Proof: $\eta(k) = \dfrac{g(y+k) - g(y) - g'(y)k}{\|k\|} \to 0$

as $\|k\| \to 0$.

Since, $y = f(x)$, f is cont at x, Set

$k = f(x+h) - f(x)$.

Then $\|h\| \to 0 \Rightarrow \|k\| \to 0$. Also,

$$\|k\| = \|f(x+h) - f(x)\|$$
$$= \|f'(x)h + \|h\| \epsilon(h)\| \quad (\because f \text{ is diff at } x)$$
$$\leq \|f'(x)\| \|h\| + \|h\| \|\epsilon(h)\|.$$

ie $\dfrac{1}{\|h\|} \leq \dfrac{1}{\|k\|}\{\|f'(x)\| + \|\epsilon(h)\|\}$.

Now,

~~$\mu(h) = gof(x+h) - fog(x) - f'$~~

$$\mu(h) = \frac{gof(x+h) - gof(h) - g'(f(x))f'(x)h}{\|h\|}$$
$$= \frac{g(y+k) - g(y) - g'(y)(k - \|h\|\epsilon(h))}{\|h\|}$$
$$= \frac{\|k\|\eta(h) - \|h\| g'(y)\epsilon(h)}{\|h\|}$$

$\|\mu(h)\| \leq \|\eta(h)\| \cdot \{\|f'(x)\| + \|\epsilon(h)\|\} + \|g'(y)\| \|\epsilon(h)\|$

$\to 0$ as $\|h\| \to 0$. Hence

$(gof)'(x)$ exists & $= g'(f(x))f'(x)$.

Ex. Let $f : \mathbb{R} \to \mathbb{R}$ be diff & $F : \mathbb{R}^n \to \mathbb{R}$ be defined by $F(x) = f(\|x\|^2)$. Then F is differentiable & $F'(x) = 2f'(\|x\|^2)x$.

Soln. Let $g(x) = \|x\|^2 = x_1^2 + \cdots + x_n^2$

$$g'(x) = (2x_1 \; 2x_2 \; \cdots \; 2x_n).$$

$F(x) = (f \circ g)(x)$. By chain rule, F is diff & $F'(x) = f'(g(x))\, g'(x)$

i.e. $F'(x) = 2f'(\|x\|^2)\, x$.

Ex. Let $F(x) = f(\|x\|^{2k})$. Show that

$$F'(x) = 2k\,(\|x\|)^{2(k-1)} f'(\|x\|^{2k})\, x.$$

__Euler formula__: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be diff. & $f(rx) = r^{\alpha} f(x)$, $\forall r > 0$. and some $\alpha \in \mathbb{R}$. Then $f'(x)x = \alpha f(x)$.

__Proof__: Since $f(rx) = r^{\alpha} f(x)$, $\forall r > 0$. differentiable both sides w.r.t r,

$$f'(rx) \frac{d}{dr}(rx) = \alpha r^{\alpha-1} f(x)$$

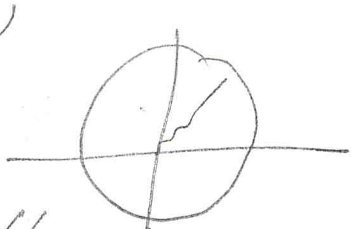
$$f'(rx)\,x = \alpha\, r^{\alpha-1} f(x),$$

put $r=1 \Rightarrow f'(x)\,x = \alpha f(x)$.

For $n=2$, $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \alpha f(x,y).$$

Ex. If $\alpha>0$, f is cont at (0). If $\alpha>1$, f is diff at 0.

(i) $\alpha>0$, $f(0+h)-f(0) = f(h)$



$h = \|h\|\, v$, $\|v\|=1$.

$$\|f(0+h)-f(0)\| = \|h\|^{\alpha}\, \|f(v)\| \to 0.$$
as $\|h\|\to 0$.

