We know that $m^* : P(\mathbb{R}) \longrightarrow [0, \infty]$.

Restrict m^* to M. Then for $E \in M$, we write $m^*(E) = m(E)$.

i.e. $m^*|_M = m$ (say).

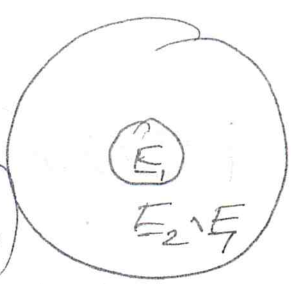
<u>Result</u>: Let $(E_n) \subset M$ be a $\uparrow$ seq^n of sets.

Then $\lim_{n\to\infty} m(E_n) = m\left(\bigcup_{n=1}^{\infty} E_n\right)$. —— (*)

<u>Proof</u>: Let $E = \bigcup E_n$. If $m(E) = \infty$, then some of $m(E_{n_0}) = \Phi$. Hence (*) holds. Therefore, suppose $m(E_n) < \infty$, $\forall\, n \geq 1$. Since $m(E_n)$ $\uparrow$ seq^n.

$$\lim m(E_n) = \sup m(E_n) \leq m(E).$$

Now, $\bigcup_{n=1}^{\infty} E_n = E_1 \,\dot{\cup}\, \dot{\bigcup_{n=1}^{\infty}} (E_{n+1} \setminus E_n)$



$$\therefore m(E) = m(E_1) + \sum_{n=1}^{\infty} m(E_{n+1} \setminus E_n)$$

$$= m(E_1) + \lim_{k\to\infty} \sum_{n=1}^{k} \left(m(E_{n+1}) - m(E_n)\right)$$

$$= \lim_{k\to\infty} m(E_{k+1}).$$

<u>Result</u>: Let $(E_n) \subset M$ be a $\downarrow$ seq^n of sets s.t. $m(E_1) < \infty$. Then

$$\lim m(E_n) = m\left(\bigcap_{n=1}^{\infty} E_n\right).$$

Proof: Since $m(E_n) \geq m(E_{n+1}) \geq m(\bigcap E_n)$

$\lim m(E_n) = \inf m(E_n) \geq m(\bigcap E_n)$. 133

$$E_1 \setminus \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E_n \setminus E_{n+1}) \quad (\text{Ex.})$$

$$m\left(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n \setminus E_{n+1})$$

$$m(E_1) - m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{k\to\infty} \sum_{n=1}^{k} \left(m(E_k) - m(E_{k+1})\right)$$

$$= m(E_1) - \lim_{k\to\infty} m(E_{k+1}).$$

$$\Rightarrow m(\bigcap E_n) = \lim_{k\to\infty} m(E_{k+1}).$$

Alternative: $E_1 \setminus E_n \uparrow$ seq n.

$$\lim m(E_1 \setminus E_n) = m(\bigcup (E_1 \setminus E_n))$$

$$m(E_1) - \lim m(E_n) = m(E_1 \setminus \bigcap E_n)$$

$$= m(E_1) - m(\bigcap E_n).$$

so $\lim m(E_n) = m(\bigcap E_n)$.

Ex: $E \in \mathcal{M}$ iff $E \cap [a,b] \in \mathcal{M}$, $\forall\, a,b \in \mathbb{R}$.

Solution: If $E \in \mathcal{M}$, obviously $E \cap (a,b) \in \mathcal{M}$ for any $a,b \in \mathbb{R}$, because $(a,b) \in \mathcal{M}$.

Suppose $E \cap (a,b) \in \mathcal{M}$, $\forall\, a,b \in \mathbb{R}$.

Then $E \cap (k, k+1] = E \cap (k, k+1) \cup (E \cap \{k+1\})$
$\in \mathcal{M}$ $(\because m^*(E \cap \{k+1\}) = 0)$

But $E = \bigcup_{k \in \mathbb{Z}} (E \cap (k, k+1]) \in \mathcal{M}$. (134)

Result: $E \in \mathcal{M}$ iff $\forall A \subset \mathbb{R}$, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E) \quad \text{--- (1)}.$$

But proving (*), it is enough to prove

$$m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E).$$

Proof: ~~If~~ $m^*(A) = \infty$, then (1) is true.

