MA15010H: Multi-variable Calculus

(Lecturenote 2: Sequential continuity and vector differentiability)
July - November, 2025

Sequential criterion of closed set.

Theorem 0.1. *A set $S \subseteq \mathbb{R}^m$ is closed in $\mathbb{R}^m$ if and only if $x \in S$ for every sequence (x_n) in S with $x_n \to x$.*

Proof. Let S be closed in $\mathbb{R}^m$ and let (x_n) be a sequence in S such that $x_n \to x \in \mathbb{R}^m$. If possible, let $x \notin S$. Then $x \in \mathbb{R}^m \setminus S$ and since $\mathbb{R}^m \setminus S$ is open in $\mathbb{R}^m$, there exists $r > 0$ such that $B_r(x) \subseteq \mathbb{R}^m \setminus S$. Now, since $x_n \to x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B_r(x)$ for all $n \geq n_0$. In particular, $x_{n_0} \in \mathbb{R}^m \setminus S$, which is a contradiction. Hence $x \in S$.

Conversely, let $x \in S$ for every $x \in \mathbb{R}^m$ and for every sequence (x_n) in S with $x_n \to x$. If possible, let S be not closed in $\mathbb{R}^m$. Then $\mathbb{R}^m \setminus S$ is not open in $\mathbb{R}^m$ and hence there exists $x \in \mathbb{R}^m \setminus S$ such that $x \notin (\mathbb{R}^m \setminus S)^0$. So $B_{1/n}(x) \nsubseteq \mathbb{R}^m \setminus S$ for each $n \in \mathbb{N}$. Hence for each $n \in \mathbb{N}$, there exists $x_n \in B_{1/n}(x)$ such that $x_n \notin \mathbb{R}^m \setminus S$, i.e., $x_n \in S$. We have $\|x_n - x\| < \dfrac{1}{n} \to 0$, which gives $x_n \to x$. Thus we get a contradiction. Therefore S must be closed in $\mathbb{R}^m$.

□

Example 0.2. (a) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is a closed set but not an open set in $\mathbb{R}^2$.

(b) More generally, if $x_0 \in \mathbb{R}^m$ and $r > 0$, then $B_r[x_0]$ is a closed set but not an open set in $\mathbb{R}^m$.

(b) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is an open set but not a closed set in $\mathbb{R}^2$.
More generally, if $x_0 \in \mathbb{R}^m$ and $r > 0$, then $B_r(x_0)$ is an open set but not a closed set in $\mathbb{R}^m$.

(c) $\{(x, y) \in \mathbb{R}^2 : 1 < x < 2\}$ is neither open nor a closed set in $\mathbb{R}^2$.

(d) $\mathbb{R}^m$ is both an open set and a closed set in $\mathbb{R}^m$.

Theorem 0.3. *Let S be a nonempty closed and bounded subset of $\mathbb{R}^m$. If $f : S \to \mathbb{R}^p$ is continuous, then $f(S) = \{f(x) : x \in S\}$ is a closed and bounded subset of $\mathbb{R}^p$.*

Proof. Let $x_n \in S$ for all $n \in \mathbb{N}$ and let $y \in \mathbb{R}^p$ such that $f(x_n) \to y$. Since S is bounded, (x_n) is a bounded sequence in S and hence by the Bolzano-Weierstrass theorem in $\mathbb{R}^m$, there exist $x_0 \in \mathbb{R}^m$ and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x_0$. Again, since S is closed in $\mathbb{R}^m$, $x_0 \in S$. Now, since f is continuous at x_0, $f(x_{n_k}) \to f(x_0)$. Also, $f(x_n) \to y$ and so $y = f(x_0) \in f(S)$. Therefore $f(S)$ is closed in $\mathbb{R}^p$.

If possible, let $f(S)$ be not bounded. Then for each $n \in \mathbb{N}$, there exists $x_n \in S$ such that $\|f(x_n)\| > n$. Since S is bounded, (x_n) is a bounded sequence in S and hence by the Bolzano-Weierstrass theorem in $\mathbb{R}^m$, there exist $x_0 \in \mathbb{R}^m$ and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x_0$. Again, since S is closed in $\mathbb{R}^m$, $x_0 \in S$. Now, since f is continuous at x_0, $f(x_{n_k}) \to f(x_0)$. Thus the sequence $(f(x_{n_k}))$ must be bounded. However, $\|f(x_{n_k})\| > n_k$ for all $k \in \mathbb{N}$ and so $(f(x_{n_k}))$ is not bounded. Thus we get a contradiction. Therefore $f(S)$ must be bounded.

