}, x) \to (x, x)$ but $f(x + \frac{1}{n}, x) = nx^2 + x \neq 0 = f(x,x)$. So f is not continuous at (x,x). Again, $(\frac{1}{n} + \frac{1}{n^2}, \frac{1}{n}) \to (0,0)$ but $f(\frac{1}{n} + \frac{1}{n^2}, \frac{1}{n}) = 1 + \frac{1}{n} \to 1 \neq 0 = f(0,0)$. So f is not continuous at $(0,0)$. Therefore the set of points of continuity of f is $\{(x,y) \in \mathbb{R}^2 : x \neq y\}$.*

Question 0.45. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, where for all $(x, y) \in \mathbb{R}^2$,

$$f(x,y) = \begin{cases} xy & \text{if } xy \in \mathbb{Q}, \\ -xy & \text{if } xy \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

solution 0.46. *Let $(x, y) \in \mathbb{R}^2$ such that $xy = 0$ and let $((x_n, y_n))$ be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x, y)$. Then $x_n \to x$ and $y_n \to y$. We have $|f(x_n, y_n)| = |x_n y_n| \to |xy| = 0$ and so $f(x_n, y_n) \to 0 = f(x, y)$. Hence f is continuous at (x, y). Again, let $(x, y) \in \mathbb{R}^2$ such that $xy \neq 0$. We consider the following two possible cases.*
Case (i): $xy \in \mathbb{R} \setminus \mathbb{Q}$.
We can find two sequences (x_n) and (y_n) in $\mathbb{Q}$ such that $x_n \to x$ and $y_n \to y$. Then $((x_n, y_n))$ is a sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x, y)$ but $f(x_n, y_n) = x_n y_n \to xy \neq -xy = f(x, y)$. Hence f is not continuous at (x, y).
Case (ii): $xy \in \mathbb{Q}$.
Since $x \neq 0$, we can find a sequence (x_n) in $\mathbb{Q} \setminus \{0\}$ and a sequence (y_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to x$ and $y_n \to y$. Then $((x_n, y_n))$ is a sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x, y)$ but $f(x_n, y_n) = -x_n y_n \to -xy \neq xy = f(x, y)$. Hence f is not continuous at (x, y). Therefore the set of points of continuity of f is $\{(x,y) \in \mathbb{R}^2 : xy = 0\}$.

Question 0.47. Let α, β be positive real numbers and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{|x|^\alpha |y|^\beta}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is continuous iff $\alpha + \beta > 2$.

solution 0.48. *Let $\alpha+\beta > 2$ and let $((x_n, y_n))$ be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (0,0)$. Then $x_n \to 0$ and $y_n \to 0$. For all $n \in \mathbb{N}$ for which $(x_n, y_n) \neq (0,0)$, we have*

$$0 \leq f(x_n, y_n) \leq \frac{(x_n^2+y_n^2)^{\alpha/2}(x_n^2+y_n^2)^{\beta/2}}{x_n^2+y_n^2} = \frac{(x_n^2+y_n^2)^{(\alpha+\beta)/2}}{x_n^2+y_n^2} = (x_n^2+y_n^2)^{(\alpha+\beta-2)/2}$$

and since $f(0,0) = 0$, we have $0 \leq f(x_n, y_n) \leq 2^{(\alpha+\beta-2)/2}(x_n^2+y_n^2)^{(\alpha+\beta-2)/2}$ for all $n \in \mathbb{N}$. Since $2^{(\alpha+\beta-2)/2}(x_n^2+y_n^2)^{(\alpha+\beta-2)/2} \to 0$, we get $f(x_n, y_n) \to 0 = f(0,0)$. This shows that f is continuous at $(0,0)$. Also, it is clear (by similar arguments given in other examples) that f is continuous at each $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Therefore f is continuous. Conversely, let f be continuous and if possible, let $\alpha + \beta \leq 2$. We have $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ but $f(\frac{1}{n}, \frac{1}{n}) = \frac{1}{2}n^{2-(\alpha+\beta)} \not\to 0 = f(0,0)$ (because for $\alpha + \beta = 2$, $f(\frac{1}{n}, \frac{1}{n}) \to \frac{1}{2}$ and for $\alpha+\beta < 2$, the sequence $f(\frac{1}{n}, \frac{1}{n})$ is unbounded). Hence f is not continuous at $(0,0)$, which is a contradiction. Therefore $\alpha + \beta > 2$.