Suppose $m^*(A) < \infty$, & $E \in \mathcal{M}$. Then $\exists$ G_δ-set $G \supseteq A$ st $m^*(A) = m^*(G)$. $(\because G = \bigcap_{n=1}^{\infty} O_n)$

$$\therefore m^*(A \cap E) + m^*(A \setminus E) \leq m^*(G \cap E) + m^*(G \setminus E).$$
$$= m^*((G \cap E) \cup (G \setminus E))$$
$$= m^*(G) = m^*(A).$$

Now, let (1) holds. Claim $E \in \mathcal{M}$.

First consider $m^*(E) < \infty$. Then $\exists$ G_δ-set G st $E \subseteq G$ & $m^*(G) = m^*(E) < \infty$.

Since (1) is true for all $A \subset \mathbb{R}$,

$$m^*(G) = m^*(G \cap E) + m^*(G \setminus E)$$
$$\text{ie } m^*(G) = m^*(G) + m^*(G \setminus E)$$
$$\text{ie } m^*(G \setminus E) = 0 \Rightarrow G \setminus E \in \mathcal{M}.$$

But $G \setminus (G \setminus E) = E \Rightarrow E \in M$. If $m^*(E) = \infty$, then, write $E = \bigcup_{n \in \mathbb{Z}} (E \cap (n, n+1]) = \dot{\bigcup}_{n \in \mathbb{Z}} E_n$.

We claim that $E \in M$. For this, we all need to prove that if E_1 & E_2 satisfy (1), then $E_1 \cap E_2$ satisfies (1). From the bounded case $(n, n+1] \in M \Leftrightarrow (n, n+1]$ satisfies (1). Thus,

$$m^*(A) = m^*(A \cap E_n) + m^*(A \setminus E_n).$$

Since $E_n = E \cap (n, n+1]$. Hence, by the bounded case $E_n \in M$. Since $E = \dot{\bigcup} E_n$ $\Rightarrow E \in M$.

Now, $m^*(A) = m^*(E_1 \cap A) + m^*(A \setminus E_1)$ —(2)

$m^*(A) = m^*(E_2 \cap A) + m^*(A \setminus E_2)$ —(3)

Replace A in (3) by $A \cap E_1$ & $A \setminus E_1$ & use them in (2), then

$$\begin{aligned} \text{RHS of (1)} &= m^*(E_1 \cap E_2 \cap A) + m^*(A \cap E_1 \setminus E_2) \\ &\quad + m^*(E_2 \cap (A \setminus E_1)) + m^*(A \setminus E_1 \setminus E_2) \\ &\geq m^*\big((E_1 \cap E_2 \cap A) \cup (A \cap E_1 \setminus E_2) \cup (E_2 \cap (A \setminus E_1)) \\ &\qquad \cup (A \setminus E_1 \setminus E_2)\big) \\ &\geq m^*(A) \quad (\text{verify (})\end{aligned}$$

Thus, $(E_1 \cup E_2)^c = E_1^c \cap E_2^c$ will satisfying (1) as (1) is closed under complement. (1) is called Caratheodory criteria of measurability.

Lebesgue measurable functions:

Let $\mathcal{J}_u$ = Collection of all open subsets of $\mathbb{R}$ w.r.t usual metric u on $\mathbb{R}$.

$= \{ O \subseteq \mathbb{R} : O = \bigcup_{n=1}^{\infty} I_n ,\ I_n = (a_n, b_n) \}$

& $\mathcal{M}$ = Class of all L-measurable subsets of $\mathbb{R}$.

$\mathcal{J}_{d_0}$ = Collection of all open sets of $\mathbb{R}$ w.r.t d_0 - the discrete metric on $\mathbb{R}$ $= P(\mathbb{R})$.

$\Rightarrow \quad \mathcal{J}_u \subsetneq \mathcal{M} \subsetneq \mathcal{J}_{d_0} = P(\mathbb{R})$.

Since $\mathcal{J}_u$ is not closed under countable intersection (& complement) of open sets.

$\Rightarrow \mathcal{J}_u \subsetneq \mathcal{M}$, and $\mathcal{M} \subsetneq \mathcal{J}_{d_0}$, because every subset need not be L-mble.

Consider $f : (\mathbb{R}, \mathcal{J}_u) \xrightarrow{\text{continuous}} (\mathbb{R}, \mathcal{J}_u)$.

