MA15010H: Multi-variable Calculus

(Lecturenote 1: Limits and continuity)

July - November, 2025

The Space $\mathbb{R}^m$. For each $m \in \mathbb{N}$, let

$$\mathbb{R}^m = \{(x_1, \ldots, x_m) : x_j \in \mathbb{R} \text{ for } j = 1, \ldots, m\}.$$

If $m > 1$, an element of $\mathbb{R}^m$ is called a **vector** and is denoted by x, y, etc. If $m = 1$, we identify $\mathbb{R}^m$ with $\mathbb{R}$ and write an element as x, α, etc., which is then called a **scalar**.

Structures on $\mathbb{R}^m$.

(a) **Addition:** For $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, define

$$x + y = (x_1 + y_1, \ldots, x_m + y_m).$$

(b) **Multiplication by a scalar:** For $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$, define

$$\alpha x = (\alpha x_1, \ldots, \alpha x_m).$$

(c) **Dot product (scalar product):** For $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, define

$$x \cdot y = x_1 y_1 + \cdots + x_m y_m.$$

(d) **Norm:** For $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, define

$$\|x\| = \left(x_1^2 + \cdots + x_m^2\right)^{\frac{1}{2}} = \sqrt{x \cdot x}.$$

(e) **Distance:** For $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, define

$$\|x - y\| = \left(\sum_{j=1}^{m} (x_j - y_j)^2\right)^{\frac{1}{2}}.$$

With these structures, $\mathbb{R}^m$ is called **Euclidean m-space**.

Notations. We write $0 = (0, \ldots, 0)$ and, for each $j \in \{1, \ldots, m\}$, $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the j-th position. If $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, then $x = x_1 e_1 + \cdots + x_m e_m$.

Properties. For all $x, y, z \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$:

(a) $\|x\| = 0 \iff x = 0$

(b) $\|\alpha x\| = |\alpha| \|x\|$

(c) $|x_j| \leq \|x\|$ for all $j \in \{1, \ldots, m\}$, where $x = (x_1, \ldots, x_m)$

(d) $x \cdot (y + z) = x \cdot y + x \cdot z$

(e) $\|x + y\|^2 = \|x\|^2 + 2x \cdot y + \|y\|^2$ and $\|x - y\|^2 = \|x\|^2 - 2x \cdot y + \|y\|^2$

Cauchy-Schwarz Inequality.

$$|x \cdot y| \leq \|x\|\|y\| \qquad \text{for all } x, y \in \mathbb{R}^m.$$

Proof: If $y = 0$, then $|x \cdot y| = 0 = \|x\|\|y\|$. Now assume $y \neq 0$. Then $\|y\| \neq 0$. Let $\alpha = \frac{x \cdot y}{\|y\|^2}$.

$$\begin{aligned} 0 &\leq \|x - \alpha y\|^2 \\ &= \|x\|^2 - 2\alpha x \cdot y + \alpha^2 \|y\|^2 \\ &= \|x\|^2 - 2\frac{(x \cdot y)^2}{\|y\|^2} + \frac{(x \cdot y)^2}{\|y\|^2} \\ &= \|x\|^2 - \frac{(x \cdot y)^2}{\|y\|^2} \end{aligned}$$

So $(x \cdot y)^2 \leq \|x\|^2\|y\|^2$, hence $|x \cdot y| \leq \|x\|\|y\|$.

Remark. If $x, y \in \mathbb{R}^m$, then $|x \cdot y| = \|x\|\|y\|$ iff $y = 0$ or $x = \alpha y$ for some $\alpha \in \mathbb{R}$.
Proof: If $y = 0$, then $|x \cdot y| = 0 = \|x\|\|y\|$. If $x = \alpha y$ for $\alpha \in \mathbb{R}$, then $|x \cdot y| = |\alpha|\|y\|^2$ and $\|x\|\|y\| = |\alpha|\|y\|^2$, so $|x \cdot y| = \|x\|\|y\|$.

Conversely, if $|x \cdot y| = \|x\|\|y\|$ and $y \neq 0$, then for $\alpha := \frac{x \cdot y}{\|y\|^2} \in \mathbb{R}$,

$$\|x - \alpha y\|^2 = \|x\|^2 - \frac{|x \cdot y|^2}{\|y\|^2} = 0$$

implying $x = \alpha y$.