Question 0.49. Let S be a nonempty subset of $\mathbb{R}^m$ and let $f_j : S \to \mathbb{R}$ for each $j \in \{1, \ldots, k\}$. If $f(x) = (f_1(x), \ldots, f_k(x))$ for all $x \in S$, then show that $f : S \to \mathbb{R}^k$ is continuous at $x_0 \in S$ iff f_j is continuous at x_0 for each $j \in \{1, \ldots, k\}$.

solution 0.50. *We first assume that f is continuous at x_0 and let (x_n) be any sequence in S such that $x_n \to x_0$. Then $(f_1(x_n), \ldots, f_k(x_n)) = f(x_n) \to f(x_0) = (f_1(x_0), \ldots, f_k(x_0))$ and hence $f_j(x_n) \to f_j(x_0)$ for each $j \in \{1, \ldots, k\}$. Consequently f_j is continuous at x_0 for each $j \in \{1, \ldots, k\}$. Conversely, let f_j be continuous at x_0 for each $j \in \{1, \ldots, k\}$ and let (x_n) be any sequence in S such that $x_n \to x_0$. Then $f_j(x_n) \to f_j(x_0)$ for each $j \in \{1, \ldots, k\}$ and hence*

$$f(x_n) = (f_1(x_n), \ldots, f_k(x_n)) \to (f_1(x_0), \ldots, f_k(x_0)) = f(x_0).$$

Therefore f is continuous at x_0.

Question 0.51. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}^2$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,

$$f(x,y) = \begin{cases} \left(\dfrac{x^3}{x^2+y^2}, \sin(x^2+y^2)\right) & \text{if } (x,y) \neq (0,0), \\ (0,0) & \text{if } (x,y) = (0,0). \end{cases}$$

solution 0.52. *For all $(x,y) \in \mathbb{R}^2$, let $\varphi(x,y) = \sin(x^2+y^2)$ and*

$$\psi(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \textit{if } (x,y) \neq (0,0), \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$$

Since $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a composition of a polynomial function and the sine function, both of which are continuous, φ is continuous at $(0,0)$. Again, let $\varepsilon > 0$. Then for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have

$$|\psi(x,y) - \psi(0,0)| = \left|\frac{x^3}{x^2+y^2}\right| \leq |x| \leq \sqrt{x^2+y^2}.$$

Since $\psi(0,0) = 0$, we get $|\psi(x,y) - \psi(0,0)| \leq \sqrt{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2$. Let $\delta = \varepsilon$. Then $\delta > 0$ and for all $(x,y) \in \mathbb{R}^2$ with $\|(x,y) - (0,0)\| = \sqrt{x^2+y^2} < \delta$, we have

$$|\psi(x,y) - \psi(0,0)| < \varepsilon.$$

Therefore ψ is continuous at $(0,0)$. Consequently (by Ex.17 of Practice Problem Set - 1) f is continuous at $(0,0)$.

Question 0.53. If $f, g : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ are continuous at $x_0 \in S$ and if $\varphi(x) = f(x) \cdot g(x)$ for all $x \in S$, then show that $\varphi : S \to \mathbb{R}$ is continuous at x_0.

solution 0.54. *Let (x_n) be any sequence in S such that $x_n \to x_0$. Since f and g are continuous at x_0, $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$. Hence (by Ex. 9 of Practice Problem Set - 1) $f(x_n) \cdot g(x_n) \to f(x_0) \cdot g(x_0) = \varphi(x_0)$. Therefore φ is continuous at x_0.*

Question 0.55. Let $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ be continuous at $x_0 \in S^0$ and let $f(x_0) \neq 0$. Show that there exists $r > 0$ such that $f(x) \neq 0$ for all $x \in B_r(x_0)$.

solution 0.56. *Since $x_0 \in S^0$, there exists $s > 0$ such that $B_s(x_0) \subseteq S$. Again, since $f(x_0) \neq 0$, $\frac{1}{2}\|f(x_0)\| > 0$. By the continuity of f at x_0, there exists $\delta > 0$ such that*

$$\|f(x) - f(x_0)\| < \frac{1}{2}\|f(x_0)\|$$

for all $x \in S$ satisfying $\|x - x_0\| < \delta$. Taking $r = \min\{s, \delta\} > 0$, we find that $\|f(x) - f(x_0)\| < \frac{1}{2}\|f(x_0)\|$ for all $x \in B_r(x_0)$. If possible, let $f(x) = 0$ for some $x \in B_r(x_0)$. Then from above, we get $\|f(x_0)\| < \frac{1}{2}\|f(x_0)\|$, which is not true. Therefore $f(x) \neq 0$ for all $x \in B_r(x_0)$.

Question 0.57. Let S be an open subset of $\mathbb{R}^m$ and let $f : S \to \mathbb{R}^k$ and $g : S \to \mathbb{R}^k$ be continuous at $x_0 \in S$. If for each $\varepsilon > 0$, there exist $x, y \in B_\varepsilon(x_0)$ such that $f(x) = g(y)$, then show that $f(x_0) = g(x_0)$.

solution 0.58. *By the given condition, for each $n \in \mathbb{N}$, there exist $x_n, y_n \in B_{\frac{1}{n}}(x_0)$ such that $f(x_n) = g(y_n)$. So $\|x_n - x_0\| < \frac{1}{n} \to 0$ and $\|y_n - x_0\| < \frac{1}{n} \to 0$. Hence $x_n \to x_0$ and $y_n \to x_0$. Since f and g are continuous at x_0, $f(x_n) \to f(x_0)$ and $g(y_n) \to g(x_0)$. Therefore $f(x_0) = g(x_0)$.*

Question 0.59. If $S = \{(x,y) \in \mathbb{R}^2 : x + y \geq 2\}$, then determine (with justification) S^0.