□

Example 0.4. We know that $S_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is a closed and bounded set in $\mathbb{R}^2$. Also, $\mathbb{R}$ is not a bounded set in $\mathbb{R}$ and $S_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is not a closed set in $\mathbb{R}^2$. Hence there cannot exist any continuous function from S_1 onto $\mathbb{R}$ or onto S_2.

Theorem 0.5. *Let S be a nonempty closed and bounded subset of $\mathbb{R}^m$. If $f : S \to \mathbb{R}$ is continuous, then there exist $x_0, y_0 \in S$ such that $f(x_0) = \sup f(S)$ and $f(y_0) = \inf f(S)$.*

Proof. We have proved above that $f(S)$ is a (nonempty) bounded set in $\mathbb{R}$. Hence $\sup f(S), \inf f(S) \in \mathbb{R}$. Let $\alpha = \sup f(S)$. Then for each $n \in \mathbb{N}$, there exists $x_n \in S$ such that $\alpha - \frac{1}{n} < f(x_n) \leq \alpha$. Hence $f(x_n) \to \alpha$. Since $f(x_n) \in f(S)$ for all $n \in \mathbb{N}$ and $f(S)$ is closed in $\mathbb{R}$ (as proved above), $\alpha \in f(S)$. So there exists $x_0 \in S$ such that $\alpha = f(x_0)$. Similarly we can show that there exists $y_0 \in S$ such that $f(y_0) = \inf f(S)$ □

Remark 0.6. A function $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^p$ is called bounded if $f(S)$ is a bounded subset of $\mathbb{R}^p$. We note that for a bounded function $f : S \subseteq \mathbb{R}^m \to \mathbb{R}$, it is not necessary that $\max f(S)$ and $\min f(S)$ exist.

For example, if $f(x,y) = \frac{x^2+y^2}{1+x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2$, then $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded, because $0 \leq f(x,y) \leq 1$ for all $(x,y) \in \mathbb{R}^2$, but there is no $(x_0,y_0) \in \mathbb{R}^2$ such that $f(x,y) \leq f(x_0,y_0)$ for all $(x,y) \in \mathbb{R}^2$.

Definition 0.7 (Limit Point)**.** A point $x \in \mathbb{R}^m$ is said to be a limit point of $S \subset \mathbb{R}^m$ if for every $r > 0$, $(B_r(x) \setminus \{x\}) \cap S \neq \emptyset$ (i.e. if for every $r > 0$, there exists $y \in S$ such that $0 < \|x - y\| < r$). For example, $(0,0)$ and $(1,0)$ are limit points of $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \subset \mathbb{R}^2$ but $(1,1)$ is not a limit point of S. We note that $(0,0) \in S$ but $(1,0) \notin S$.

Definition 0.8 (Limit)**.** Let x_0 be a limit point of $S \subset \mathbb{R}^m$. Then $y \in \mathbb{R}^k$ is said to be a limit of $f : S \to \mathbb{R}^k$ as x approaches x_0 if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(x) - y\| < \varepsilon$ for all $x \in S$ satisfying $0 < \|x - x_0\| < \delta$.

Such an y, if it exists, is unique because if $z \in \mathbb{R}^k$ with $z \neq y$ is also a limit of f as x approaches x_0, then $\varepsilon = \frac{1}{2}\|y - z\| > 0$ and so there exist $\delta_1, \delta_2 > 0$ such that $f(x) \in B_\varepsilon(y)$ for all $x \in (B_{\delta_1}(x_0) \setminus \{x_0\}) \cap S$ and $f(x) \in B_\varepsilon(z)$ for all $x \in (B_{\delta_2}(x_0) \setminus \{x_0\}) \cap S$. If $\delta = \min\{\delta_1, \delta_2\}$, then $\delta > 0$ and since $(B_\delta(x_0) \setminus \{x_0\}) \cap S \neq \emptyset$, we can choose $x \in (B_\delta(x_0) \setminus \{x_0\}) \cap S$. Then $f(x) \in B_\varepsilon(y) \cap B_\varepsilon(z)$, which contradicts the fact that $B_\varepsilon(y) \cap B_\varepsilon(z) = \emptyset$, proved earlier.

