Metric Spaces:

Let X be a non-empty set. A map
$d: X\times X \longrightarrow R_+ = [0,\infty)$ s.t.
(i) $d(x,y)=0$ if and only if $x=y$.
(ii) $d(x,y) = d(y,x)$ (Symmetry)
(iii) $d(x,z) \le d(x,y) + d(y,z)$
(triangle inequality)
is called a metric on X. The pair (X,d) is called metric space.

Ex. If $X=R^n$, then for $x,y\in R^n$,
(i) $d_1(x,y) = \sum_{j=1}^{n} |x_j - y_j|$;
(ii) $d_2(x,y) = \left(\sum_{j=1}^{n} |x_j - y_j|^2\right)^{1/2}$;
(iii) $d_\infty(x,y) = \sup_{1\le j\le n} |x_j - y_j|$;
define metrics on R^n.

Ex. If $X = C[0,1]$, the space of all continuous functions on $[0,1]$ to R (or $\mathbb{C}$). Then
$d_\infty(f,g) = \sup_{0\le t\le 1} |f(t) - g(t)|$ defines a metric on X.
(Hint: $|f(t) - h(t)| \le |f(t) - g(t)| + |g(t) - h(t)|$;

Ex. If $X \neq \phi$, then for $x, y \in X$,

$$d_0(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

defines a metric on X. This is called discrete metric on X. (X, d_0) is called discrete metric space. For $x, y, z \in X$,

(i) $x = y$ or $y = z$ or $z = x$

(ii) all of x, y, z are distinct.

In either case $d_0(x,z) \leq d_0(x,y) + d_0(y,z)$ holds. Thus, (X, d_0) is a metric space.

Ex. Let (X, d) be a metric space, then $\left(X, \frac{d}{1+d}\right)$ is also a metric space.

For this, consider $f(t) = \frac{t}{1+t}$, $t \in [0, \infty)$.

Then $f'(t) = \frac{1}{(1+t)^2} > 0$. Hence f is an strictly increasing function & $f(0) = 0$.

Thus $\frac{t+s}{1+t+s} < \frac{t}{1+t} + \frac{s}{1+s}$

put $t = d(x,y)$, $s = d(y,z)$. Then

$t+s \geq d(x,z)$ & $f \nearrow \Rightarrow$

$f \circ d(x,z) \leq f(t+s) < \frac{t}{1+t} + \frac{s}{1+s} = f \circ d(x,y) + f \circ d(y,z)$.

Ex. Let (X, d) be a metric space &

$f : [0,\infty) \longrightarrow [0,\infty)$ s.t. $f(t) = 0$ iff $t=0$.

and $f'(t) \geq 0$. Then $f \circ d$ is a metric on X.

Ex. Let H^∞ (Hilbert Cube) be the space of seq^{n}s $x = (x_n) = (x_1, x_2, \cdots)$ s.t. $|x_n| \leq 1$. Then $d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}$, defines a metric on H^∞.

(i) $d(x,y) \leq \sum_{n=1}^{\infty} \frac{2}{2^n} = 2 < \infty$.

(ii) $|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n|$

$$\Rightarrow \sum_{n=1}^{k} \frac{|x_n - z_n|}{2^n} \leq \sum_{n=1}^{k} \frac{|x_n - y_n|}{2^n} + \sum_{n=1}^{k} \frac{|x_n - y_n|}{2^n}$$

$$\leq d(x,y) + d(y,z) < \infty.$$

Since LHS is an $\uparrow$ seq^n which bounded above. $\Rightarrow$ LHS is convergent seq^n.

$$\lim_{k\to\infty} \sum_{n=1}^{k} \frac{|x_n - z_n|}{2^n} \leq d(x,y) + d(y,z)$$

i.e. $d(x,z) \leq d(x,y) + d(y,z)$.

Ex. Show that $d(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right|$ defines a metric on $(0,\infty)$.

