Ex. Let $\nabla f = (f_x, f_y)$, as long as $f_x(X_0)$ & $f_y(X_0)$ just exist (f need not differentiable at X_0).

Now, if f is differentiable

$$D_v f(X_0) = f'(X_0) = (f_x(X_0), f_y(X_0)) = \nabla f(X_0).$$

Ex. $$f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & \text{if } y \neq 0 \\ 0 & \text{o.w.} \end{cases}$$

Then f is cont at (0,0) &

$$D_v f(0,0) = \frac{v_2}{|v_2|} \text{ or } 0 \text{ if } v_2 = 0$$

But f is not diff at (0,0).

$$\epsilon(h,k) = \frac{f(h,k) - f(0,0) - 0.h - 1.k}{\sqrt{h^2+k^2}}$$

$$= \frac{\frac{k}{|k|}\sqrt{h^2+k^2} - k}{\sqrt{h^2+k^2}} = \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}}$$

For $h = mk$, $m, k > 0$,

$$\epsilon(mk,k) = 1 - \frac{1}{\sqrt{1+m^2}} \not\to 0 \text{ as } k \to 0.$$

Ex. Show that

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{o.w.} \end{cases}$$

is diff at $(0,0)$ & $f'(0,0) = (0,0)$. But non of f_x & f_y is cont. at $(0,0)$.

Result: Let D be an open set in $\mathbb{R}^2$. Suppose f_x & f_y are cont in a nbd of $(x_0,y_0) \in D$. Then f is diff at (x_0,y_0).

Pf: Since $(x_0,y_0) \in D$ & D is open, $\exists\ \delta > 0$ st $B_\delta((x_0,y_0)) \subset D$. Let $(x_0+h, y_0+k) \in B_\delta((x_0,y_0))$

~~Then~~ Consider

$$e(h,k) = \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - h f_x(x_0,y_0) - k f_y(x_0,y_0)}{\sqrt{h^2+k^2}}$$

Since f_x & f_y exist in a $B_\delta(x_0,y_0)$ (say), we can apply MVT co-ordinate wise.

Thus,

$$e(h,k) = \frac{h f_x(x_0+\theta_1 h, y_0+k) + k f_y(x_0, y_0+k\theta_2) - h f_x(x_0,y_0) - k f_y(x_0,y_0)}{\sqrt{h^2+k^2}}$$

$$|e(h,k)| \le \frac{\sqrt{h^2+k^2}\left[\left\{ f_x(x_0+h\theta_1, y_0+k) - f_x(x_0,y_0)\right\}^2 + \cancel{\left(f_y(x_0, y_0 \right.} + \left(f_y(x_0, y_0+\theta_2 k) - f_y(x_0,y_0)\right)^2 \right]}{\sqrt{h^2+k^2}}$$

Since f_x & f_y are cont in $B_\delta((x_0,y_0))$,

$|e(h,k)| \to 0$ as $\sqrt{h^2+k^2} \to 0$

Thus f is diff. at (x_0,y_0).

Geometric interpretation of ~~derivative~~ for function from $\mathbb{R}^n \to \mathbb{R}$.

Let $z = f(x_0) + f'(x_0)(x - x_0)$

For $n=1$, $y = f(x_0) + f'(x_0)(x - x_0)$

line passing through $(x_0, f(x_0))$.

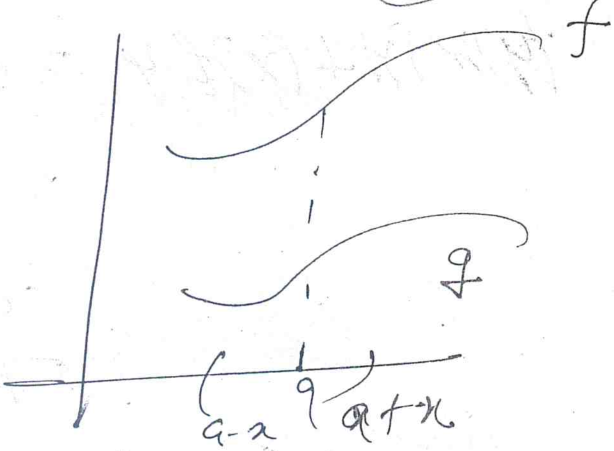
For $n=2$

$$z = f(x_0) + f'(x_0)(x - x_0)$$

$$z = f(x_0,y_0) + (x-x_0) f_x(x_0,y_0) + (y-y_0) f_y(x_0,y_0)$$

is a plane passing through $(x_0, y_0, f(x_0,y_0))$

Chain rule:



$$I \subset \mathbb{R} \xrightarrow{g} J \xrightarrow{f} \mathbb{R}$$

$$F = f \circ g$$

If f & g both are differentiable, then $f \circ g$ is diff.

Pf: Since f is diff. at $y = g(x)$.

$$f(y+k) - f(y) - f'(y)k = k\eta(k) \quad \text{— (1)}$$

when $\eta(k) \to 0$ as $k \to 0$.

Since g is diff at x, g is cont and set $k = g(x+h) - g(x)$. then

~~hence~~ $h \to 0 \Rightarrow k \to 0$.

Since g is diff.,

$$k = g(x+h) - g(x) = hg'(x) + h\mu(h).$$

when $\mu(h) \to 0$ as $h \to 0$.

Consider

$$e(h) = \frac{f \circ g(x+h) - f \circ g(x) - f'(g(x))\, g'(x)\, h}{h}$$

$$= \frac{f(y+k) - f(y) - f'(y)\,(k - h\mu(h))}{h}$$

Since $\frac{k}{h} = \frac{g'(x) + \mu(h)}{1}$,

$\epsilon(h) = \eta(k)\{g'(x)+\mu(h)\} + f'(y)\mu(h)$

Since $h \to 0 \Rightarrow k \to 0 \Rightarrow \eta(k) \to 0$.

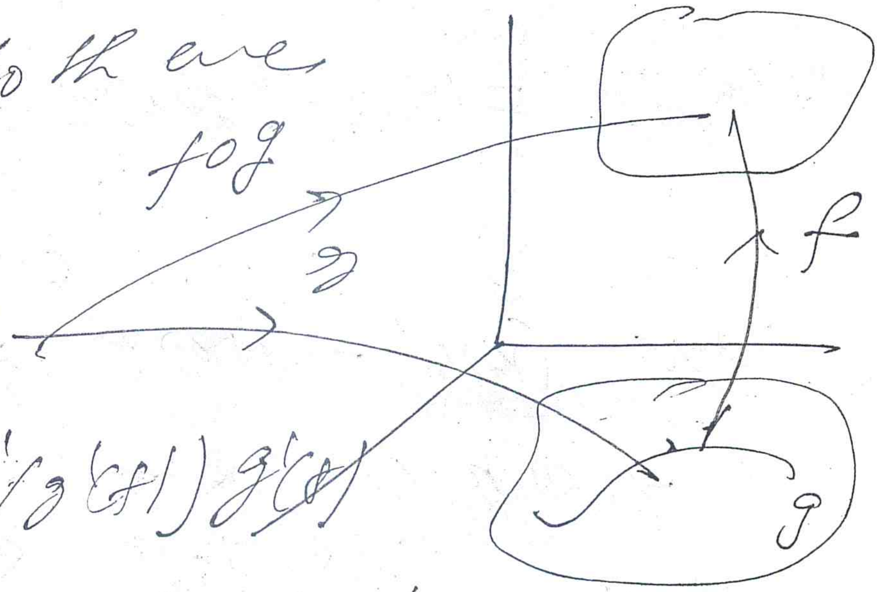
$\epsilon(h) \to 0$. Thus fog is diff &

$(fog)'(x) = f'(g(x))\, g'(x)$.

Chain rule for $R^2 \to R$:

If f & g both are differentiable, fog then fog is diff &

$[a,b]$



$(fog)'(x) = f'(g(x))\, g'(x)$

Pf:

$\eta(k) = \dfrac{f(y+k) - f(y) - f'(y)h}{\|k\|} \to 0$ as $\|k\| \to 0$.