(ii) $\alpha>1$, $$\frac{\partial f}{\partial x_j}(0) = \lim_{h\to 0} \frac{f(0+h_j e_j)-f(0)}{h_j}$$

$$= \lim_{h\to 0} \frac{|h_j|^{\alpha} f(e_j)}{h_j} \to 0 \text{ as } h_j \to 0$$
$(\because \alpha>1)$.

$\Rightarrow J_f(0) = 0$. (m×n matrix)

$$e(h) = \frac{f(0+h)-f(0)-J_f(0)\,h}{\|h\|}$$

$$= \frac{\|h\|^{\alpha} f(v)}{\|h\|},\quad \|v\|=1,\ h=\|h\|\,v.$$

$\to 0$ as $\|h\|\to 0$.

Mixed derivatives:

Let $D \subset \mathbb{R}^n$ (or $\mathbb{R}^2$) be an open set.

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial^2 f}{\partial y \partial x}$$

Ex. $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \end{cases}$

$$f_{yx}(0,0) = \frac{\partial f_y}{\partial x}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}$$

But $f_y(h,0) = \lim_{k\to 0} \dfrac{f(h,k) - f(h,0)}{k} = h.$

$$\therefore f_{yx}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1$$

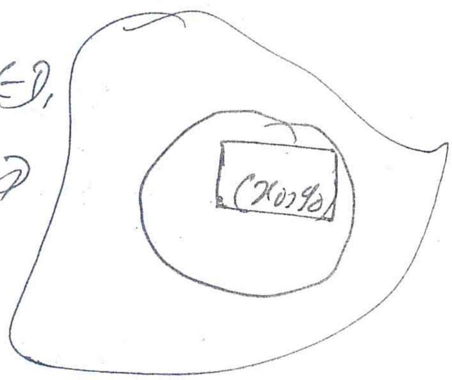
Similarly, $f_{xy}(0,0) = -1 \neq f_{yx}(0,0)$.

Notations: $C^1(D)$ — Set of all differentiable functions and whose derivative is continuous.
(or f_x & f_y both are cont)

$C^2(D)$ - set of all functions on D whose p.d. upto 2nd order are conti. ($f_x, f_y, f_{xy}, f_{yx}, f_{xx}, f_{yy}$ are cont).

Result: If D is open & $f \in C^2(D)$, then $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.

proof: Since D is open & $(x_0,y_0) \in D$, $\exists$ an open ball $B_\delta(x_0,y_0) \subset D$ & we can ~~draw~~ draw a rectangle

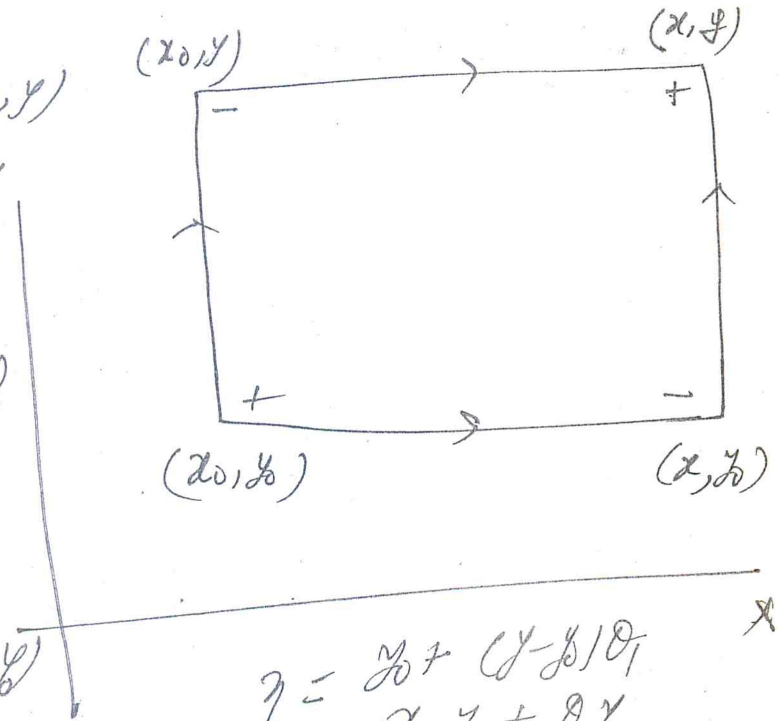


Let $F(x,y) = f(x,y) - f(x_0,y) + f(x_0,y_0) - f(x,y_0)$ —(1)