Norm linear spaces:

Let X be a vector space over the field $\mathbb{R}$ or ($\mathbb{C}$). A map $\|\cdot\|: X \longrightarrow [0,\infty)$ is called norm

if (i) $\|x\| = 0$ iff $x = 0$.

(ii) $\|\alpha x\| = |\alpha|\,\|x\|$, $\forall\, x \in X$, $\forall\, \alpha \in \mathbb{R}$ (or $\mathbb{C}$).

(iii) $\|x+y\| \le \|x\| + \|y\|$, $\forall\, x, y \in X$.

If we write, $d(x,y) = \|x-y\|$. Then d is a metric on the vector space X. But all metric on a vector space can not be obtained by norm.

Ex. Let $X = \mathbb{R}$ be a vector space. Then the discrete metric cannot be induced by any norm on X. For this, if so then $d_0(x,y) = \|x-y\|$. Then for $x \neq 0$,

$\|x\| = d(x,0) = 1 = d_0(\alpha x, 0) = \|\alpha x\| = |\alpha|\,\|x\|$, $\forall \alpha$.

However, if d is a metric on a vector space X s.t. $d(x,y) = d(x-y,0)$ & $d(\alpha x, \alpha y) = |\alpha| d(x,y)$.

Then $d(x,0) = \|x\|$ defines a norm on X.

(i) ~~def~~ $\|x\| = 0$ iff $x = 0$.

(ii) $\|\alpha x\| = |\alpha|\,\|x\|$

(iii) $\|x+y\| = d(x+y, 0) = d(x, -y)$
$\le d(x,0) + d(-y,0)$.

Ex. Let l^1 denotes the space of all the sequences of real (or complex) s.t.

$$\sum_{n=1}^{\infty} |x_n| < \infty.$$

Then $\|\cdot x\|_1 = \sum_{n=1}^{\infty} |x_n|$, defines a norm on l^1. The pair $(l^1, \|\cdot\|_1)$ is a n.l.s. For simplicity, we write l^1 for $(l^1, \|\cdot\|_1)$.

(Hint: $\sum_{n=1}^{k} |x_n + y_n| \le \sum_{n=1}^{k} |x_n| + \sum_{n=1}^{k} |y_n|$

$\le \|x\|_1 + \|y\|_1$).

Ex. l^2 denotes the space of all seq^n on $\mathbb{R}$ (or $\mathbb{C}$) s.t. $\sum_{n=1}^{\infty} |x_n|^2 < \infty$.

$\|x\|_2 := \left(\sum |x_n|^2\right)^{1/2} < \infty$ defines a norm on l^2. (Hint: $\sum_{n=1}^{k} |x_n + y_n|^2 \le \left(\left(\sum_{n=1}^{k} |x_n|\right)^{1/2} + \left(\sum_{n=1}^{k} |y_n|\right)^{1/2}\right)^2$)

Ex. l^{∞} = space of all seq^{ns} on $\mathbb{R}$ (or $\mathbb{C}$) s.t. $\sup_{n \in \mathbb{N}} |x_n| < \infty$. The function

$\|x\|_{\infty} = \sup_{n \in \mathbb{N}} |x_n|$, defines a norm on l^{∞}.

Ex. C_0 = space of all seq^{ns} on $\mathbb{R}$ (or $\mathbb{C}$) s.t. $\lim_{n \to \infty} x_n = 0$. Then (x_n) must be bounded. Hence $\|x\|_{C_0} = \sup_{n \in \mathbb{N}} |x_n| < \infty$.

Thus, $(C_0, \|\cdot\|_{\infty})$ is a n.l.s.