Since g is cont, set

$k = g(x+h) - g(x)$. Then

~~$= h\, g'(x+h)$~~

$\|k\| \to 0$ as $|h| \to 0$.

Since g is diff of x.

$k = g(x+h) - g(x) = h\,g'(x) + |h|\,\mu(h)$.

i.e. $\|k\| \leq |h| \, \|g'(x)\| + |h| \, \|\mu(h)\|$ (41)

now,

$$e(h) = \frac{f\circ g(x+h) - f\circ g(x) - f'(g(x))\, g'(x) h}{|h|}$$

$$|e(h)| \leq |\eta(k)| \left(\|g'(x)\| + \|\mu(h)\| \right) + \|f'(y)\| \, \|\mu(h)\|$$

$\to 0$ as $h \to 0$, because $h \to 0 \Rightarrow k \to 0$.

Thus $(f\circ g)'(x) = \underbrace{f'(g(x))}_{1\times 2} \underbrace{g'(x)}_{2\times 1}$

MVT for Convex domain:

Let D be an open & convex set in $\mathbb{R}^2$. Suppose $f: D \to \mathbb{R}$ is diff. Then for any $X, Y \in D$, $\exists$ $C \in D$ st

$$f(X) - f(Y) = (X - Y) f'(C),$$

where $C \in (X, Y) = \{\lambda X + (1-\lambda) Y : 0 < \lambda < 1\}$.

Pf: Consider

$\phi(t) = f((1-t)X + tY)$. Then by chain rule ϕ is diff on $(0, 1)$. and

$\phi'(t) = f'((1-t)X + tY) \cdot (Y - X)$

By MVT for one variable,

$$\phi(1)-\phi(0) = \phi'(\lambda)(1-0)$$

$$f(Y)-f(X) = f'((1-\lambda)X+\lambda Y)(Y-X).$$

Function from $\mathbb{R}^n \longrightarrow \mathbb{R}^m$:

Let D be an open set in $\mathbb{R}^n$ & $f: D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be ~~differ~~ differentiable. Then

$$f'(X_0) = \left(\frac{\partial f_i(X_0)}{\partial x_j}\right)_{m\times n}$$

Pf:

We know that $f: D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is diff at X_0 if $\exists$ $A_{m\times n}$ matrix

s.t. $$E(H) = \frac{f(X_0+H) - f(X_0) - AH}{\|H\|} \to 0 \quad \text{--- (1)}$$

as $\|H\| \to 0$.

Let $\{e_1, \dots, e_n\}$ & $\{u_1, u_2, \dots, u_m\}$ be the S.B. for $\mathbb{R}^n$ & $\mathbb{R}^m$ resp.

$$f = (f_1, \dots, f_m)$$

then $f_i(X) = f(X) \cdot u_i$

In (1) substitute $H = h_j e_j$, $\|H\| = |h_j|$.

$$e(h_j e_j) = \frac{f(x_0 + h_j e_j) - f(x_0) - h_j f'(x_0) e_j}{|h_j|} \longrightarrow 0 \text{ as } h_j \to 0.$$

$\Longleftrightarrow$

$$\lim_{h_j \to 0} \frac{f(x_0+h_j e_j) - f(x_0)}{h_j} = f'(x_0) e_j$$

$$\left(\frac{\partial f_1}{\partial x_j}(x_0), \ldots, \frac{\partial f_m}{\partial x_j}(x_0)\right) = f'(x_0) e_j$$

$\Rightarrow \dfrac{\partial f_i}{\partial x_j}(x_0)$ exists &

$$f'(x_0) = \left(\frac{\partial f_i}{\partial x_j}(x_0)\right)_{m \times n}$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{pmatrix}_{m \times n} \cancel{(x_0)}.$$

write $J_f(x_0) = \left(\frac{\partial f_i(x_0)}{\partial x_j}\right)_{m\times n}$. Then

$J_f(x_0)$ is called Jacobian matrix of f.