solution 0.60. *Let $(x_0, y_0) \in S$ with $x_0 + y_0 > 2$. Let $r = \frac{x_0+y_0-2}{\sqrt{2}} > 0$ and let $(x,y) \in B_r((x_0,y_0))$. Then $\|(x,y) - (x_0,y_0)\| = \sqrt{(x-x_0)^2 + (y-y_0)^2} < r$. By Cauchy-Schwarz inequality, we have $x_0 - x + y_0 - y \leq \sqrt{(x_0-x)^2 + (y_0-y)^2} \cdot \sqrt{2} < \sqrt{2}r = x_0 + y_0 - 2$.*

Hence $x+y>2$ and so $(x,y)\in S$. Thus $B_r((x_0,y_0))\subseteq S$ and therefore $(x_0,y_0)\in S^0$. Now, let $(x_0,y_0)\in S$ such that $x_0+y_0=2$ and if possible, let $(x_0,y_0)\in S^0$. Then there exists $r>0$ such that $B_r((x_0,y_0))\subseteq S$. Since $\|(x_0-\frac{r}{2},y_0)-(x_0,y_0)\|=\|(-\frac{r}{2},0)\|=\frac{r}{2}<r$, $(x_0-\frac{r}{2},y_0)\in B_r((x_0,y_0))$. However, $(x_0-\frac{r}{2},y_0)\notin S$, since $x_0-\frac{r}{2}+y_0=x_0+y_0-\frac{r}{2}=2-\frac{r}{2}<2$. Thus we get a contradiction. Hence $(x_0,y_0)\notin S^0$.

Therefore $S^0=\{(x,y)\in\mathbb{R}^2 : x+y>2\}$.

Question 0.61. If $S=\{(x_1,\dots,x_m)\in\mathbb{R}^m : x_m=1\}$, then determine (with justification) S^0.

solution 0.62. *If possible, let $S^0\neq\emptyset$. Then there exists $x=(x_1,\dots,x_m)\in S^0$ and hence there exists $r>0$ such that $B_r(x)\subseteq S$. If $y=(x_1,\dots,x_{m-1},x_m+\frac{r}{2})$, then $\|y-x\|=|\frac{r}{2}|<r$ and so $y\in B_r(x)$. But $y\notin S$, because $x_m+\frac{r}{2}=1+\frac{r}{2}\neq 1$. Thus we get a contradiction. Therefore $S^0=\emptyset$.*

Question 0.63. If $x\in\mathbb{R}^m$ and $r>0$, then determine (with justification) all the interior points of $B_r[x]$.

solution 0.64. *Let $y\in B_r(x)$. Then $\|y-x\|<r$. If $s=r-\|y-x\|$, then $s>0$. Let $z\in B_s(y)$. Then $\|z-y\|<s$ and so $\|z-x\|=\|z-y+y-x\|\leq\|z-y\|+\|y-x\|<s+\|y-x\|=r$. Hence $z\in B_r[x]$ and so $B_s(y)\subseteq B_r[x]$. Therefore $y\in(B_r[x])^0$. Again, let $y\in B_r[x]$ such that $\|y-x\|=r$. If possible, let $y\in(B_r[x])^0$. Then there exists $s>0$ such that $B_s(y)\subseteq B_r[x]$. Now, $y+\frac{s}{2r}(y-x)\in B_s(y)$, since*

$$\left\|y+\frac{s}{2r}(y-x)-y\right\|=\frac{s}{2r}\|y-x\|=\frac{s}{2}.$$

But $y+\frac{s}{2r}(y-x)\notin B_r[x]$, because

$$\|y+\frac{s}{2r}(y-x)-x\|=(1+\frac{s}{2r})\|y-x\|=r+\frac{s}{2}>r.$$

Thus we get a contradiction. Hence $y\notin(B_r[x])^0$. Therefore $(B_r[x])^0=B_r(x)$.

Question 0.65. Examine whether $\{(x,y)\in\mathbb{R}^2 : 0<x<y\}$ is an open set in $\mathbb{R}^2$.

solution 0.66. *Let $S=\{(x,y)\in\mathbb{R}^2 : 0<x<y\}$ and let $(x_0,y_0)\in S$. If $r=\min\left\{x_0,\frac{y_0-x_0}{\sqrt{2}}\right\}$, then $r>0$. Let $(x,y)\in B_r((x_0,y_0))$. Then $\|(x,y)-(x_0,y_0)\|=\sqrt{(x-x_0)^2+(y-y_0)^2}<r$. Hence $x_0-x\leq|x-x_0|<r\leq x_0$ and so $x>0$. Also, using Cauchy-Schwarz inequality, we have $x-x_0+y_0-y\leq\sqrt{(x-x_0)^2+(y_0-y)^2}\cdot\sqrt{2}<\sqrt{2}r\leq y_0-x_0$ and hence $x-y<0$, i.e. $x<y$. Thus $(x,y)\in S$ and so $(x_0,y_0)\in S^0$. Since $(x_0,y_0)\in S$ is arbitrary, it follows that S is an open set in $\mathbb{R}^2$.*