Ex. If $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ (or $\mathbb{C}^n$), then

$$\|x\|_\infty \le \|x\|_1 \le \sqrt{n}\,\|x\|_2 \le n\|x\|_\infty .$$

Ex. If $x=(x_n)\in \ell^1$ then $x\in\ell^\infty$ (~~$\|x\|_\infty<\infty$~~).

$$\sum_{n=1}^{\infty}|x_n|^2 < \sum_{n=1}^{\infty}\|x\|_\infty |x_n|$$

$$\Rightarrow \quad \|x\|_2 \le \|x\|_\infty \, \|x\|_1 .$$

Thus, $\ell^1 \subsetneq \ell^2 \subsetneq c_0 \subsetneq \ell^\infty$.

Ex. If $1<p<\infty$, then for $\sum_{p=1}^{\infty}|a_n|^p<\infty$, we can defines a norm $\|\cdot\|_p$ on ℓ^p via

$$\|x\|_p = \left(\sum_{n=1}^{\infty}|x_n|^p\right)^{1/p}.$$

To prove this, we need some inequalities.

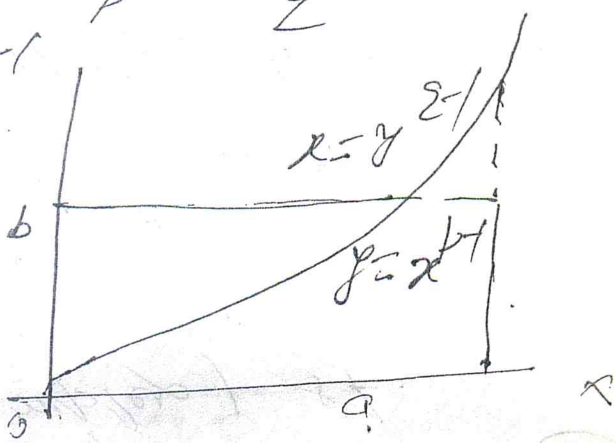
Young's inequality:

Let $1<p<\infty$ and $a,b\ge 0$. Then for $\frac{1}{p}+\frac{1}{q}=1$, $ab\le \frac{a^p}{p}+\frac{b^q}{q}$ — (*)

Let $y=x^{p-1}$, then $x=y^{q-1}$.

$$ab\le \int_0^a x^{p-1}\,dx+\int_0^b y^{q-1}\,dy$$

$$\frac{a^p}{p}+\frac{b^q}{q}.$$



Note that equality in (*) holds iff $a^p = b^q$ (or $a = b^{q-1}$) (78)

Consider $ab = \frac{a^p}{p} + \frac{b^q}{q}$, $\frac{1}{p}+\frac{1}{q}=1$

replace $a \to a^{\frac{1}{p}}$, $b \to b^{\frac{1}{q}}$, $\frac{1}{p}=\alpha$

$$a^\alpha b^{1-\alpha} = \alpha a + (1-\alpha) b$$

$$\text{or } \left(\frac{a}{b}\right)^\alpha = \alpha\left(\frac{a}{b}\right) + 1-\alpha$$

put $\frac{a}{b} = t$, then $t^\alpha - \alpha t - (1-\alpha) = 0$

$f(t) = t^\alpha - \alpha t - (1-\alpha)$, $t \in [0, \infty)$.

$f(1) = 0$, $f'(t) = \alpha(t^{\alpha-1} - 1) = 0$ iff $t=1$.

$\Rightarrow f$ attain its maximum at $t=1$. &

$f(t) \leq f(1) = 0$.

$\Rightarrow$ $f(t)=0$ iff $t=1$.