Angle Between Vectors. If $x, y \in \mathbb{R}^m \setminus \{0\}$, by Cauchy-Schwarz,

$$-1 \leq \frac{x \cdot y}{\|x\|\|y\|} \leq 1$$

so there exists a unique $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}.$$

θ is called the angle between x and y.

Vectors x, y are **orthogonal** if $x \cdot y = 0$, and are **parallel** if there exists $\alpha \in \mathbb{R}$ such that $x = \alpha y$.

Triangle Inequality.

$$\|x + y\| \leq \|x\| + \|y\| \qquad \text{for all } x, y \in \mathbb{R}^m.$$

Proof:

$$\|x + y\|^2 = \|x\|^2 + 2x \cdot y + \|y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

so $\|x + y\| \leq \|x\| + \|y\|$.

Open Ball and Closed Ball. Let $x_0 \in \mathbb{R}^m$, $r > 0$. Define

$$B_r(x_0) = \{x \in \mathbb{R}^m : \|x - x_0\| < r\}, \qquad B_r[x_0] = \{x \in \mathbb{R}^m : \|x - x_0\| \leq r\}$$

as the open ball and closed ball in $\mathbb{R}^m$ with center x_0 and radius r.

Examples:

- In $\mathbb{R}^2$: $B_1((0,0)) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
- In $\mathbb{R}^3$: $B_1((0,0,0)) = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$

Remark: If $x, y \in \mathbb{R}^m, x \neq y$ and $r = \frac{1}{2}\|x - y\| > 0$, then $B_r(x) \cap B_r(y) = \emptyset$.

Suppose there existed $z \in B_r(x) \cap B_r(y)$. Then $\|z - x\| < r$, $\|z - y\| < r$, so

$$2r = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = \|z - x\| + \|z - y\| < r + r = 2r$$

which is a contradiction.

Sequences in $\mathbb{R}^m$. A sequence in $\mathbb{R}^m$ is a function $f : \mathbb{N} \to \mathbb{R}^m$, denoted (x_n), with $x_n := f(n)$.

A sequence (x_n) in $\mathbb{R}^m$ is called **convergent** if there exists $x \in \mathbb{R}^m$ such that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ satisfying $\|x_n - x\| < \varepsilon$ for all $n \geq n_0$ (i.e., $x_n \in B_\varepsilon(x)$ for those n).

The x in the above definition is unique: If $y \neq x$ also satisfies the definition, then $\varepsilon = \frac{1}{2}\|x - y\| > 0$, so there exist n_1, n_2 such that $x_n \in B_\varepsilon(x)$ for $n \geq n_1$, $x_n \in B_\varepsilon(y)$ for $n \geq n_2$. For $n_0 = \max\{n_1, n_2\}$, $x_{n_0} \in B_\varepsilon(x) \cap B_\varepsilon(y)$, a contradiction. Thus the limit x is unique, denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Note: $x_n \to x$ in $\mathbb{R}^m$ if and only if $\|x_n - x\| \to 0$ in $\mathbb{R}$.

Theorem: Let (x_n) be a sequence in $\mathbb{R}^m$, $x_n = (x_1^{(n)}, \ldots, x_m^{(n)})$. Then (x_n) converges in $\mathbb{R}^m$ if and only if, for each $j \in \{1, \ldots, m\}$, the sequence $\left(x_j^{(n)}\right)$ converges in $\mathbb{R}$; that is, if $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, then $x_n \to x$ in $\mathbb{R}^m$ iff $x_j^{(n)} \to x_j$ for each j.

Proof: Assume $(x_n) \to x$ in $\mathbb{R}^m$. Given $\varepsilon > 0$, $\|x_n - x\| < \varepsilon$ for $n \geq n_0$. For each j,

$$|x_j^{(n)} - x_j| \leq \|x_n - x\| < \varepsilon,$$

so $\left(x_j^{(n)}\right) \to x_j$ in $\mathbb{R}$ for each j.

Conversely, let $\left(x_j^{(n)}\right) \to x_j$ in $\mathbb{R}$ for each j. For any $\varepsilon > 0$, choose n_j so that for $n \geq n_j$,

$$|x_j^{(n)} - x_j| < \frac{\varepsilon}{\sqrt{m}}.$$

Let $n_0 = \max\{n_1, \ldots, n_m\}$, then for $n \geq n_0$,

$$\|x_n - x\| = \left(\sum_{j=1}^{m} (x_j^{(n)} - x_j)^2\right)^{1/2} < \left(m\left(\frac{\varepsilon^2}{m}\right)\right)^{1/2} = \varepsilon.$$

Examples:

(a) The sequence $\left((2n+3n)^{1/n}, \frac{1}{n}\cos n\right)$ in $\mathbb{R}^2$ converges to $(3,0)$ since $(2n+3n)^{1/n} \to 3$ and $\frac{1}{n}\cos n \to 0$ in $\mathbb{R}$.
(b) The sequence $(n, \frac{1}{n})$ in $\mathbb{R}^2$ is not convergent since (n) does not converge in $\mathbb{R}$.