The y appearing in the above definition is called the limit of f as x approaches x_0 and we write

$$\lim_{x \to x_0} f(x) = y.$$

Sequential criterion of limit.

Theorem 0.9. *Let x_0 be a limit point of $S \subset \mathbb{R}^m$ and let $f : S \to \mathbb{R}^k$.*

$$\lim_{x \to x_0} f(x) = y \in \mathbb{R}^k \iff \text{for every sequence } (x_n) \text{ in } S \setminus \{x_0\} \text{ converging to } x_0,$$

the sequence $(f(x_n))$ converges to y.

Proof. Let $\lim_{x\to x_0} f(x) = y$ and let (x_n) be a sequence in $S \setminus \{x_0\}$ such that $x_n \to x_0$. If $\varepsilon > 0$, then there exists $\delta > 0$ such that $\|f(x) - y\| < \varepsilon$ for all $x \in S$ satisfying $0 < \|x - x_0\| < \delta$. Also, since $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$, there exists $n_0 \in \mathbb{N}$ such that $0 < \|x_n - x_0\| < \delta$ for all $n \geq n_0$. Hence $\|f(x_n) - y\| < \varepsilon$ for all $n \geq n_0$. Thus $(f(x_n)) \to y$.

Conversely, let $f(x_n) \to y$ for every sequence (x_n) in $S \setminus \{x_0\}$ with $x_n \to x_0$. If possible, let $\lim_{x\to x_0} f(x) \neq y$. Then there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, there exists $x_n \in S$ with $0 < \|x_n - x_0\| < \frac{1}{n}$ and $\|f(x_n) - y\| \geq \varepsilon$. Thus (x_n) is a sequence in $S \setminus \{x_0\}$ such that $\|x_n - x_0\| \to 0$, i.e. $x_n \to x_0$ but $f(x_n) \not\to y$. This is a contradiction. Therefore $\lim_{x\to x_0} f(x) = y$. □

Example 0.10.

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + y^2} = 0$$

but

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} \text{ does not exist (in } \mathbb{R}).$$

A method for showing the non-existence of limit.

Theorem 0.11. *Let (x_0, y_0) be a limit point of $S \subset \mathbb{R}^2$ and let $f : S \to \mathbb{R}$. Let $D \subseteq \mathbb{R}$ such that x_0 is a limit point of D and let $\varphi : D \to \mathbb{R}$ such*

that $(x, \varphi(x)) \in S$ for all $x \in D$ and $\lim_{x\to x_0} \varphi(x) = y_0$. If $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \ell \in \mathbb{R}$, then

$$\lim_{x\to x_0} f(x, \varphi(x)) = \ell.$$

Proof. Let (x_n) be any sequence in $D \setminus \{x_0\}$ such that $x_n \to x_0$. Since $\lim_{x\to x_0} \varphi(x) = y_0$, we get $\varphi(x_n) \to y_0$. Now, $((x_n, \varphi(x_n)))$ is a sequence in $S \setminus \{(x_0, y_0)\}$ and $((x_n, \varphi(x_n))) \to (x_0, y_0)$. Since $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \ell$, we have $f(x_n, \varphi(x_n)) \to \ell$. Consequently

$$\lim_{x\to x_0} f(x, \varphi(x)) = \ell.$$

□

Example 0.12. **(a)** If $f(x, y) = \frac{xy}{x^2+y^2}$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, then

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

does not exist (in $\mathbb{R}$) because if $m \in \mathbb{R}$, then $\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$, which gives more than one value if we vary m.

(b) If $f(x, y) = \frac{x^2 y}{x^2+y^2}$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, then $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist (in $\mathbb{R}$) because if $m \in \mathbb{R}$, then $\lim_{x\to 0} f(x, mx^2) = \lim_{x\to 0} \frac{mx^4}{x^2+m^2x^4} = \lim_{x\to 0} \frac{mx^4}{x^2+m^2x^4} = 0$ for all $m \in \mathbb{R}$. Note that in this case $\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{mx^3}{x^2+m^2x^2} = \frac{mx^3}{x^2(1+m^2)} = 0$ for all $m \in \mathbb{R}$.