Again, let

$A(x,y) = f(x,y) - f(x_0,y)$.

From (1), we get



$$F(x,y) = A(x,y) - A(x_0,y_0)$$

$$= \frac{\partial A}{\partial y}(x,\eta)(y-y_0). \qquad \eta = y_0 + (y-y_0)\theta_1$$

$$\cancel{\eta = y - y_0 + \theta y_0}$$

$$= \left(\frac{\partial f}{\partial y}(x,\eta) - \frac{\partial f}{\partial y}(x_0,\eta)\right)(y-y_0).$$

$$F(x,y) = \frac{\partial^2 f}{\partial x \partial y}(\xi,\eta)(x-x_0)(y-y_0), \quad \cancel{\xi = x+x_0+\ldots} \quad \xi = x_0+(x-x_0)\theta_2$$

$$\frac{F(x,y)}{(x-x_0)(y-y_0)} = \frac{\partial^2 f}{\partial x \partial y}(\xi,\eta).$$

Since $(x,y) \to (x_0,y_0) \Rightarrow (\xi,\eta) \to (x_0,y_0)$.

and $\dfrac{\partial^2 f}{\partial x \partial y}$ is cont at (x_0,y_0),

$$\lim_{(x,y)\to(x_0,y_0)} \frac{F(x,y)}{(x-x_0)(y-y_0)} = \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0). \qquad \text{—— (2)}$$

Similarly, let $B(x,y) = f(x,y) - f(x_0,y)$.

Then $F(x,y) = B(x,y) - B(x_0,y)$.

It is easy to see that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{F(x,y)}{(x-x_0)(y-y_0)} = \frac{\partial^2 f}{\partial y \partial x}(x_0,y_0). \qquad \text{— (3)}$$

From (2) & (3), $\dfrac{\partial^2 f}{\partial x \partial y}(x_0,y_0) = \dfrac{\partial^2 f}{\partial y \partial x}(x_0,y_0)$.

~~Note that if $f: D \subset \mathbb{R}^n \to \mathbb{R}$,~~

~~Note that if $D \subset \mathbb{R}^n$ & $f \in C^2$~~

Note that if $f \in C^2(\mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^n$,
then $$\frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j} \quad \forall j, k = 1, 2, \ldots n.$$

<u>Taylor's theorem:</u>

Let $\mathcal{D}$ be an open set in $\mathbb{R}^2$ & $f \in C^2(\mathcal{D})$. Then $\exists$ $\lambda \in (0,1)$ s.t.
$$f(X+H) = f(X) + f'(X)H + H^t f''(C)H,$$
where $C = X + \lambda H$. & $\|H\| < \delta$.

<u>proof:</u> Let $g(t) = f(X+tH)$, $\varphi(t) = X + tH$.
Then $g(t) = f \circ \varphi(t)$. Then
$$g'(t) = f'(\varphi(t))\varphi'(t) = f'(\varphi(t))H$$
$$= h f_x(\varphi(t)) + k f_y(\varphi(t)).$$
$$g''(t) = h (f_x)'(\varphi(t))\varphi'(t) + k (f_y)'(\varphi(t))\varphi'(t)$$
$$= h\left(f_{xx}(\varphi(t)) \; f_{xy}(\varphi(t))\right)H$$
$$+ k\left(f_{yx}(\varphi(t)) \; f_{yy}(\varphi(t))\right)H$$
$$= H^t \begin{pmatrix} f_{xx}(\varphi(t)) & f_{xy}(\varphi(t)) \\ f_{yx}(\varphi(t)) & f_{xy}(\varphi(t)) \end{pmatrix} H$$
where $H^t = (h \; k)$ [row vector].