Bounded Sets. A subset $S \subseteq \mathbb{R}^m$ is called **bounded** if there exists $r > 0$ so that $\|x\| \leq r$ for all $x \in S$ (i.e., $S \subseteq B_r[0]$).

Examples:

- $\{(x,y,z) \in \mathbb{R}^3 : |x| + 2|y| + 3z^2 < 1\}$ is bounded in $\mathbb{R}^3$.
- $\{(x,y) \in \mathbb{R}^2 : x + y \leq 1\}$ is unbounded in $\mathbb{R}^2$.

A sequence (x_n) in $\mathbb{R}^m$ is **bounded** if its set $\{x_n : n \in \mathbb{N}\}$ is bounded.

Theorem: Every convergent sequence in $\mathbb{R}^m$ is bounded.

Proof: Let (x_n) be a convergent sequence in $\mathbb{R}^m$, with $x_n = (x_1^{(n)}, \ldots, x_m^{(n)})$. For each j, $(x_j^{(n)})$ is convergent in $\mathbb{R}$, hence bounded. So, for each j, there is $r_j > 0$ so that $|x_j^{(n)}| \leq r_j$ for all n. Thus,

$$\|x_n\| = \left(\sum_{j=1}^{m} |x_j^{(n)}|^2\right)^{1/2} \leq \left(\sum_{j=1}^{m} r_j^2\right)^{1/2}$$

so (x_n) is bounded.

Remark: An unbounded sequence cannot converge in $\mathbb{R}^m$. The converse is false; for example, $(1,1), (0,0), (1,1), (0,0), \ldots$ is bounded but not convergent in $\mathbb{R}^2$.

Bolzano-Weierstrass Theorem in $\mathbb{R}^m$. Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence.

Proof: Will be shown for $m = 2$. General case is similar.

Let (x_n, y_n) be bounded in $\mathbb{R}^2$. There is $r > 0$ so that $\|(x_n, y_n)\| \leq r$ for all n. So $|x_n| \leq r$ and $|y_n| \leq r$, so (x_n) is bounded. By Bolzano-Weierstrass in $\mathbb{R}$, there exist $x \in \mathbb{R}$ and a subsequence $x_{n_k} \to x$. Then (y_{n_k}) is also bounded, so there is a further subsequence $y_{n_{k_l}} \to y$. Thus $(x_{n_{k_l}}, y_{n_{k_l}})$ is a subsequence converging to $(x,y) \in \mathbb{R}^2$.

Example: The sequence $(\sin n, \cos n)$ in $\mathbb{R}^2$ has no limit because $(\sin n)$ doesn't converge, but it has a convergent subsequence because $\|(\sin n, \cos n)\| = 1$ for all n.

Continuity. Let $\emptyset \neq S \subseteq \mathbb{R}^m$. A function $f : S \to \mathbb{R}^k$ is **continuous at** $x_0 \in S$ if for every $\varepsilon > 0$, there exists $\delta > 0$ so that $\|f(x) - f(x_0)\| < \varepsilon$ whenever $x \in S$ and $\|x - x_0\| < \delta$.

f is **continuous on** S iff it is continuous at all $x_0 \in S$.

Examples:

(1) If $y_0 \in \mathbb{R}^k$ and $f(x) = y_0$ for all $x \in \mathbb{R}^m$, then $f : \mathbb{R}^m \to \mathbb{R}^k$ is continuous.
(2) $f(x) = x_j$ for $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}$ is continuous.
(3) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Sequential Criterion of Continuity: A function $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ is continuous at $x_0 \in S$ iff for every sequence (x_n) in S converging to x_0, $(f(x_n))$ converges to $f(x_0)$ in $\mathbb{R}^k$.
Proof: Suppose f is continuous at x_0 and $(x_n) \to x_0$ in S. For $\varepsilon > 0$, there exists $\delta > 0$ so that $\|f(x) - f(x_0)\| < \varepsilon$ whenever $x \in S$, $\|x - x_0\| < \delta$. Since $x_n \to x_0$, for some n_0, $\|x_n - x_0\| < \delta$ for $n \geq n_0$, hence $\|f(x_n) - f(x_0)\| < \varepsilon$. Thus $f(x_n) \to f(x_0)$.