Note that existence of $\frac{\partial f_i}{\partial x_j}(x_0)$ does not imply that $f'(x_0)$ exists.

Ex. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$f(x,y) = \begin{cases} \left(\frac{x^2y}{x^2+y^2}, \frac{xy^2}{x^2+y^2}\right) & \text{if } x^2+y^2 \neq 0 \\ (0,0) & \text{o.w.} \end{cases}$$

Then $f = (g, h)$

$$J_f(0,0) = \begin{pmatrix} g_x & g_y \\ h_x & h_y \end{pmatrix}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

But f is not differentiable at $(0,0)$.

$$\|E(h,k)\| = \frac{\left\| \left(\frac{h^2k}{h^2+k^2}, \frac{hk^2}{h^2+k^2}\right) - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} \right\|}{\sqrt{h^2+k^2}}$$

$$\|E(h,k)\| = \frac{|hk|}{h^2+k^2} \not\to 0 \text{ as } \sqrt{h^2+k^2} \to 0.$$

$\Rightarrow f$ is not diff at $(0,0)$. (45)

Ex. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) = (e^x \cos y,\ e^x \sin y)$$

$\det(J_f(x,y)) = e^{2x} \neq 0$, $\Rightarrow J_f(x,y)$ is non singular matrix, $\forall (x,y) \in \mathbb{R}^2$, but f is not 1-1 on $\mathbb{R}^2$

$$f(x, 2\pi + y) = f(x,y).$$

Norm of a matrix (or linear map)

Let $A : \mathbb{R}^n \xrightarrow{\text{linear}} \mathbb{R}^m$ Then

$A = (R_1 \cdot R_2 \cdots R_m)^T$, where R_i's are rows of A. Let $X \in \mathbb{R}^n$.

Then $AX = (R_1 \cdot X,\ R_2 \cdot X,\ \ldots,\ R_m \cdot X) \in \mathbb{R}^m$

$$\|AX\| = \sqrt{\sum |R_i \cdot X|^2}$$

$$\leq \left(\sqrt{\sum \|R_i\|^2}\right) \|X\|$$

If $X \neq 0$, $\dfrac{\|AX\|}{\|X\|} \leq \sqrt{\sum \|R_i\|^2}$

i.e. $\left\{ \dfrac{\|AX\|}{\|X\|} : X \neq 0 \right\}$ is bounded in $\mathbb{R}$.

Hence, it has supremum. Let

$\|A\| := \sup_{X \neq 0} \frac{\|AX\|}{\|X\|} < \infty$. Then (46)

(i) $\|AX\| \leq \|A\| \|X\|, \quad \forall X \in \mathbb{R}^n$.

(ii) $\|A\| = \sup_{\|X\|=1} \|AX\|$.

Ex. $A : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad A(x,y) = 4x+3y$.

$\|A\| = \sup_{x^2+y^2=1} |4x+3y| = \sup_{-1 \leq x \leq 1} |4x+3\sqrt{1-x^2}|$.

Ex. $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad A(x,y) = (3x, 4y)$.

$\|A\| = \sup_{x^2+y^2=1} \|(3x,4y)\| = \sup_{x^2+y^2} \sqrt{9x^2+16y^2}$

$= \sup_{0 \leq x \leq 1} \sqrt{9x^2+16(1-x^2)}$

Chain rule for function from $\mathbb{R}^n \rightarrow \mathbb{R}^m$.

Let D be an open set in $\mathbb{R}^n$ &

$f : D \subset \mathbb{R}^n \xrightarrow{\text{diff}} \mathbb{R}^m$ &

$g : f(D) \xrightarrow{\text{diff}} \mathbb{R}^l$.

Then $g \circ f : D \longrightarrow \mathbb{R}^l$ is diff

and $(g \circ f)'(X) = \underbrace{g'(f(X))}_{l \times m} \underbrace{f'(X)}_{m \times n}$.