Since $g(0) = f(X)$, $g(1) = f(X+H)$. By 2nd MVT, for one variable

$$g(1) = g(0) + g'(0)\cdot 1 + \frac{1}{2} g''(\lambda)\, 1^2.$$

$$\Rightarrow f(X+H) = f(X) + f'(X)H + \frac{1}{2} H^T f''(c) H$$

$$c = X + \lambda H.$$

Result: Let $f : [a,b] \longrightarrow \mathbb{R}^n$ be diff on (a,b) & cont on $[a,b]$. Then $\exists\ \lambda \in (a,b)$ st

$$\|f(b) - f(a)\| \le \|f'(\lambda)\| (b-a).$$

proof:

Let $g(t) = (f(b) - f(a)) \cdot f(a + (b-a)t)$

Then $g'(t) = (f(b) - f(a)) \cdot f'(a + (b-a)t)(b-a)$ (By chain rule)

Since $g : [a,b] \xrightarrow{\text{diff}} \mathbb{R}$. by MVT, $\exists\ \lambda \in (a,b)$ st

$$g(b) - f(a) = g'(\lambda)(b-a).$$

$$\|f(b) - f(a)\|^2 = (f(b) - f(a)) \cdot f'(\lambda)(b-a)$$

$$\le \|f(b) - f(a)\| \, \|f'(\lambda)\| (b-a)$$

$$\Rightarrow \|f(b) - f(a)\| \le \|f'(\lambda)\| (b-a).$$

Result: Let D be open in $\mathbb{R}^n$ &

$f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be diff at $X \in D$.

Then $\exists\ \lambda \in (0,1)$ st

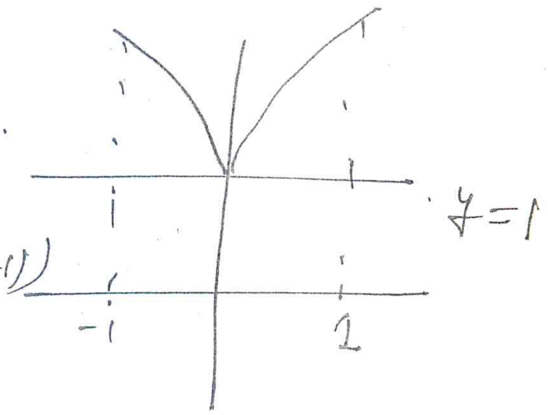
$\| f(X+H) - f(X) \| \leq \| f'(C) \| \, \| H \|$,

where $C = X + \lambda H$, $\|H\| < \delta$ (for some $\delta > 0$).

Note: Equality need not holds. For

$g : (-1,1) \longrightarrow \mathbb{R}^2$

$g(t) = (t^3, 1-t^2)$.



Suppose $g(1) - g(-1) = g'(\lambda)(1-(-1))$

$(2,0) = 2(3\lambda^2, -2\lambda)$

$\lambda = 0, \pm \frac{1}{\sqrt{3}}$.

But $x = t^3$, $y = 1-t^2$, $x^2 = (1-y)^3$.

has no tangent parallel to x-axis.

Proof: Let $g(t) = f(X + tH)$. Then

$g : [0,1] \longrightarrow \mathbb{R}^m$ be diff.

By previous MVT, $\exists\ \lambda \in (0,1)$ st

$\| g(1) - g(0) \| \leq \| g'(\lambda) \| (1-0)$.

$\| f(X+H) - f(X) \| \leq \| g'(\lambda) \| \leq \| f'(C) \| \, \| H \|$

$g'(\lambda) = f'(X + \lambda H) H$, $C = X + \lambda H$.