Then $f^{-1}(O) \in \mathcal{J}_u$, $\forall\, O \subset \mathcal{J}_u$ (from range side)

Now, if $f : (\mathbb{R}, \mathcal{M}) \longrightarrow (\mathbb{R}, \mathcal{J}_u)$, what happen to $f^{-1}(O)$? If f is continuous on $(\mathbb{R}, \mathcal{J}_u)$, then $f^{-1}(O)$ is open & hence $f^{-1}(O) \in M$.

In addition, consider $f(x) = \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$, then f cannot be made continuous at 0, ~~and~~ ~~hence~~ set $f(x) = \infty$ iff $x = 0$. (Important!) (137)

If we want to take $f(x) = \frac{1}{x}$ into consideration, we have to ~~exlam~~ extend the range $(-\infty, \infty)$ to $[-\infty, \infty]$

let $\mathbb{R} = (-\infty, \infty)$ & $\overline{\mathbb{R}} = [-\infty, \infty]$.

Therefore, the sets $[-\infty, a)$ & $(b, \infty]$ for $a, b \in \mathbb{R}$ should be added to J_4. That is $\overline{J_4} = J_4 \cup \{[\infty, a) \cup (a, \infty] : a, b \in \mathbb{R}\}$.

Def^n: $f : (\mathbb{R}, M) \longrightarrow (\overline{\mathbb{R}}, \overline{J_4})$ is said to be L-measurable if $f^{-1}(O) \in M$, $\forall\, O \in \overline{J_4}$.

Since $O \in \overline{J_4}$, can be expressed as the countable union / intersection of sets of the form $[-\infty, a)$ & $(b, \infty]$, and M is closed under countable union/intersection it is enough to consider $O = (b, \infty]$ or $[-\infty, a)$.

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Thus $f : (\mathbb{R}, M) \longrightarrow (\overline{\mathbb{R}}, \overline{J_4})$ or $\overline{\mathbb{R}}$ is L-msble if $f^{-1}\{(\alpha, \infty]\} \in M$, $\forall \alpha \in \mathbb{R}$.

<u>Result</u>: If $f : (\mathbb{R}, M) \longrightarrow \overline{\mathbb{R}} = [-\infty, \infty]$.

Then FAE

(i) $f^{-1}\{(\alpha, \infty]\} \in M$, $\forall \alpha \in \mathbb{R}$.

(ii) $f^{-1}\{[\alpha, \infty]\} \in M$ $\forall \alpha \in \mathbb{R}$.

(iii) $f^{-1}\{[-\infty, \alpha)\} \in M$, $\forall \alpha \in \mathbb{R}$.

(iv) $f^{-1}\{[-\infty, \alpha]\} \in M$, $\forall \alpha \in \mathbb{R}$

(v) $f^{-1}\{\pm\infty\} \in M$ & $f^{-1}\{(a, b)\} \in M$ $\forall a, b \in \mathbb{R}$.

<u>Proof</u>: (i) $\Rightarrow$ (ii):

$$[\alpha, \infty] = \bigcap_{n=1}^{\infty} (\alpha - \tfrac{1}{n}, \infty] \ni x, \text{ let}$$

~~but~~ $x \notin [\alpha, \infty] \Rightarrow \alpha > x > \alpha - \frac{1}{n}, \forall n \geq 1$

$\Rightarrow \alpha = x = \alpha$

which is a contradiction.

Since M is closed under complement, (ii) $\Rightarrow$ (iii). Now, (iii) $\Rightarrow$ (iv), because $[-\infty,\alpha] = \bigcap_{n=1}^{\infty} [-\infty, \alpha+\frac{1}{n})$. 139

(iv) $\Rightarrow$ (i) as M is closed under complement ($\because M^c = M$)

Thus, (i) to (iv) are equivalent.

Here $f^{-1}\{\infty\} = \bigcup f^{-1}\{(n,\infty]\} \in M$ (by (i))

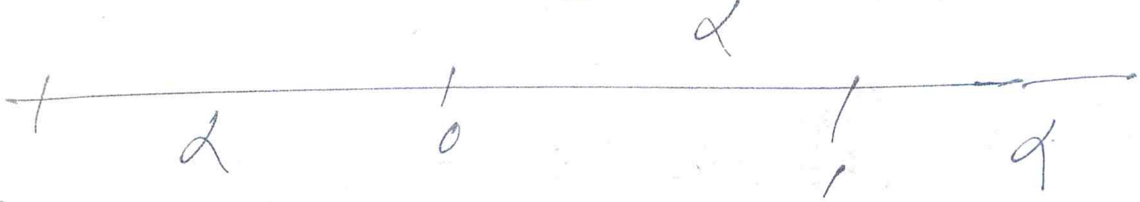
$f^{-1}\{-\infty\} = \bigcup f^{-1}\{[-\infty,-n)\} \in M$ (by (iii))

$(a,b) = (a,\infty] \cap [b,\infty]$

$\Rightarrow f^{-1}\{(a,b)\} \in M, \ \forall\, a,b \in \mathbb{R}$.