Holder's inequality:

let $1 < p < \infty$ & $\frac{1}{p}+\frac{1}{q}=1$. Then for $x \in \ell^p$, $y \in \ell^q$ implies $x \cdot y \in \ell^1$ and

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q.$$

Proof: let $1<p<\infty$. Then $1<q<\infty$. By young's inequality with $a_j = \frac{|x_j|^p}{\|x\|_p^p}$ &

$b_j = \frac{|y_j|^q}{\|y\|_q^q}$, we get

$$\sum_{j=1}^{n} \frac{|x_j y_j|}{\|x\|_p \|y\|_q} \leq \sum_{j=1}^{n} \frac{|x_j|^p}{p\|x\|_p^p} + \sum_{j=1}^{n} \frac{|y_j|^q}{q\|y\|_q^q} < \frac{1}{p} + \frac{1}{q} = 1.$$

ie $\displaystyle\sum_{j=1}^{n} |x_j y_j| < \|x\|_p \|y\|_q$, $\forall\ n \geq 1$.

$$\Rightarrow \sum_{j=1}^{\infty} |x_j y_j| \leq \|x\|_p \|y\|_q.$$

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q.$$

Notice that if $p = 1$, $q = \infty$. ($p = \infty \Rightarrow q = 1$)

$$|x_j y_j| < |x_j| \, \|y\|_\infty$$

$$\Rightarrow \|x_j y\|_1 \leq \|x\|_1 \|y\|_\infty.$$

Note that $\|x \cdot y\|_1 = \|x\|_p \|y\|_q$ iff $\dfrac{|x_j|^p}{\|x\|_p^p} = \dfrac{|y_j|^q}{\|y\|_q^q}$

Minkowski's inequality:

If $1 \leq p \leq \infty$, then for $x, y \in l^p$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p. \quad \text{--- } (*)$$

Proof: Let $1 < p < \infty$. Then

$$\|x + y\|_p = \left(\sum |x_j + y_j|^p\right)^{1/p}$$

$$\leq \left(\sum (|x_j| + |y_j|)^p\right)^{1/p}. \quad \text{--- } (1)$$

$$\therefore (|x_j| + |y_j|)^p = (|x_j| + |y_j|)^{p-1}|x_j| + (|x_j| + |y_j|)^{p-1}|y_j|.$$

Then $\displaystyle\sum (|x_j| + |y_j|)^{p-1}|x_j| \leq \left(\sum (|x_j| + |y_j|)^{(p-1)q}\right)^{1/q} \left(\sum |x_j|^p\right)^{1/p}$

$$\sum (|x_j|+|y_j|)^p \leq \left(\sum \left((|x_j|+|y_j|)^p\right)\right)^{1/q} (\|x\|_p + \|y\|_p)$$

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$$\Rightarrow \left(\sum (|x_j|+|y_j|)^p\right)^{1-\frac{1}{q}} \leq \|x\|_p + \|y\|_p.$$

From (1), $\|x+y\|_p \leq \left(\sum \left(|x_j|+|y_j|\right)^p\right)^{1/p} \leq \|x\|_p + \|y\|_p.$

Note that as similar to above cases, it can be shown that equality in (*) holds iff $x = \frac{\|x\|_p}{\|y\|_p} y$.

Now, if $x, y \in l^p$, then $x+y \in l^p$.

Because $a, b > 0$, $(a+b)^p \leq \{2 \max\{a,b\}\}^p$

i.e. $(a+b)^p \leq 2^p (a^p + b^p)$.

$$\sum |x_j+y_j|^p \leq 2^p \left(\sum |x_j|^p + \sum |y_j|^p\right) < \infty.$$

Thus, l^p is closed under $\|\cdot\|_p$. Hence $(l^p, \|\cdot\|_p)$ is a n.l.s.

<u>Result</u>: If $f, g \in R[a,b]$, then

for $\|f\|_p = \left(\int |f|^p\right)^{1/p}$, we get

(i) $\|fg\|_1 \leq \|f\|_p \|g\|_q$, $\frac{1}{p}+\frac{1}{q}=1$.

(ii) $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, ~~$1 \leq p \leq \infty$~~ $1 \leq p < \infty$.

For $p=\infty$, $\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|$, where (81)

$f\in R[a,b]$. Then $(R[a,b], \|\cdot\|_\infty)$

is a n.l.s.

Def^n: (i) $B_r(x_0) = \{y\in X : d(x_0,y) < r\}$ is

called open ball.