Remark 0.13. The polar coordinates in $\mathbb{R}^2$ can also be used in the evaluation of certain limits of functions $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$. For example, taking $x = r\cos\theta, y = r\sin\theta$, where $r > 0$ and $\theta \in [0, 2\pi)$, we find that

$$\left|\frac{x^2}{x^2+y^2}\right| = |r|\cos^2\theta$$

$|s| = r \to 0$ as $r \to 0$ and hence we can conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = 0.$$

However, while using this method we should not assume θ to be a constant while taking limit as $r \to 0$.

Theorem 0.14. *Let x_0 be a limit point of $S \subseteq \mathbb{R}^m$ and let $f : S \to \mathbb{R}^k$. If $f(x) = (f_1(x_1), ..., f_k(x_k))$ for all $x \in S$, where $f_j : S \to \mathbb{R}$ for $j = 1, \ldots, k$, then $\lim_{x\to x_0} f(x)$ exists in $\mathbb{R}^k$ iff $\lim_{x\to x_0} f_j(x)$ exists in $\mathbb{R}$ for each $j \in \{1, ..., k\}$, and in that case*

$$\lim_{x\to x_0} f(x) = (\lim_{x\to x_0} f_1(x), ..., \lim_{x\to x_0} f_k(x))$$

.

Proof. Let us first assume that $\lim_{x\to x_0} f(x) = y = (y_1, ..., y_k) \in \mathbb{R}^k$ and let (x_n) be any sequence in $S \setminus \{x_0\}$ such that $x_n \to x_0$. Then $f(x_n) = (f_1(x_n), ..., f_k(x_n)) \to (y_1, ..., y_k)$ and hence $f_j(x_n) \to y_j$ for each $j \in \{1, ..., k\}$. Consequently $\lim_{x\to x_0} f_j(x) = y_j$ for each $j \in \{1, ..., k\}$.

Conversely, let $\lim_{x\to x_0} f_j(x) = y_j \in \mathbb{R}$ for each $j \in \{1, ..., k\}$ and let (x_n) be any sequence in $S \setminus \{x_0\}$ such that $x_n \to x_0$. Then $f_j(x_n) \to y_j$ for each $j \in \{1, ..., k\}$ and hence $f(x_n) = (f_1(x_n), ..., f_k(x_n)) \to (y_1, ..., y_k) \in \mathbb{R}^k$. Therefore $\lim_{x\to x_0} f(x) = (y_1, ..., y_k) \in \mathbb{R}^k$. □

Remark 0.15. The limit rules for combinations of functions can be given similar to those for continuity.

Relation between limit and continuity. Let $S \subseteq \mathbb{R}^m$ and $x_0 \in S$ If x_0 is also a limit point of S, then from the definitions of continuity and limit, it follows immediately that $f : S \to \mathbb{R}^k$ is continuous at x_0 iff $\lim_{x\to x_0} f(x) = f(x_0)$.

On the other hand, if x_0 is not a limit point of S, then there exists $\delta > 0$ such that $B_\delta(x_0) \cap S = \{x_0\}$ and so $||f(x) - f(x_0)|| = 0$ for all $x \in S$ satisfying $||x - x_0|| < \delta$. Consequently f is continuous at x_0.

Infinite limits: Let x_0 be a limit point of $S \subseteq \mathbb{R}^n$ and let $f : S \to \mathbb{R}$. Then we write $\lim_{x\to x_0} f(x) = \infty$ if for every $r > 0$, there exists $\delta > 0$ such that $f(x) > r$ for all $x \in S$ satisfying $\|x - x_0\| < \delta$.
It can be shown that $\lim_{x\to x_0} f(x) = \infty$ if and only if for every sequence (x_n) in $S \setminus \{x_0\}$ converging to x_0, $f(x_n) \to \infty$.
We can also define $\lim_{x\to x_0} f(x) = -\infty$ analogously and obtain its sequential criterion.

Example 0.16.

