Indian Institute of Technology Guwahati

Mid-Semester Examination, Even Semester, 2014-15

Time: 2 Hours | *MA 102 Mathematics-II* | Marks: 30

1. For $x \in \mathbb{R}$, the greatest integer less than or equal to x is denoted by $[x]$. If $a \in (0,1)$, what is the limit of the sequence $X_n = \left(a^n, \frac{[na]}{n}\right)$ in $\mathbb{R}^2$? [1]

2. Write the set of interior points of the set $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ in $\mathbb{R}^2$. [1]

3. List out the points of (i) local maxima (ii) local minima and (iii) saddle points of the function $f(x,y) = x - y$ defined on the domain $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. [1]

4. Let S be the unit sphere $x^2 + y^2 + z^2 = 1$ and $\hat{n}$ be the unit outward normal to S. If F is the vector field given by $F(x,y,z) = xy\hat{i} + yz\hat{j} + zx\hat{k}$, then the value of $\iint_S \text{curl}F \cdot \hat{n} d\sigma$ is [1]

5. Let $D = [0,1] \times [0,1]$ and consider the function $f : D \longrightarrow \mathbb{R}$ given by [1]

$$f(x,y) = \begin{cases} 1 & \text{if } y = \frac{1}{3} \text{ and } x \text{ is rational,} \\ -1 & \text{if } y = \frac{1}{3} \text{ and } x \text{ is irrational,} \\ 0 & \text{otherwise.} \end{cases}$$

Does the double integral $\iint_D f(x,y)dxdy$ exist? Justify.

6. Let $D = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$. Consider the function $f : D \longrightarrow \mathbb{R}$ given by [2]

$$f(x,y) = \begin{cases} x & \text{if } |y| > |x|, \\ -x & \text{otherwise.} \end{cases}$$

Find the set of all points in D where f is discontinuous.

7. Using the method of Lagrange multipliers, find all the points where the function [2]

$$f(x,y) = x^2 + 2y^2 - 4y$$

has (absolute) maximum or (absolute) minimum subject to the constraint $x^2 + y^2 = 9$.

8. Let D be the region bounded by the lines joining the points $(0,0)$, $(1,0)$ and $(1,2)$ in $\mathbb{R}^2$. Find the area of the surface given by the graph of the function $f(x,y) = x^2 + 2y$ over the domain D. [2]

9. Prove or disprove: If directional derivatives of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ exist at a point along all directions, then the function must be continuous at that point. [2]

10. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a differentiable function. A vector $\alpha \in \mathbb{R}^2$ is said to be a *derivative* of f at $X_0 \in \mathbb{R}^2$, if $e(H) = \frac{f(X_0+H) - f(X_0) - \alpha \cdot H}{\|H\|} \to 0$ as $\|H\| \to 0$. Show that derivative of f at X_0 is unique. [2]

11. Let S be the boundary of the region in $\mathbb{R}^3$ bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$. Let $\hat{n}$ be the unit outward normal to the surface S. Evaluate the integral $\iint_S (y\hat{i} + x\hat{j} + z^2\hat{k}) \cdot \hat{n} d\sigma$. [3]

12. If C is any simple closed and smooth curve in $\mathbb{R}^2$ which is not passing through the point $(1, 0)$, then evaluate the integral $\oint\limits_C \frac{-ydx + (x-1)dy}{(x-1)^2 + y^2}$. [3]

13. Using second derivative test, classify the critical points of the function [3]

$$f(x, y) = \frac{x^4}{4} - x^3 + x^2 - y^2 + 1.$$

14. Evaluate the integral $\iint\limits_D e^{(x-2y)} dxdy$ over the domain D bounded by the lines $x - 2y = 0$, $2x - y = 0$ and $x + y = 1$ in $\mathbb{R}^2$. [3]

15. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function. If there exists $\alpha > 1$ such that [3]

$$f(rx, ry) = r^\alpha f(x, y),$$

for all $r > 0$ and $x, y \in \mathbb{R}$, then show that f is differentiable at $(0, 0)$.

$* * *$ End $* * *$