(ii) $\bar{B}_r(x_0) = \{y\in X : d(x_0,y) \le r\}$ is

called closed ball.

Def^n: A sequence $(x_n)\in (X,d)$ is

said to be convergent if $\forall \epsilon>0$,

$\exists N\in\mathbb{N}$ ~~st~~ and $x_0\in X$ st

$n\ge N \Rightarrow d(x_n, x_0) < \epsilon$.

$\Leftrightarrow x_n \in B_\epsilon(x_0)$, $\forall n\ge N$.

Def^n: A sequence $(x_n)\in (X,d)$ is said

to be Cauchy sequence if $\forall \epsilon>0$,

$\exists N\in\mathbb{N}$ st

$n,m \ge N \Rightarrow d(x_n, x_m) < \epsilon$.

Ex. Let $X=(0,1)$ & $d(x,y) = |x-y|$.

Then $\{\frac{1}{n}\}$ is a Cauchy sequence

because $|x_n - x_m| = |\frac{1}{n} - \frac{1}{m}| \to 0$

as $n,m\to\infty$.

But $\lim x_n = 0 \notin X$. Hence not conv.

However, every conv. seq^n is a C.S.

Def^n: A set $A \subseteq (X, d)$ is said to be bounded if $\exists$ $x_0 \in X$ & $M > 0$ s.t.

$$d(a, x_0) \leq M, \ \forall a \in A$$

$$\Rightarrow a \in \bar{B}_M(x_0), \ \forall a \in A.$$

i.e. A is bounded iff A is contained in a ball.

Result: Every Cauchy seq^n is bdd.

Pf: Since $(x_n) \subset (X, d)$ is C.S., for $\epsilon = 1$, $\exists$ $N \in \mathbb{N}$ s.t.

$$d(x_n, x_m) < 1, \ \forall m, n \geq N.$$

$$\& \ d(x_n, x_N) < 1, \ \forall n \geq N$$

Let $M = \max\{1, d(x_i, x_N) : i = 1, 2, \ldots N-1\}$

Then $d(x_n, x_N) \leq M, \ \forall n \geq \cancel{N}$

$$\Rightarrow x_n \in \bar{B}_M(x_n).$$

But converse need not be true.

For $(\mathbb{R}, U)$, U - usual metric

$x_n = \{-1, 1, -1, \ldots\}$ is bdd but not C.S.

Result: Let (x_n) be a C.C. in (X, d). If (x_{n_k}) is a subsequence which converges to x. Then $x_n \to x$. 83

Pf: For $\epsilon > 0$, $\exists\, N_1 \in \mathbb{N}$ s.t.

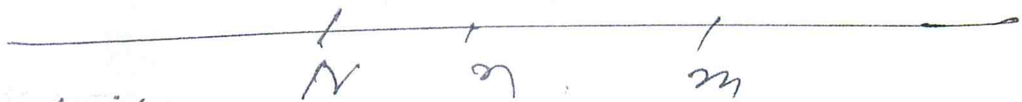
$$d(x_n, x_m) < \epsilon/2, \quad \forall\, n, m \geq N.$$

Also for the same $\epsilon > 0$, $\exists\, N_2 \in \mathbb{N}$ s.t.

$$d(x_{n_k}, x) < \epsilon/2, \quad \forall\, n_k \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then

$$\left.\begin{aligned} d(x_n, x_m) &< \epsilon/2 \\ d(x_{n_k}, x) &< \epsilon/2 \end{aligned}\right\} \quad \forall\, n, m, n_k \geq N_2$$



$$d(x_{n_k}, x_m) < \epsilon/2, \quad \forall\, n_k, m \geq N_2$$

Thus,

$$d(x, x_m) \leq d(x, x_{n_k}) + d(x_{n_k}, x_m) < \epsilon$$

$$\forall\, m \geq N.$$

$\Rightarrow \quad x_m \to x.$

Remark: If $X = (0,1)$ & $d(x,y) = |x-y|$. Then $x_n = \frac{1}{n}$ is C.C, but it has no conv. subsequence.