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = \infty \qquad \text{but} \qquad \lim_{(x,y)\to(0,0)} \frac{1}{x+y} \neq \infty.$$

0.1. **Differentiability of vector valued function of one real variable.** A function $F : S \subset \mathbb{R} \to \mathbb{R}^k$ is said to be differentiable at $t_0 \in S^0$ if $\lim_{t\to t_0} \frac{1}{t-t_0}(F(t) - F(t_0))$ exists (in $\mathbb{R}^k$) and in that case the derivative of F at t_0 is defined as $F'(t_0) = \lim_{t\to t_0} \frac{1}{t-t_0}(F(t) - F(t_0))$.
If $F(t) = (f_1(t), \dots, f_k(t))$ for all $t \in S$, then F is differentiable at t_0 ($t_0 \in S^0$) if and only if $f_j : S \to \mathbb{R}$ is differentiable at t_0 for each $j \in \{1, \dots, k\}$, and in such case $F'(t_0) = (f_1'(t_0), \dots, f_k'(t_0))$.
We say that $F : S \subset \mathbb{R} \to \mathbb{R}^k$ is differentiable (on S) if F is differentiable at every point of S.

Example 0.17. If $F(t) = (\cos t, \sin t, t)$ for all $t \in \mathbb{R}$, then $F : \mathbb{R} \to \mathbb{R}^3$ is differentiable (since each component function of F is differentiable) and $F'(t) = (-\sin t, \cos t, 1)$ for all $t \in \mathbb{R}$.

Remark 0.18. Let $F : S \subset \mathbb{R} \to \mathbb{R}^k$ and let $t_0 \in S \setminus \partial S$ be one point of an interval contained in S. Then the differentiability and the derivative of F at t_0 are defined as in the above definition by considering $t \to t_0^+$ or $t \to t_0^-$, whichever is applicable.

Differentiation of composite functions. Differentiable functions can be combined (in meaningful ways) to produce new differentiable functions. We illustrate this with the following results. Let $F, G : S \subset \mathbb{R} \to \mathbb{R}^k$ be differentiable at $t_0 \in S^0$. Then

(a) $F + G : S \to \mathbb{R}^k$ is differentiable at t_0 and $(F+G)'(t_0) = F'(t_0) + G'(t_0)$.
(b) If $F : G : S \to \mathbb{R}^k$ is differentiable at t_0 and $(F \cdot G)'(t_0) = F'(t_0) \cdot G(t_0) + F(t_0) \cdot G'(t_0)$.
(c) $\varphi F : S \to \mathbb{R}^k$ is differentiable at t_0 and $(\varphi F)'(t_0) = \varphi'(t_0)F(t_0) + \varphi(t_0)F'(t_0)$, where $\varphi : S \to \mathbb{R}$ is differentiable at t_0.

We prove (b). The other two can be similarly proved.

Proof of (b). Let $F(t) = (f_1(t), \dots, f_k(t))$ and $G(t) = (g_1(t), \dots, g_k(t))$ for all $t \in S$, where $f_j, g_j : S \to \mathbb{R}$ for each $j \in \{1, \dots, k\}$. Then $(F \cdot G)(t) = F(t) \cdot G(t) = \sum_{j=1}^k f_j(t)g_j(t)$ for all $t \in S$.
Since F and G are differentiable at t_0, f_j and g_j are differentiable at t_0 for each $j \in \{1, \dots, k\}$ and hence $F \cdot G$ is differentiable at t_0. Also,

$$(F \cdot G)'(t_0) = \sum_{j=1}^k f_j'(t_0)g_j(t_0) + \sum_{j=1}^k f_j(t_0)g_j'(t_0) = F'(t_0) \cdot G(t_0) + F(t_0) \cdot G'(t_0).$$

□

0.2. **Chain rule.**

Theorem 0.19. *Let $\varphi : D \subset \mathbb{R} \to \mathbb{R}$ and $F : S \subset \mathbb{R} \to \mathbb{R}^k$ be such that $\varphi(D) \subset S$. If φ is differentiable at $s_0 \in D^0$ and F is differentiable at $t_0 = \varphi(s_0) \in S^0$, then $F \circ \varphi : D \to \mathbb{R}^k$ is differentiable at s_0 and $(F \circ \varphi)'(s_0) = \varphi'(s_0)F'(\varphi(s_0))$.*