If $f(x_n) \to f(x_0)$ for all sequences $(x_n) \to x_0$, suppose f isn't continuous there. Then for some $\varepsilon > 0$, for all n there is $x_n \in S$ so that $\|x_n - x_0\| < 1/n$ and yet $\|f(x_n) - f(x_0)\| \geq \varepsilon$. But $\|x_n - x_0\| \to 0$, so $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$, a contradiction.

Examples:

(a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} 1, & x^2 + y^2 \leq 1 \\ 2, & x^2 + y^2 > 1 \end{cases}$ is continuous at (x,y) iff $x^2 + y^2 \neq 1$.

(b) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is not continuous at $(0,0)$.

(c) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} x^2 + y^2 & x, y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$ is continuous only at $(0,0)$.

Combination of Continuous Functions. Continuous functions may be combined in meaningful ways to produce new continuous functions.

(a) If $f, g : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ are continuous at $x_0 \in S$, then $f + g : S \to \mathbb{R}^k$ is continuous at x_0, where $(f+g)(x) = f(x) + g(x)$.
(b) If $f, g : S \subseteq \mathbb{R}^m \to \mathbb{R}$ are continuous at $x_0 \in S$, then $fg : S \to \mathbb{R}$ is continuous at x_0.
(c) If $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$, $g : S \to \mathbb{R}$ are continuous at $x_0 \in S$ and $g(x) \neq 0$ for all $x \in S$, then $\dfrac{f}{g} : S \to \mathbb{R}^k$ is continuous at x_0.
(d) If $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ and $g : T \subseteq \mathbb{R}^k \to \mathbb{R}^p$ with $f(S) \subseteq T$, f is continuous at $x_0 \in S$, and g is continuous at $f(x_0)$, then $g \circ f : S \to \mathbb{R}^p$ is continuous at x_0.

Proof of (2): Given $(x_n) \to x_0$, since f and g are continuous, $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$. Then $(fg)(x_n) = f(x_n)g(x_n) \to f(x_0)g(x_0) = (fg)(x_0)$. So, fg is continuous at x_0.

Proof of (4): Given $(x_n) \to x_0$, $f(x_n) \to f(x_0)$ and g is continuous at $f(x_0)$. So $g(f(x_n)) \to g(f(x_0))$, thus $(g \circ f)(x_n) \to (g \circ f)(x_0)$.

Examples:

(a) Let $p : \mathbb{R}^m \to \mathbb{R}$ be a polynomial,

$$p(x_1, \ldots, x_m) = \sum_{j_1=0}^{k_1} \cdots \sum_{j_m=0}^{k_m} a_{j_1,\ldots,j_m} x_1^{j_1} \cdots x_m^{j_m},$$

where $a_{j_1,\ldots,j_m} \in \mathbb{R}$, $k_1, \ldots, k_m$ are non-negative integers. Then p is continuous.

(b) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{x+y} & x + y \neq 0 \\ 0 & x + y = 0 \end{cases}$ is continuous at (x,y) iff $x + y \neq 0$.

(c) If $f(x, y) = e^{\sin(x^2+y^2)}$ for all $(x, y) \in \mathbb{R}^2$, then f is continuous.

Interior Point. If $S \subseteq \mathbb{R}^m$, $x_0 \in S$ is an **interior point** if there exists $r > 0$ with $B_r(x_0) \subseteq S$. Let S^0 denote the set of all interior points of S.

E.g., if $S = \{(x, y) \in \mathbb{R}^2 : x + y \leq 0\} \subseteq \mathbb{R}^2$, then $(-1, 0) \in S^0$ but $(0, 0) \notin S^0$.

Open and Closed Sets. $S \subseteq \mathbb{R}^m$ is **open** in $\mathbb{R}^m$ if every point of S is interior.

$S \subseteq \mathbb{R}^m$ is **closed** in $\mathbb{R}^m$ if $\mathbb{R}^m \setminus S$ is open.

Example: $S = \{(x, y) \in \mathbb{R}^2 : x + y < 0\}$ is open in $\mathbb{R}^2$, so $\mathbb{R}^2 \setminus S = \{(x, y) : x + y \geq 0\}$ is closed in $\mathbb{R}^2$.