Defn: A set $O \subset (X,d)$ is said to be open if $\forall\, x \in O$, $\exists\, r>0$ st $B_r(x) \subset O$.

Result: If $\{O_i : i \in I\}$, I is any index set.

Then (i) $\bigcup_{i\in I} O_i$ is open (arbitrary union of open sets is open).

(ii) $\bigcap_{i=1}^{n} O_i$ is open (finite intersection of open sets is open).

Rk. Arbitrary ~~union of~~ intersection of open sets need not be open.

Ex. $X = \mathbb{R}$, $d(x,y) = |x-y|$.

$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ is not open.

Ex. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be cont. Then

$A = \{x \in \mathbb{R} : f(x) > 0\}$ is open.

Let $x \in A \Rightarrow f(x) > 0$. for $\epsilon = f(x) > 0$, $\exists\, \delta > 0$ st $\forall\, y \in (-\delta, \delta) + x = (x-\delta, x+\delta)$,

$$|f(y) - f(x)| < f(x)$$

$$0 < f(y) < 2f(x), \ \forall\, y \in (x-\delta, x+\delta)$$

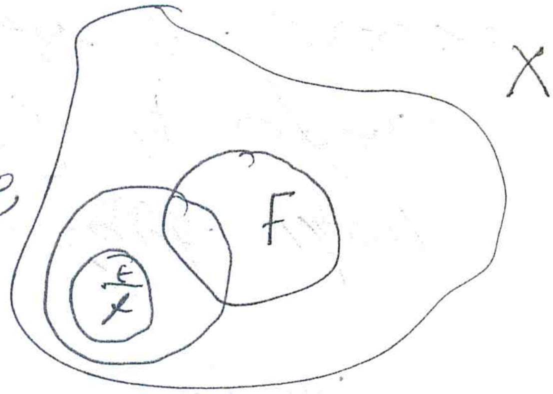
$$\Rightarrow (x-\delta, x+\delta) \subset A \Rightarrow A \text{ is open.}$$

Closed Sets in (X,d):

Def^n: A set $F \subset (X,d)$ is said to be closed if F^c is open.

i.e. $\forall\, x \in F^c = X \setminus F$, $\exists\, \epsilon > 0$ s.t.

$$B_\epsilon(x) \subseteq F^c.$$



On the other hand, if $\forall \epsilon > 0$,

$$B_\epsilon(x) \cap F = \emptyset \Rightarrow x \in F.$$

Theorem: Let (X,d) be a metric space. Then TFAE

(i) F is closed set (F^c - open)

(ii) $\forall \epsilon > 0$, $B_\epsilon(x) \cap F \neq \emptyset \Rightarrow x \in F$.

(iii) $\forall$ seq^n $(x_n) \subset F$ st $x_n \to x \Rightarrow x \in F$.

Proof: (i) $\Rightarrow$ (ii). Claim $B_\epsilon(x) \cap F \neq \emptyset$, $\forall \epsilon > 0$ $\Rightarrow x \in F$. If $x \notin F$, then $x \in F^c$. But F^c is open, $\exists\, \epsilon_0 > 0$ st

$$B_{\epsilon_0}(x) \subset F^c \Rightarrow B_{\epsilon_0}(x) \cap F = \emptyset$$

$\times$.

(ii) $\Rightarrow$ (iii) ~~Sim:~~ Let $(x_n) \subset F$ & $x_n \to x$.

Then $\forall\, \epsilon > 0$, $x_n \in B_\epsilon(x)$, $\forall\, n \geq N$.

$$\Rightarrow x_n \in B_\epsilon(x) \cap F \neq \Phi, \ \forall\, \epsilon > 0.$$

By (ii), it follows that $x \in F$.