Proof. Let $F(t) = (f_1(t), \dots, f_k(t))$ for all $t \in S$, where $f_j : S \to \mathbb{R}$ for each $j \in \{1, \dots, k\}$. Since F is differentiable at t_0, f_j is differentiable at t_0 for each $j \in \{1, \dots, k\}$. Hence by the chain rule of calculus of one real variable, $f_j \circ \varphi : D \to \mathbb{R}$ is differentiable at s_0 and $(f_j \circ \varphi)'(s_0) = \varphi'(s_0)f_j'(\varphi(s_0))$. Thus $F \circ \varphi$ is differentiable at s_0 and

$$(F \circ \varphi)'(s_0) = (\varphi'(s_0)f_1'(\varphi(s_0)), \dots, \varphi'(s_0)f_k'(\varphi(s_0))) = \varphi'(s_0)F'(\varphi(s_0)).$$

for each $j \in \{1, \dots, k\}$. Since

$$(F \circ \varphi)(s) = F(\varphi(s)) = (f_1(\varphi(s)), \dots, f_k(\varphi(s))) = ((f_1 \circ \varphi)(s), \dots, (f_k \circ \varphi)(s))$$

for all $s \in D$, $F \circ \varphi$ is differentiable at s_0 and $(F \circ \varphi)'(s_0) = \big(f_1 \circ \varphi\big)'(s_0), \dots, \big(f_k \circ \varphi\big)'(s_0)$

$$= (\varphi'(s_0)f_1'(t_0), \dots, \varphi'(s_0)f_k'(t_0)) = \varphi'(s_0)(f_1'(t_0), \dots, f_k'(t_0)) = \varphi'(s_0)F'(t_0).$$

□

Example 0.20. Let $\varphi(s) = 2s^4 + 3s - 3$ for all $s \in \mathbb{R}$ and $F(t) = (2t^3, t^6 + 9, 5t^4 + 1)$ for all $t \in \mathbb{R}$. Then $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable at 1 and $F : \mathbb{R} \to \mathbb{R}^3$ is differentiable at $\varphi(1) = 2$. Hence by chain rule, $F \circ \varphi : \mathbb{R} \to \mathbb{R}^3$ is differentiable at 1 and

$$(F \circ \varphi)'(1) = \varphi'(1)F'(2) = 11(24, 192, 160) = (264, 2112, 1760).$$

However, since $(F \circ \varphi)(s) = (2(\varphi(s))^3, (\varphi(s))^6 + 9, 5(\varphi(s))^4 + 1)$ for all $s \in \mathbb{R}$, in this case without using the above chain rule also we can directly obtain that $F \circ \varphi$ is differentiable at 1 and

$$(F \circ \varphi)'(1) = (264, 2112, 1760).$$

Partial derivative. Let $f : S \subset \mathbb{R}^2 \to \mathbb{R}$ and let $(x_0, y_0) \in S^0$. The partial derivative of f with respect to x (the first variable) at (x_0, y_0) is defined as

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}\bigg|_{x_0, y_0} = f_x(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t},$$

provided this limit exists (in $\mathbb{R}$).

Similarly, the partial derivative of f with respect to y (the second variable) at (x_0, y_0) is defined as

$$\frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial y}\bigg|_{x_0, y_0} = f_y(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0, y_0 + t) - f(x_0, y_0)}{t},$$

provided this limit exists (in $\mathbb{R}$).

Thus if $f : \{x \in \mathbb{R} : (x, y_0) \in S\}, B = \{y \in \mathbb{R} : (x_0, y) \in S\}, \varphi(x) = f(x, y_0)$ for all $x \in A$ and $\psi(y) = f(x_0, y)$ for all $y \in B$, then $f_x(x_0, y_0) = \varphi'(x_0)$ and $f_y(x_0, y_0) = \psi'(y_0)$.

More generally, if $f : S \subset \mathbb{R}^m \to \mathbb{R}$, $x_0 \in S^0$ and $j \in \{1, \ldots, m\}$, then the partial derivative of f with respect to x_j at x_0 is

$$\frac{\partial f}{\partial x_j}(x_0) = \left.\frac{\partial f}{\partial x_j}\right|_{x_0} = f_{x_j}(x_0) = \lim_{t \to 0} \frac{f(x_0 + te_j) - f(x_0)}{t},$$

provided this limit exists (in $\mathbb{R}$).