Ex. Let $E \in M$, define

$$f(x) = \chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$



$$f^{-1}\{(\alpha,\infty]\} = \begin{cases} E & \alpha = 0 \\ E & 1 > \alpha > 0 \\ \phi & \alpha > 1 \\ \mathbb{R} & \alpha < 0 \end{cases}$$

Ex. $f: \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, $f(x) = K$ is L-m'sble.

$$f^{-1}\{(\alpha, \infty]\} = \begin{cases} \phi & \alpha \geq K \\ \mathbb{R} & \alpha < K \end{cases} \quad \bigg\} \ K\text{-finite}$$

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if $K = \infty$, $f(x) = \infty$, $\forall x \in \mathbb{R}$. Then

$$f^{-1}\{(\alpha, \infty]\} = \mathbb{R}.$$

Notice that for $\alpha \in \mathbb{R}$, $\exists$. $r_j \in \mathbb{Q}$ s.t. $r_j \uparrow \alpha$.

$f(x) \geq \alpha \Rightarrow f(x) \geq \alpha \geq r_j$, $\forall j$ $\quad$ [r_j increases to α, $r_j \to \alpha$]

$$\{x : f(x) > \alpha\} = \bigcap_{j=1}^{\infty} \{x : f(x) > r_j\}$$

Thus, f is L-m'sble iff $f^{-1}\{(r_j, \infty]\} \in M$, $\forall r_j \in \mathbb{Q}$

Ex. If $f, g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ be L-m'sble

s.t. $f(x) + g(x) \neq \infty - \infty$, for any $x \in \mathbb{R}$.

Then $f+g$ is L-m'sble.

Thus, we need to show the following sets to L-m'sble.

$A = \{x \in \mathbb{R} : f(x) + g(x) = \pm\infty\}$ & f

$B = \{x \in \mathbb{R} : \infty > f(x) + g(x) > \alpha\}$, $\forall \alpha \in \mathbb{R}$.

$A = \{x \in \mathbb{R} \ f(x) = \pm\infty + g(x)\}$ if $g(x)$ are finite. (or otherwise)

for $x \in B$, $\infty > f(x) + g(x) > \alpha$, $\exists$ r_x st

$$f(x) > r_x > \alpha - g(x)$$

$$x \in \bigcup_{r \in \mathbb{Q}} \{x : f(x) > r\} \cap \{x : g(x) > \alpha - r\}$$

$\Rightarrow B = \bigcup_{r\in\mathbb{Q}} \left\{ \{x\in\mathbb{R} : f(x) > r\} \cap \{x\in\mathbb{R} : g(x) < \alpha - r\} \right\}$

(41)

$\Rightarrow B \in \mathcal{M}.$

~~Ex $4fg = (f+g)^2 - (f-g)^2 \Rightarrow fg$ is L-msble iff f^2 is.~~

Ex. $\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : -f(x) > \sqrt{\alpha}\}$
$\in \mathcal{M}.$

Ex $4fg = (f+g)^2 - (f-g)^2 \Rightarrow$ If f, g are L-msble, then f^2, fg are L-msble.

Defn: A property P is called "holding almost everywhere" if the places (& pts) ~~which~~ where it false has L-measure zero.
ie P is true almost everywhere.

$m^*\{x\in\mathbb{R} : P \text{ is false}\} = 0.$

If $f = g$ a.e on $\mathbb{R}$, then
$m^*\{x\in\mathbb{R} : f(x) \neq g(x)\} = 0.$

Ex. If $f : \mathbb{R} \to \overline{\mathbb{R}}$ & $f(x) = 0$ for a.e $x\in\mathbb{R}$, then f is L-msble.

let $E = \{x\in\mathbb{R} : f(x) \neq 0\}$, then
$m^*(E) = 0 \Rightarrow E, E^c \in \mathcal{M}$, etc.