(iii) $\Rightarrow$ (i): ~~Eta~~ Claim F^c is open.

Let $x \in F^c \Rightarrow x \notin F$. By (iii), $\exists\, \epsilon_0 > 0$

s.t $B_{\epsilon_0}(x) \cap F = \Phi \Rightarrow B_{\epsilon_0}(x) \subset F^c$

Ex. Let $f : \mathbb{R} \xrightarrow{\text{cont}} \mathbb{R}$. Then $A = \{x : f(x) = 0\}$ is closed.

Since $x_n \in A$ & $x_n \to x$. ~~& f is cont~~

$$f(x_n) = 0, \ \forall\, n \geq 1 \Rightarrow \lim f(x_n) = 0$$

$$\Rightarrow f(x) = 0.$$

Interiors in (X, d): Let $A \in (X, d)$. Then

Interior (A) or $\operatorname{Int}(A)$ or A° is the largest open set contained in A.

i.e. $$A^\circ = \bigcup \{O \subset X, \ O \text{ is open } \& \ O \subset A\}$$

$$= \bigcup \{B_\epsilon(x) \subset A : \text{for } x \in A \text{ some } \epsilon > 0\}$$

Closure in (X,d): Let $A \subset (X,d)$.

The closure of A is $Cl(A)$ or $\bar{A}$ is the smallest closed set that contains A.

i.e. $\bar{A} = \bigcap \{ F \subset X : F \text{ is closed } \& F \supset A \}$.

Ex. $A = \{ (n, \frac{1}{n}) : n \in \mathbb{N} \}$. Then $\bar{A} = A$.

$A^{0} = \phi$. (why??).

Ex. let c_{00} = space of all seq^{ns} having finitely many non-zero terms

$= \{ x = (x_1, x_2, \ldots x_n, 0, 0, \ldots) : x_i \in \mathbb{R} \}$

$\|x\|_\infty = \max_{1 \le i \le n} |x_i| < \infty$.

$\Rightarrow c_{00} \subsetneq \ell^{\infty}$ (proper subspace).

Let $X^{n} = (1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, \ldots) \in c_{00}$.

Let $X = (1, \frac{1}{2}, \frac{1}{n}, \frac{1}{n+1}, \ldots) \in \ell^{\infty}$.

~~But~~ $\|X - X^{n}\|_\infty = \sup_{k \ge n} \frac{1}{k+1} = \frac{1}{n+1} \to 0$.

(88)

But $X \notin c_{00}$. Hence c_{00} is not closed in l^∞. In addition, c_{00} is not open in l^∞. For this, let $\epsilon > 0$ be arbitrary. Then $(\epsilon/2, \epsilon/2, \cdots \epsilon/2, \cdots) \in B_\epsilon(0)$.

But $(\epsilon/2, \epsilon/2, \cdots \epsilon/2, \cdots) \notin c_{00}$.

i.e. $B_\epsilon(0) \not\subseteq c_{00}$, for any $\epsilon > 0$.

Notice that $c_{00} \subsetneq l^p$, $1 \le p \le \infty$. But neither closed nor open in any l^p.

Consider $x_n = \left(\frac{\epsilon^p}{2^{n+1}}\right)^{\frac{1}{p}}$; $1 \le p < \infty$.

$x = (x_1, \cdots x_n, \cdots) \in l^p$ &

$x \in B_\epsilon(0)$, because $\|x\|_p = \frac{\epsilon}{2^p} < \epsilon$.

But $x \notin c_{00}$

$\Rightarrow c_{00}$ is not open in l^p.

Let $X^n = (x_1, \cdots x_n, 0, 0, \cdots) \in c_{00}$.

Then $X^n \to x$ in l^p, since

$$\|X^n - x\|_p^p = \sum_{k=n+1}^{\infty} \frac{\epsilon^p}{2^{k+1}} \to 0 \text{ as } n \to \infty.$$

But $x \notin c_{00}$.